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# Applications of Conformable Fractional Pareto Probability Distribution

**Duha Abu Judeh and Ma'mon Abu Hammad**

Department of Mathematics Al-Zaytoonah University of Jordan
E-mail: duhajudeh@yahoo.com
E-mail: m.abuhammad@zuj.edu.jo

***Abstract***

***In this paper looks at fractional isotopes conformable to some basic concepts linked to the probability distribution of random variables, which is density, cumulative distribution, survival, and hazard function. Furthermore, it introduces conformable fractional isotopes with the expected values, $r^{th}$ moments, mean, variance, skewness, and kurtosis. As well, it introduces conformable fractional isotopes with measures of entropy such as Shannon, Renyi, and Tsallis and characteristic function.***

**Keywords:** ***Entropy, conformable distribution, Pareto, and probability measures.***

## 1 Introduction

The Pareto distribution was named after Swiss economist Pareto (1848-1923) who established it while studying distributions for modeling income in Switzerland [3]. Since that time, the Pareto distribution has been widely used in modeling heavy-tailed distributions, such as income distribution. Many applications of the Pareto distribution in economics, biology and physics can be found throughout the literature. Discussed applications of the Pareto distribution in modeling earthquakes, forest fire areas oil-gas field sizes. Presented an application of the Pareto distribution in modeling disk drive sector errors. To add flexibility to the Pareto distribution, various generalizations of the distribution have been derived, including the generalized Pareto distribution, the beta-Pareto distribution [14] and the beta generalized Pareto distribution [15]. Conformable fractional derivative is a very contemporary definition. This definition appeared first time in 2014. [12]

Let $\alpha \epsilon (0,1]$ and $f: N \subseteq (0,\infty) \rightarrow \mathbb{R}$. For $u \in N$ let

$$D^{\alpha}(f)(u) = \lim_{\varepsilon \to 0} \frac{f(u+\epsilon u^{1-\alpha}) - f(u)}{\varepsilon}.$$

If the limit exists, then the above formula is called $\alpha -$ conformable fractional derivative of $f\ at\ u$. Observe that for $u = 0, D^{\alpha} f(0) = \lim_{u \to 0} D^{\alpha} f(0)$ if such limit exists. This definition satisfies the following properties:

$$D^{\alpha}(nf + mg) = nD^{\alpha}(f) + mD^{\alpha}(g), \text{for all } n, m \in \mathbb{R}.$$

Further, for $\alpha \in (0,1]$, let $f, g$ be the $\alpha$-differentiable functions at a point $s$ with $g(s) \neq 0$. Then we can have the following properties:

1) $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$ .
2) $D^{\alpha}\left(\frac{f}{g}\right) = \frac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}$.
3) $D^{\alpha}(x^{w}) = wx^{w-\alpha}\ for\ all\ w \epsilon \mathbb{R}$ .
4) $D^{\alpha} \sin\left(\frac{1}{\alpha} x^{\alpha}\right) = \cos\left(\frac{1}{\alpha} x^{\alpha}\right)$.
5) $D^{\alpha} \cos\left(\frac{1}{\alpha} x^{\alpha}\right) = -\sin\left(\frac{1}{\alpha} x^{\alpha}\right)$.
6) $D^{\alpha}\left(e^{\left(\frac{1}{\alpha}\right) x^{\alpha}}\right) = e^{\frac{1}{\alpha} x^{\alpha}}$.

Many studies use the definition of fractional derivative [8], [5], [6], [16], and [9].

## 2 Applications to Conformable $\alpha$ −Pareto Distribution

The conformable differential equation of $\alpha$-Pareto distribution can be expressed as follows: [1], [3], and [7]

$$x^{\theta+\alpha} D^{(\alpha)} y + \theta(\theta - \alpha + 2)\beta^{\theta} x^{-2+\alpha} = 0, \text{ where } x > \beta\ ; 0 < \alpha < 1.$$

By considering the function $f_{\alpha}(x) = \frac{A\theta\beta^{\theta}}{x^{\theta-\alpha+2}}$ to be a conformable fractional probability distribution function CFPDF with support $[\beta, \infty)$, we need to verify $\int_{\beta}^{\infty} f_{\alpha}(x)\, d^{\alpha}x = 1$. Hence, we have:

$$f_{\alpha}(x) = \beta^{\theta-2\alpha+2}(2\alpha + \theta + 2)\, x^{\alpha-\theta-2}. \tag{1}$$

It is conformable fractional probability density function (CFPDF) of a random variable X with support $(\beta, \infty)$. It is interesting to note that: [2] and [17]

$$\lim_{\alpha \to 1^{-}} f_{\alpha}(x) = \frac{\theta\beta^{\theta}}{x^{\theta+1}} \ . \tag{2}$$

which is a probability distribution function of Pareto distribution with parameters $(\beta, \theta)$. Actually, the conformable fractional probability distribution function (or simply CFPDF) can be plotted to be shown on in Figure 1 according to $\beta = 2.5$ , $\theta = 1.2$ and to different value of $\alpha$.

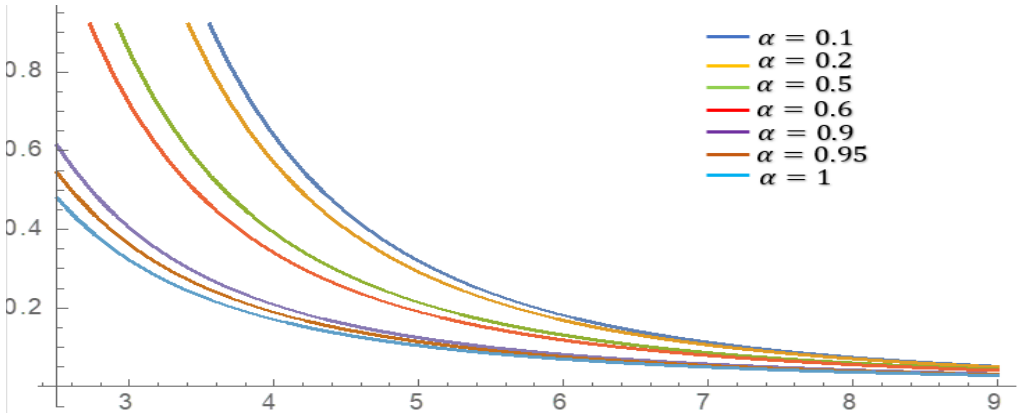


**Fig 1**. The CFPDF of $\alpha$ −Pareto distribution

Conclusion from the Fig.1:The larger the value of $\alpha$, the closer it is to the Pareto distribution.

## 2.1 The mode of the conformable fractional Pareto distribution

The (CFPDF) of X is given by (1). The logarithm to base e of $f_\alpha(x)$ is ( [1] and [13])

$$\log\big(f_\alpha(x)\big) = (\theta - 2\alpha + 2)\log(\beta) + \log(\theta - 2\alpha + 2) - (\theta - \alpha + 2)\log(x). \quad (3)$$

The conformable fractional derivative of $\log(f_\alpha(x))$ is then as:

$$D^{(\alpha)}(\log(f_\alpha(x))) = (\theta + 2 - \alpha)x^{-\alpha}. \quad (4)$$

One can set this derivative to be equal zero to get the critical point $x_0 = 0$. We note that the function is decreasing on $(\beta, \infty)$, Therefore, the smallest value of $f_\alpha$ is $\beta$, which represents the value of mode.

## 2.2 The conformable fractional cumulative distribution function (CFCDF) $\boldsymbol{\alpha}$CDF

In the some regard, the conformable fractional cumulative distribution function CFCDF for the $\alpha$ −Pareto distribution will be as follows:

$$F_\alpha(x) = \beta^{\theta-2\alpha+2}\big(\beta^{2\alpha-\theta-2} - x^{2\alpha-\theta-2}\big). \quad (5)$$

Proof $F_\alpha(x) = P_\alpha(X \le x)$.

$$F_\alpha(x) = \int_\beta^x \beta^{\theta-2\alpha+2}(\theta - 2\alpha + 2)\, t^{(\alpha-\theta-2)} d^\alpha t.$$

This gives:

$$F_\alpha(x) = \beta^{\theta-2\alpha+2}\left(\beta^{2\alpha-\theta-2} - x^{2\alpha-\theta-2}\right) x \geq \beta.$$

It should be noted that $\lim_{x\to-\infty} F_\alpha(x) = 0$ and $\lim_{x\to\infty} F_\alpha(x) = 1$.
This function can be plotted as shown in Figure 2 according to β=3 , $\theta$=1.5 and to different value of $\alpha$.

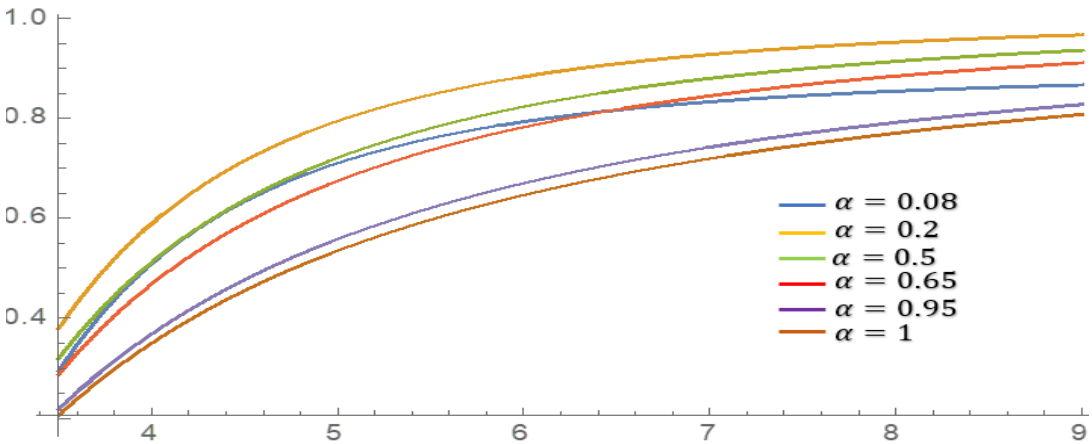


**Fig 2**. The CFCDF of $\alpha$ −Pareto distribution

## 2.3 The conformable fractional reliability function (CFRF) $S_\alpha$

The conformable fractional reliability function of X is defined as, ([4] and [11])

$$S_\alpha(x) = 1 - F_\alpha(x),$$

$$= \beta^{\theta-2\alpha+2} x^{2\alpha-\theta-2}. \tag{6}$$

## 2.4 The conformable fractional hazard function (CFHF) $h_\alpha$

The conformable fractional hazard function of X is defined as, $h_\alpha(x) = \frac{f_\alpha(x)}{S_\alpha(x)}$, where $f_\alpha(x)$ is the CFPDF reported in (1) and $S_\alpha(x)$ reported in (6).

Hence,

$$h_\alpha(x) = (\theta - 2\alpha + 2)x^{-\alpha}. \tag{7}$$

To show this function, we plot Figure 3 that represents the plot of such function according to $\theta$=2.5

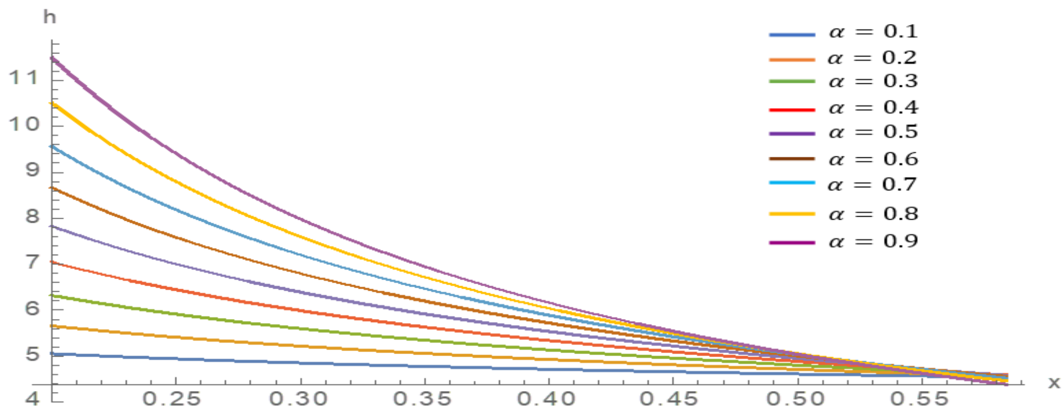


**Fig 3:** The conformable fractional hazard function of $\alpha$ −Pareto Distribution.

## 2.5 The conformable fractional expectation $E_\alpha$

### 2.5.1 $r^{th}$ $\alpha - Moment\left(E_\alpha(X^r)\right)$

The conformable fractional expectation $E_\alpha$ of a function $(X)$ of continuous random variable X whose CFPDF $f_\alpha(x)$ is defined as, [1]

$$E_\alpha\left(L(X)\right) = \int_a^d L(x) f_\alpha(x) d^\alpha x,$$

$$E_\alpha(X^r) = \int_\beta^\infty x^r f_\alpha(x)\, d^\alpha x\,,$$

$$= \frac{\beta^r(\theta-2\alpha+2)}{\theta-r-2\alpha+2}\ . \tag{8}$$

The first and second moments can be given below as follows:

$$\mu = E_\alpha(X) = \frac{\beta(\theta-2\alpha+2)}{\theta-2\alpha+1}\ ,\theta > 1 \tag{9}$$

$$E_\alpha(X^2) = \frac{\beta^2(\theta-2\alpha+2)}{\theta-2\alpha}\ ,\theta > 2\alpha \tag{10}$$

### 2.5.2 $r^{th}$ conformable fractional central moment $E_\alpha(X-\mu)^r$

The $r^{th}$_conformable fractional central moment $E_\alpha(X-\mu)^r$ of X is defined as

$$E_\alpha(X-\mu)^r = \int_\beta^\infty (x-\mu)^r\, f_\alpha(x)\, d^\alpha x.$$

Based on the aforesaid discussion, we can list the following assertions:

$$E_\alpha(X-\mu) = 0.$$

$$E_\alpha(X-\mu)^2 = \frac{\beta^2(2\alpha-\theta-2)}{(2\alpha-\theta)(\theta-2\alpha+1)^2} \qquad ,2\alpha < \theta. \tag{11}$$

$$E_\alpha(X-\mu)^3 = \frac{2\beta^3(2\alpha-\theta-3)(2\alpha-\theta-2)}{(2\alpha-\theta)(2\alpha-\theta+1)(\theta-2\alpha+1)^3}\ , 1+2\alpha<\theta. \tag{12}$$

$$E_\alpha(X-\mu)^4 = \frac{3\beta^4(2\alpha-\theta-2)\left(12\alpha^2+\theta(3\theta+13)-2\alpha(6\theta+13)+16\right)}{(2\alpha-\theta)(2\alpha-\theta+1)(2\alpha-\theta+2)(\theta-2\alpha+1)^4}\ ,$$
$$2+2\alpha<\theta. \tag{13}$$

### 2.5.3 The conformable fractional variance

The conformable fractional variance $\sigma_\alpha{}^2(X)$ of X is defined as

$$\sigma_\alpha{}^2(X) = E_\alpha(X^2) - (E_\alpha(X))^2,$$

$$= \frac{\beta^2(2\alpha-\theta-2)}{(2\alpha-\theta)(\theta-2\alpha+1)^2}, 2\alpha<\theta. \tag{14}$$

### 2.5.4 The conformable standard deviation

The conformable fractional standard deviation $\sigma_\alpha(X)$ of X is defined as

$$\sigma_\alpha(X) = \sqrt{\sigma_\alpha{}^2(X)},$$

$$= \frac{\beta}{(\theta-2\alpha+1)}\sqrt{\frac{2\alpha-\theta-2}{2\alpha-\theta}}\ ,\ 2\alpha<\theta. \tag{15}$$

### 2.5.5 The conformable fractional quantile of X

The quantile function associates with arrange at and below a probability input the likelihood that a random variable is realized in that range for some probability distribution, is given by $F_\alpha^{-1}(k)$, $0 \le k < 1$.

$$F_\alpha^{-1}(k) = \beta\left(1-k^{\left(\frac{1}{2\alpha-\theta-2}\right)}\right) \tag{16}$$

### 2.5.6 The conformable fractional skewness $\alpha sk$ of X

The conformable fractional Skewness αsk of X is defined as:

$$\alpha sk = \left(\frac{E_\alpha(X-\mu)^3}{\left(\sigma_\alpha(X)\right)^3}\right)\ , 1+2\alpha<\theta.$$

Based on (12) and (15), we can obtain:

$$\alpha sk = \frac{2(2\alpha-\theta-3)}{2\alpha-\theta+1}\sqrt{\frac{\theta-2\alpha}{\theta-2\alpha+2}}, 1+2\alpha<\theta. \tag{17}$$

#### 2.5.7 The conformable fractional kurtosis $\alpha ku$ of X

The conformable fractional kurtosis $\alpha ku$ of X which can be defined as:

$$\alpha ku = \frac{E_\alpha (X-\mu)^4}{\left(\sigma_\alpha(X)\right)^4}$$

Referring to (13) and (15), yields the following equality:

$$\alpha ku = \frac{3(2\alpha-\theta)\left(12\alpha^2+\theta(3\theta+13)-2\alpha(6\theta+13)+16\right)}{(2\alpha-\theta-2)(2\alpha-\theta+1)(2\alpha-\theta+2)}, 2+2\alpha<\theta. \qquad (18)$$

## 2.6 The Conformable Entropy

### 2.6.1 The conformable fractional Shannon entropy $\alpha H(\theta,\beta,\alpha)$ of X

The conformable fractional Shannon entropy of a random variable X whose CFPDF $f_\alpha(x)$ is defined as (Abu Hammad, Awad, & Khalil, Properties of conformable fractional chi-square probability distribution., 2020)

$$\alpha H(\theta,\beta,\alpha) = -E_\alpha\left(\log f_\alpha(X)\right),$$
$$= \frac{\alpha-\theta-2+\alpha(2\alpha-\theta-2)\log(\beta)+(\theta-2\alpha+2)\log(\theta-2\alpha+2)}{\theta-2\alpha+2}. \qquad (19)$$

### 2.6.2 The conformable fractional Tsallis entropy $\alpha H_{T.c}(\theta,\beta,\alpha)$ of X

The conformable fractional Tsallis entropy of a random variable Y whose CFPDF $f_\alpha(x)$ is defined as (Jebril I. , Abu Hammad, Abu Judeh, Dalahmeh, & Abrikah, 2021)

$$\alpha H_{T,c}(\theta,\beta,\alpha) = \frac{1}{1-c}\left(E_\alpha\left(f_\alpha(Y)\right)^{c-1}-1\right),$$
$$= \frac{-\beta^{\alpha-\alpha c}(\theta-2\alpha+2)^c+(\theta+2)c-\alpha(c+1)}{(c-1)(\alpha(1+c)-(\theta+2)c)}. \qquad (20)$$

using L'Hopital rule we observe after some simplifications that as c $\to$ 1, the limit of conformable fractional Tsallis entropy approaches the conformable fractional Shannon entropy as expected.

$$\lim_{c\to 1} \alpha H_{T,c}(\theta,\beta,\alpha) = \frac{\alpha-\theta+\alpha(2\alpha-\theta-2)\log(\beta)+(\theta-2\alpha+2)\log(\theta-2\alpha+2)-2}{2\alpha-\theta-2}. \qquad (21)$$

### 2.6.3 The conformable fractional Renyi entropy $\alpha H_{R,c}(\theta,\beta,\alpha)$ of X

The conformable fractional Renyi entropy of a random variable Y whose CFPDF $f_\alpha(x)$ is defined as

$$\alpha H_{R,c}(\theta,\beta,\alpha) = \frac{1}{1-c}\log\left(E_\alpha\left(f_\alpha(X)\right)^{c-1}\right).$$

$$= \frac{\alpha(c-1)\log(\beta) - c\log(\theta-2\alpha+2) + \log\left((\theta+2)c - \alpha(c+1)\right)}{c-1}. \qquad (22)$$

Using L'Hopital rule we observe after some simplifications that as c → 1, the limit of conformable fractional Renyi entropy approaches the conformable fractional Shannon entropy as expected

$$\lim_{c\to 1} \alpha H_{R,c}(\theta,\beta,\alpha) = \frac{\theta-\alpha+2}{\theta-2\alpha+2} + \alpha\log(\beta) - \log(\theta - 2\alpha + 2). \qquad (23)$$

**Conclusion**

The conformable fractional $r^{th}$ central moment's case has been generalized to the special case of the Pareto distribution, when the limit goes to the one from the left. The conformable fractional entropy (Shannon,Renyi and Tsallis) cases has been generalized to the special case of the Pareto distribution, when the limit goes to the one from the left. The conformable fractional hazard and reliability functions case has been generalized to the special case of the Pareto distribution, when the limit goes to the one from the left.

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## Appendix

Percentile of the distribution classified by p in the rows and α in the columns.

The parameters of the distribution are: $\beta = 1$ and $\theta = 2.5$

| $\alpha$ \ $p$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| 0.001 | 1.501 | 1.501 | 1.501 | 1.501 | 1.501 | 1.501 | 1.501 | 1.501 | 1.501 |
| 0.005 | 1.502 | 1.502 | 1.502 | 1.503 | 1.503 | 1.503 | 1.503 | 1.503 | 1.503 |
| 0.01 | 1.504 | 1.504 | 1.504 | 1.505 | 1.505 | 1.505 | 1.505 | 1.506 | 1.506 |
| 0.025 | 1.509 | 1.51 | 1.51 | 1.511 | 1.511 | 1.512 | 1.513 | 1.514 | 1.515 |
| 0.05 | 1.519 | 1.519 | 1.52 | 1.521 | 1.523 | 1.524 | 1.526 | 1.527 | 1.529 |
| 0.1 | 1.538 | 1.54 | 1.542 | 1.544 | 1.546 | 1.549 | 1.552 | 1.556 | 1.56 |
| 0.9 | 2.563 | 2.631 | 2.708 | 2.795 | 2.897 | 3.014 | 3.153 | 3.319 | 3.52 |
| 0.95 | 3.011 | 3.115 | 3.234 | 3.371 | 3.531 | 3.719 | 3.943 | 4.215 | 4.55 |
| 0.975 | 3.538 | 3.689 | 3.863 | 4.066 | 4.304 | 4.588 | 4.931 | 5.353 | 5.881 |
| 0.99 | 4.378 | 4.613 | 4.886 | 5.208 | 5.592 | 6.056 | 6.627 | 7.341 | 8.258 |
| 0.995 | 5.143 | 5.462 | 5.836 | 6.281 | 6.816 | 7.471 | 8.287 | 9.323 | 10.674 |
| 0.999 | 7.478 | 8.088 | 8.818 | 9.703 | 10.796 | 12.167 | 13.927 | 16.24 | 19.374 |

## Contributors

***Ma'mon Abu hammad*** is Assist. Prof. of Statistics at the Department of Mathematics, Al-Zaytoonah.University of Jordan, Amman, Jordan. His main teaching and research interests include mathematical analytical methods, information theory, dynamical systems, fuzzy and fractional differential equations. He has published several research articles in international journals of mathematics**.**