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An Algorithm for Solving Linear and Non-Linear Volterra Integro-Differential Equations

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Abstract

Using Mathematica computer software, a numerical procedure called the modified Adomian decomposition method (MADM) is successfully implemented for obtaining exact solutions of some classes of Volterra integro-differential equations based on the ADM approximate series solutions, Laplace transform, and Pade approximants. The reliability and effectiveness of MADM are tested in some examples. The obtained results indicate that the implemented procedure is very effective and powerful for handling this kind of differential equation and is valid for a wide class of other types of differential equations.

Keywords: ***Integral Equations, ADM Procedure, Series Expansion, Laplace Transform, Pade Approximant.***

1. Introduction

Volterra integro-differential equations occur in many scientific fields, including astronomy, biology, biotechnology, engineering, physics, radiology, and many others. These equations are used in a variety of attractive applications, such as heat and mass transfer, diffusion processes, and cell growth. Volterra came across a situation in which both differential and integral operators occurred in the same equation while investigating a population growth model for the study of hereditary influence. The Volterra integro-differential equation was the name given to this novel type of equation. These formulas have the following form:

$$u^{(n)}(\mathrm{x}) = f(x) + \lambda \int_0^x k(x.t)u(t)dt. \tag{1.1}$$

Because this equation is a combination of differential and integral operators, it is essential to define the initial conditions $u(0). u'(0). u''(0). \dots. u^{(n-1)}(0).$ before determining the specific solution $u(\mathrm{x})$.

Due to the fact that differential equations model and describe the majority of real-world phenomena, their solutions are very essential in applied mathematics and engineering. But the solutions to these equations are not easy, especially if the equation is

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strongly nonlinear, so obtaining an exact solution or accurate approximate solutions with a high degree of accuracy is required because they enable us to study and understand the behavior of these phenomena. Regarding this purpose, various numerical and approximated methods were applied and developed for finding solutions to numerous equations such as the delay differential equations, Integral and integro differential equations and fuzzy differential equations and so on of linear and nonlinear ODEs and PDEs, for examples, the homotopy perturbation method (HPM) was employed successfully for obtaining exact solutions for linear and nonlinear integral equations of Volterra kind and in mathematical physics for solving fractional PDEs [1, 2], homotopy analysis method (HAM) which was implemented for solving fuzzy fractional two-point boundary value problems and it is applications and other types of linear and nonlinear differential equations [3-7], optimal homotopy asymptotic method (OHAM) [8-11], differential transformation method (DTM) [12-15], Galerkin method [16-20], variational iteration method (VIM) [21-24], conformable fractional approach [25, 26] and multistage OHAM [27], Adomian decomposition method (ADM) [28, 29, 32], and so on [31, 32, 33].

The Adomian decomposition method is a well-known systematic technique for solving a variety of linear and nonlinear equations, such as ordinary differential equations, partial differential equations, integral equations, integro-differential equations, and so on. The accuracy of ADM procedure depends on the given models or problem and also on the order of the approximation, and this require more efforts and calculations especially if the given problem is strongly nonlinear which means that there will be difficulties on computing the Adomian polynomial. For this purpose it very important to find a way or process that modify the ADM procedure and enhance it is efficiency and capability.

The objective of this study is to enhance the Adomian decomposition method in order to acquire accurate solutions for Volterra integro-differential equations by employing an alternative technique (MADM) that modifies the series solution for classes of Volterra integro-differential equations by applying the Laplace transformation to the truncated series obtained by ADM, then converting the transformed series into a meromorphic function by Pad'e approximants, and finally applying the inverse Laplace transform to the obtained analytic solution, which gives the exact solution or a more accurate solution than the ADM solution.

The structure of this work is as follows: The basic concepts of ADM, Pad'e approximation, and Laplace transformation are briefly described in Section 2. Some examples are provided in Section 3 to support and illustrate the applicability and effectiveness of our procedure. Section 4 provides a conclusion of this study.

2. RESEARCH METHOD

2.1 Fundamental Idea of ADM Procedure

The basic idea of the ADM procedure concerns with differential equations of the form:

$$Lu + Ru + Nu = g, \tag{2.2.1}$$

where g is the system input, and u is the system output, L is the linear operator needed to be inverted, R is the linear remainder operator, and N is the nonlinear operator, which is assumed to be analytic. We remark that this choice of the linear operator is designed to yield an easily invertible operator with resulting trivial integrations. Additionally, we concentrate that the choice for L and its inverse L^{-1} are decided by the particular equation to be solved (Adomian).

Generally, we choose $L = \frac{d^m}{dx^m}(.)$ for m^{th} $-$order differential equation and thus its inverse L^{-1} follows as m $-$fold definite integration operator from x_0 to x. We get $L^{-1}Lu = u - \psi$, where ψ incorporates the initial values as

$\psi = \sum_{v=0}^{m-1} \beta_v \frac{(x-x_0)^v}{v!}$. Applying the inverse linear operator L^{-1} to both sides of Eq.(2.2.1) gives:

$$u = g(x) - L^{-1}[Ru + Nu]. \tag{2.2.2}$$

where $g(x) = \psi + L^{-1}g$. The ADM decomposes the solution into a series:

$$u(x) = \sum_0^\infty u_n\,(\mathrm{x}). \tag{2.2.3}$$

and then decomposes the nonlinear term Nu into a series:

$$Nu = \sum_0^\infty A_n(x). \tag{2.2.4}$$

where $\mathrm{A_n}$, depending on $\mathrm{u_0.u_1.....u_n}$. are called the Adomian polynomials, and can be gained for the nonlinearity $\mathrm{Nu = f(u)}$ by the following formula (Adomian and Rach, 1983)

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\sum_{n=0}^{\infty}\lambda^n u_n\right]_{\lambda=0} \quad n = 0.1.2, \tag{2.2.5}$$

where λ is a grouping parameter of convenience.

2. 2 Padè approximation

The $\left[\frac{L}{M}\right]$ Padè approximation [14, 15] of the function u(x) is defined by:

$$\left[\frac{L}{M}\right] = \frac{P_L(t)}{Q_M(t)},$$

where $P_L(t)$ and $Q_M(t)$ are polynomials of degrees at most L and M, respectively. The general form of the power series is

$$u(t) = \sum_{i=1}^{\infty} a_i t^i.$$

The coefficients of the $P_L(t)$ and $Q_M(t)$ polynomials are obtained from:

$$u(t) - \frac{P_L(t)}{Q_M(t)} = O(t^{L+M+1}). \tag{2.2.1}$$

When the function of the numerator and denominator $\frac{P_L(t)}{Q_M(t)}$ is multiplying by a nonzero constant the functional values remain unchanged, then we can define the normalization condition as (16):

$$\mathrm{Q_M(0) = 1}. \tag{2.2.2}$$

It can be noted the $P_L(t)$ and $Q_M(t)$ have no public factors. If we represents the coefficient of $P_L(t)$ and $Q_M(t)$ as (17):

$$\begin{Bmatrix} P_L(t) = P_0 + P_1 t + P_2 t^2 + \cdots + P_L t^L \\ Q_M(t) = q_0 + q_1 t + q_2 t^2 + \cdots + q_M t^M \end{Bmatrix}, \quad (2.2.3)$$

Then, by (2.2.2) and (2.2.3) we can multiply (2.2.1) by $Q_M(t)$, to linearizes the coefficient equations. We can write out (18) in more detail as (18):

$$\begin{Bmatrix} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0 \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0 \\ . \\ . \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0 \end{Bmatrix}, \quad (2.2.4)$$

$$\begin{Bmatrix} a_0 = P_0 \\ a_0 + a_0 q_1 = P_1 \\ a_2 + a_1 q_1 + a_0 q_2 = P_2 \\ . \\ . \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = P_L \end{Bmatrix}, \quad (2.2.5)$$

The solutions of these equations will be obtained using Eq. (2.2.4), which is consider as a set of linear equations for the unknown q`s. Once the q`s are known, then (2.2.5) gives an explicit formula for the unknown p`s, which complete the solution. If (2.2.4) and (2.2.5) are non-singular, then we may solve them direct and we obtain (2.2.6) [23,33], where (2.2.6) holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\begin{bmatrix} L \\ M \end{bmatrix} = \frac{det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ . & . & \cdots & . \\ . & . & . & . \\ a_L & a_{L+1} & . & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} X^j & \sum_{j=M-1}^{L} a_{j-M+1} X^j & \cdots & \sum_{j=0}^{L} a_j X^j \end{bmatrix}}{det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ . & . & . & . \\ . & . & . & . \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ X^M & X^{M-1} & \cdots & 1 \end{bmatrix}}, \quad (2.2.6)$$

3. Numerical Examples

In this part of the discussion, we will examine two different illustrations. For both linear and nonlinear Volterra integro-differential equations, these examples are taken into account and considered to be illustrative of the technique.

Example 3.1 Consider the following Volterra integral differential equation taken from Wazwaz 2010 [31].

$$u'(x) = 1 + \frac{1}{2!}x^2 - \int_0^x u(t)dt. \quad u(0) = 1. \quad (3.3.1)$$

To obtain approximate solution of the above problem using ADM procedure, we apply the integral operator $L^{-1} = \int_0^x (.)dt$ to both sides of Eq. (3.3.1), we have

$$u(x) = 1 + x + \frac{x^3}{6} + L^{-1}(\int_0^x u(t)dt). \quad (3.3.2)$$

Based on ADM procedure, we have the following components

$$u_0(x) = 1 + x + \frac{x^3}{6}, \quad (3.3.3)$$

and

$$u_{n+1}(x) = L^{-1}(\int_0^x u_n(t)dt). \qquad (3.3.4)$$

Which gives

$$u_1(x) = -\frac{x^2}{2} - \frac{x^3}{6} - \frac{x^5}{120}, \qquad (3.3.5)$$

$$u_2(x) = \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^7}{5040}, \qquad (3.3.6)$$

$$u_3(x) = -\frac{x^6}{720} - \frac{x^7}{5040} - \frac{x^9}{362880}, \qquad (33.7)$$

$$u_4(x) = \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{11}}{39916800}, \qquad (3.3.8)$$

$$u_5(x) = -\frac{x^{10}}{3628800} - \frac{x^{11}}{39916800} - \frac{x^{13}}{6227020800}, \qquad (3.3.9)$$

Consequently, the solution of (3.3.1) in a series form given by

$$u(x) = 1 + x - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} - \frac{x^{13}}{6227020800}. \quad (3.3.10)$$

And this in the limit of infinitely many terms converge to the exact solution $u(x) = x + cosx$. Table (3.1) present numerical results for example 3.1, while Fig (3.1) represent graphically the plot of the exact and approximate solution. In order to improve the accuracy of ADM approximate solution, we will modify the ADM solution by applying the Laplace transformation on the truncated series solution of the ADM approximate solution (3.3.10) and then employing the Pade approximate and finally using to invers of the Laplace transform to get accurate results in most cases closed to the exact form, as follows:

$$L(u(t)) = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{s^5} - \frac{1}{s^7} + \frac{1}{s^9} + \cdots \qquad (3.3.11)$$

for the simplicity, let $s = \frac{1}{z}$, then

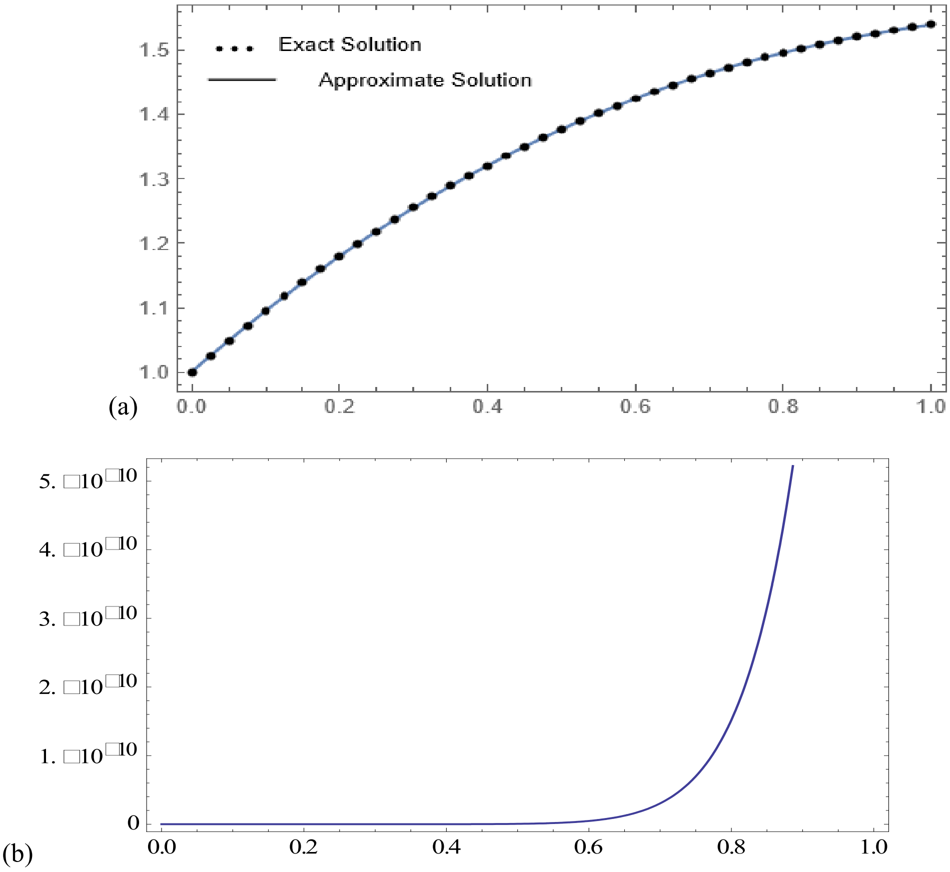
$$L(u(t)) = z + z^2 - z^3 + z^5 - z^7 + z^9 + \cdots \qquad (3.3.12)$$

The Pade approximate $\left[\frac{4}{4}\right] = \frac{z+z^2+z^4}{1+z^2}$

Recalling $= \frac{1}{s}$, we obtain Pade approximate of $\left[\frac{4}{4}\right]$ in terms of s

$$\left[\frac{4}{4}\right] = \frac{1}{(1+\frac{1}{s^2})s^4} + \frac{1}{(1+\frac{1}{s^2})s^2} + \frac{1}{(1+\frac{1}{s^2})s}. \qquad (3.3.13)$$

By using the inverse Laplace transform to the $\left[\frac{4}{4}\right]$ Pade approximate, we obtain the modified approximate solution $u(x) = x + cosx$.



Figs 3.1: (a) Plot of exact and approximate solution and (b) Plot of the absolute error for Example **3. 1**

Table 3.1: Numerical examples for example **3. 1**

x	Exact Solution	ADM	Absolute Error
0.0	)0000	1.00000000	
0.2	1.18006658	1.18006658	4.44×10^{-16}
0.4	1.32106099	1.32106099	1.02×10^{-12}
0.6	1.42533561	1.42533561	8.64×10^{-11}
0.8	1.49670671	1.49670671	2.01×10^{-9}
1.0	1.54030231	1.54030232	2.30×10^{-8}

Example 3.3.2: Consider the following Volterra integral differential equation taken from Wazwaz 2010 [31].

$$u''(x) = 1 + \int_0^x (x-t)u(t)dt. \quad u(0) = 1, u'(0) = 0. \qquad (3.3.14)$$

To obtain approximate solution of the above problem using ADM procedure, we apply the integral operator $L^{-1} = \int_0^x \int_0^x (.)\, dxdx$ to both sides of Eq. (3.4.14), which yields to

$$u(x) = 1 + \frac{x^3}{6} + L^{-1}\left(\int_0^x \int_0^x (x-t)\, dt\right). \qquad (3.3.15)$$

Based on ADM procedure, we have

$$u_0(x) = 1 + \frac{x^2}{2}, \qquad (3.3.16)$$

and

$$u_{n+1}(x) = L^{-1}(\int_0^x u_n(t)dt). \quad (3.3.17)$$

Which gives

$$u_1(x) = \frac{x^4}{24} + \frac{x^6}{720}, \quad (3.4.18)$$

$$u_2(x) = \frac{x^8}{40320} + \frac{x^{10}}{3628800}, \quad (3.4.19)$$

$$u_3(x) = \frac{x^{12}}{479001600} + \frac{x^{14}}{87178291200}, \quad (3.4.20)$$

$$u_4(x) = \frac{x^{16}}{20922789888000} + \frac{x^{18}}{6402373705728000}, \quad (3.4.21)$$

$$u_5(x) = \frac{x^{20}}{2432902008176640000} + \frac{x^{22}}{1124000727777607680000}, \quad (3.3.22)$$

Consequently, the solution of (3.3.14) in a series form given by

$$u(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40320} + \frac{x^{10}}{3628800} + \frac{x^{12}}{479001600} + \frac{x^{14}}{87178291200} + \frac{x^{16}}{20922789888000} + \frac{x^{18}}{6402373705728000} + \frac{x^{20}}{2432902008176640000} + \frac{x^{22}}{1124000727777607680000}. \quad (3.3.23)$$

And this in the limit of infinitely many terms converge to the exact solution $u(x) = Coshx$. Table (3.2) present numerical results for example 3.2, while Fig (3.2) represent graphically the plot of the exact and approximate solution. In order to improve the accuracy of ADM approximate solution, we will modify the ADM solution by applying the Laplace transformation on the truncated series solution of the ADM approximate solution (3.3.23) and the employing the Pade approximate and finally using the invers of the Laplace transform to get accurate results in most cases closed exact form, as follows:

$$L\big(u(t)\big) = \frac{1}{s^9} + \frac{1}{s^7} + \frac{1}{s^5} + \frac{1}{s^3} + \frac{1}{s} + .. \quad (3.3.24)$$

For the simplicity, let $= \frac{1}{z}$, then

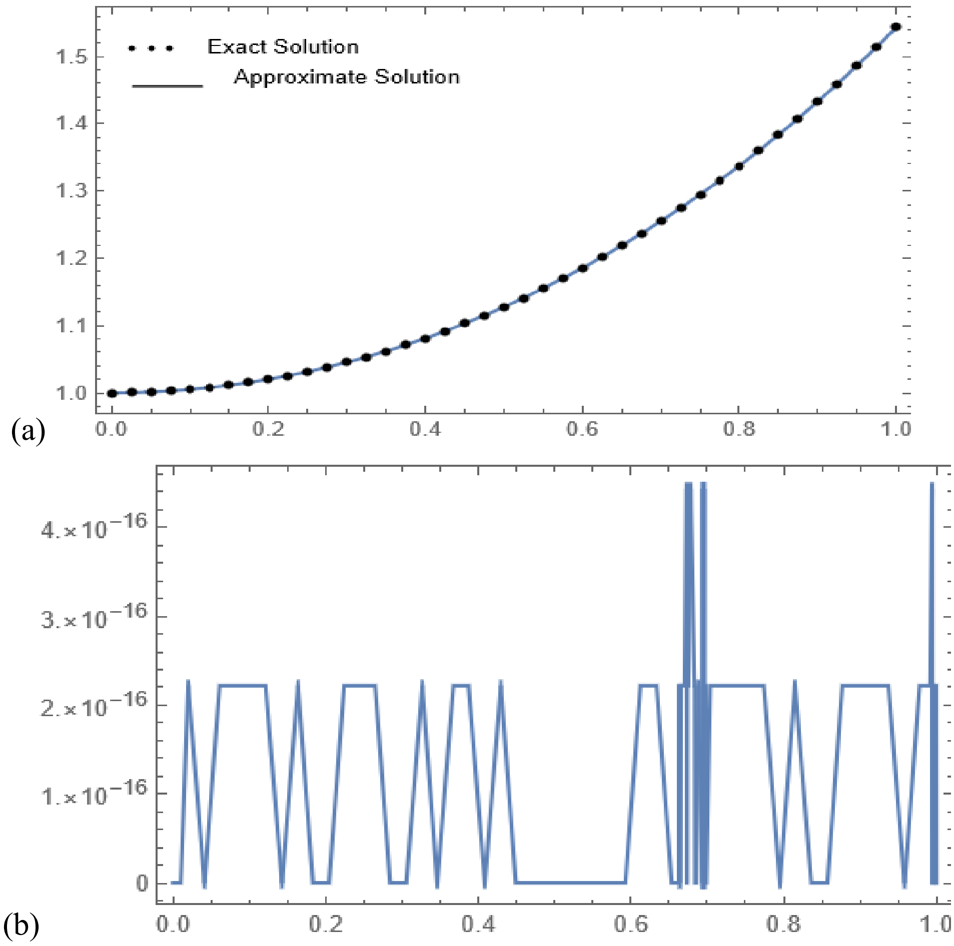
$$L\big(u(t)\big) = z + z^3 + z^5 + z^7 + z^9 + \cdots \quad (3.3.25)$$

The Pade approximate $\left[\frac{4}{4}\right] = \frac{z}{1-z^2}$.

Recalling $= \frac{1}{s}$, we obtain Pade approximate of $\left[\frac{2}{2}\right]$ in terms of s

$$\left[\frac{4}{4}\right] = \frac{s}{-1+s^2}. \quad (3.3.26)$$

By using the inverse Laplace transform to the $\left[\frac{2}{2}\right]$ Pade approximate, we obtain the modified approximate solution $u(x) = Coshx$.



Figs 3. 2: (a) Plot of exact and approximate solution and (b) Plot of the absolute error for Example 3.2.

Table 3.2 Numerical examples for example 3.2

x	Exact Solution	ADM	Absolute Error
0.0	)0000	1.00000000	)
0.2	1.02006676	1.02006676	)
0.4	1.081072371	1.081072371	2.22×10^{-16}
0.6	1.18546522	1.18546522	2.22×10^{-16}
0.8	1.33743495	1.33743495	0.00
1.0	1.54308063	1.54308063	2.22×10^{-16}

Example 3.3: In this problem, we consider the following nonlinear Volterra integro-differential equation Wazwaz 2010 [31].

$$u'(x) = \frac{1}{2} + e^{x} - \frac{1}{2}e^{2x} + \int_0^x u^2(t)dt, \;\; u(0) = 1. \qquad (3.3.27)$$

The exact solution for this problem is given by $u(x) = e^x$. The approximate solution using ADM procedure can be obtained by applying the $L^{-1} = \int_0^x (.)dx$ operators on both sides of Eq. (3.4.27) as follows:

$u(x) = L^{-1}\big(f(x)\big) + L^{-1}(\int_0^x u^2(t)\,dt).$ (3.3.28)

Then by using the decomposition series $u(x) = \sum_{n=0}^{\infty} u_n(x)$ and the concept of the Adomian polynomial on the nonlinear term $u^2(t)$, yields

$$u_n(x) = L^{-1}\big(f(x)\big) + L^{-1}(\textstyle\sum_{n=0}^{\infty} A_n(x)), \quad (3.3.29)$$

According to ADM procedure, the following iterative terms are given by $u_0(x) = 1 + x - \frac{x^3}{6} - \frac{x^4}{8} - \frac{7x^5}{120} - \frac{x^6}{48} - \frac{31x^7}{5040} - \frac{x^8}{640} - \frac{127x^9}{362880},$ (3.3.30)

and

$$u_n(x) = L^{-1}(\sum_{n=0}^{\infty} A_n(x)), \quad (3.3.31)$$

which gives $u_0(x), u_1(x)$, ...and so on. Consequently, combined the obtained results up to order four, yields the ADM series solution

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} - \frac{31x^{10}}{151200} - \frac{89x^{11}}{207900} - \frac{81733x^{12}}{239500800} - \frac{26023x^{13}}{155675520} - \frac{82651x^{14}}{1614412800} - \frac{782477x^{15}}{163459296000} + \frac{1226893x^{16}}{232475443200} + \frac{182919973x^{17}}{44460928512000} + \frac{833719x^{18}}{456855552000} + \frac{6508739x^{19}}{11467298304000} + \cdots$$
(3.3.32)

and this in the limit of infinitely many terms converge to the exact solution $u(x) = e^x$. Table (3. 3) present numerical results for example 3, while Fig (3. 3) represent graphically the plot of the exact and the ADM approximate solution of order five. In order to improve the accuracy of ADM approximate solution, we will modify the ADM solution by applying the Laplace transformation on the truncated series solution of the ADM approximate solution (3.3.32) and the employing the Pade approximate and finally using to invers of the Laplace transform to get accurate results in most cases closed exact form, as follows:

$$L\big(u(t)\big) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} + \cdots \quad (3.3.33)$$

for the simplicity, let $s = \frac{1}{z}$, then

$$L\big(u(t)\big) = z + z^2 + z^3 + z^4 + \cdots \quad (3.3.34)$$

The Pade approximate $\left[\frac{2}{2}\right] = \frac{z}{1-z}$.

Recalling $z = \frac{1}{s}$, we obtain Pade approximate of $\left[\frac{2}{2}\right]$ in terms of s

$$\left[\frac{2}{2}\right] = \frac{1}{-1+s}. \quad (3.3.35)$$

By using the inverse Laplace transform to the $\left[\frac{2}{2}\right]$ Pade approximate, we obtain the modified approximate solution $u(x) = e^x$.

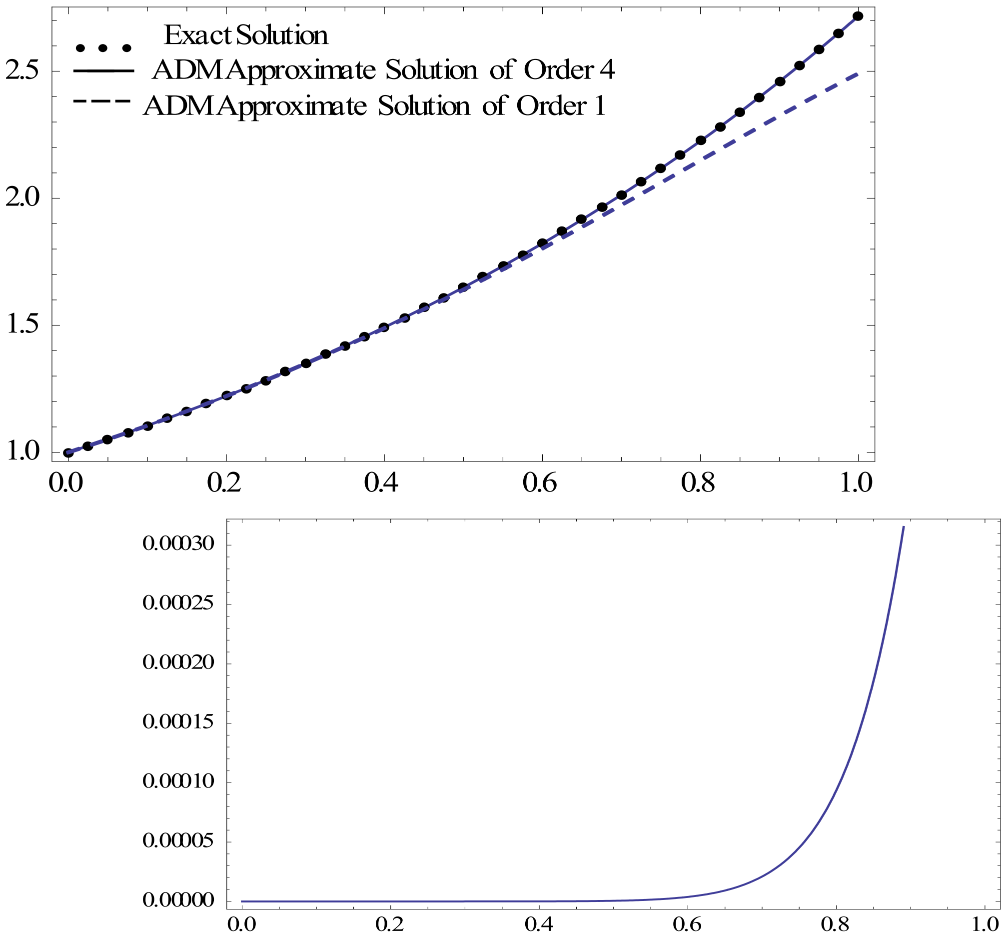


Fig 3. 3: (a) Plot of exact and approximate solution and (b) Plot of the absolute error for Example 3.3.

Table 3.3 Numerical examples for example 3.3

x	Exact Solution	ADM	Absolute Error
0.0	1.00000000	1.00000000	0
0.2	1.22140276	1.22140276	3.13×10^{-11}
0.4	1.49182470	1.49182465	4.65×10^{-8}
0.6	1.82211880	1.82211500	3.80×10^{-6}
0.8	2.22554093	2.22544733	9.36×10^{-5}
1.0	2.71828183	2.71709586	1.19×10^{-3}

Example 3.4: The following nonlinear fourth-order Volterra integro-differential equation is considered Wazwaz 2010 [31].

$$u^{(4)}(x) = -\frac{1}{4}x^2 + \sin(x) + \frac{1}{4}sin^2(x) + \int_0^x (x-t)u^2(t)dt, \tag{3.3.36}$$

with it is conditions $u(0) = u''(0) = 0,\ u'(0) = 1,$ and $u'''(0) = -1.$

The approximate solution using ADM procedure can be obtained by applying the $L^{-1} = \int_0^x \int_0^x \int_0^x \int_0^x (.)dx$ operators on both sides of Eq. (3.4.36) with it is initial conditions as follows:

$$u(x) = L^{-1}\big(f(x)\big) + L^{-1}\left(\int_0^x u^2(t)\,dt\right), \quad . \tag{3.3.37}$$

where $u_0(x) = L^{-1}\big(f(x)\big)$, and this yields to

$$u_0(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} - \frac{x^8}{20160} + \frac{x^9}{362880} + \frac{x^{10}}{453600} - \frac{x^{12}}{17107200} + \frac{x^{14}}{1383782400}, \tag{3.3.38}$$

$$u_{n+1}(x) = L^{-1}\left(\sum_{n=0}^{\infty} A_n(x)\right), \tag{3.3.39}$$

Which gives

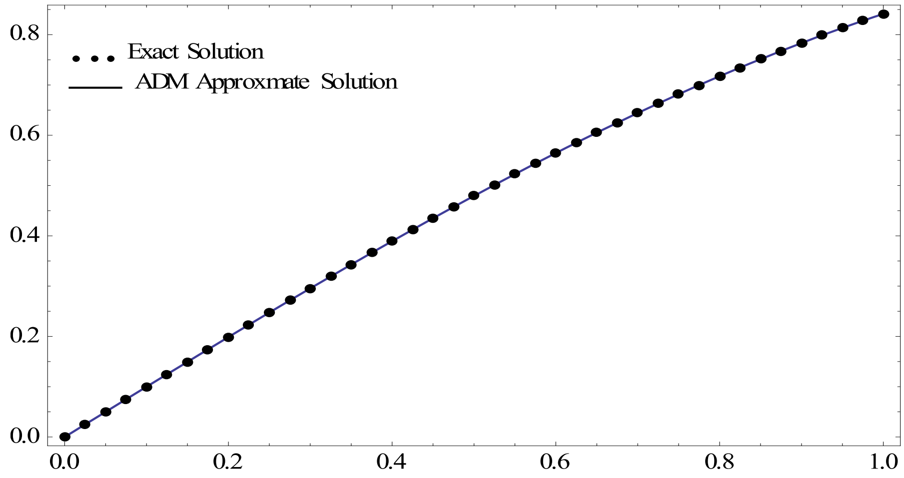
$$u_1(x) = \frac{x^{12}}{47900160} - \frac{x^{14}}{2179457280} + \frac{x^{16}}{130767436800} - \frac{x^{18}}{10003708915200}$$
$$- \frac{x^{19}}{675806113382400} + \frac{x^{20}}{950352346944000} + \frac{x^{21}}{12222713438208000}$$
$$- \frac{23x^{22}}{2554547108585472000} - \frac{67x^{23}}{3314361120369868800} + \cdots \tag{3.3.40}$$

and $u_2(x), u_3(x), \ldots$ and so on of the components, Consequently, combined the obtained results up to order four, yields the ADM series solution

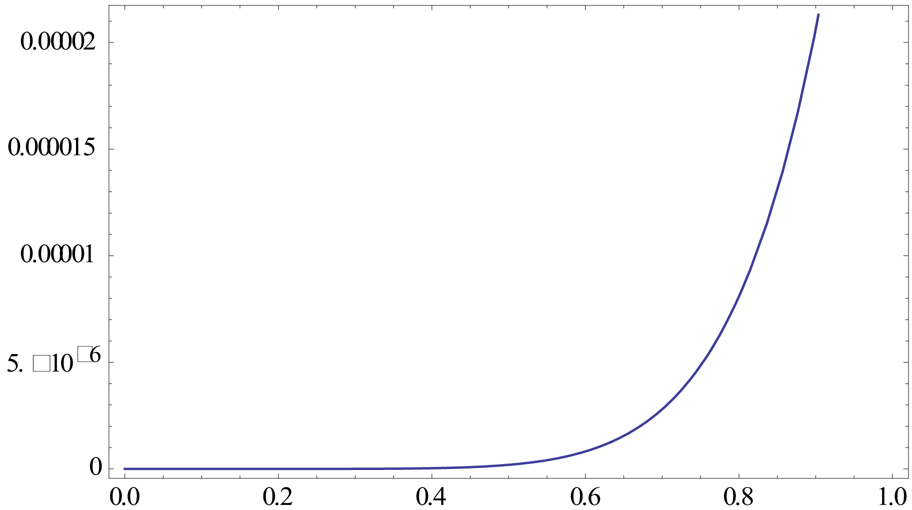
$$u(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} - \frac{x^8}{20160} + \frac{x^9}{362880} + \frac{x^{10}}{453600} - \frac{x^{12}}{26611200} + \frac{23x^{14}}{87178291200} + \frac{x^{16}}{130767436800} - \frac{x^{18}}{10003708915200} + \frac{x^{19}}{108611696793600} + \cdots \tag{3.3.41}$$

Table 3.4 Numerical examples for example 3.3

x	Exact Solution	ADM	Absolute Error
0.0	0.00000000	0.00000000	0
0.2	0.19866933	0.19866933	1.27×10^{-10}
0.4	0.38941834	0.38941831	3.23×10^{-8}
0.6	0.56464247	0.56464165	8.20×10^{-7}
0.8	0.71735609	0.71734800	8.09×10^{-6}
1.0	0.84147098	0.84142357	4.74×10^{-6}



(a)



(b)

Fig 3.4: (a) Plot of exact and approximate solution and (b) Plot of the absolute error for Example 3.4.

And this in the limit of infinitely many terms converge to the exact solution $u(x) = \sin(x)$. Table (3.4) present numerical results for example 3, while Fig (3.4) represent graphically the plot of the exact and the ADM approximate solution of order five. In order to improve the accuracy of ADM approximate solution, we will modify the ADM solution by applying the Laplace transformation on the truncated series solution of the ADM approximate solution (3.3.41) and the employing the Pade approximate and finally using to invers of the Laplace transform to get accurate results in most cases closed exact form, as follows:

$$L\big(u(t)\big) = \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} + \cdots \qquad (3.4.42)$$

for the simplicity, let $s = \frac{1}{z}$, then

$$L\big(u(t)\big) = z^2 - z^4 + z^6 - z^8 + \cdots \qquad (3.3.43)$$

The Pade approximate $\left[\frac{2}{2}\right] = \frac{z^2}{1+z^2}$.

Recalling $= \frac{1}{s}$, we obtain Pade approximate of $\left[\frac{2}{2}\right]$ in terms of s

$$\left[\frac{2}{2}\right] = \frac{1}{(1+\frac{1}{s^2})s^2}. \tag{3.3.44}$$

By using the inverse Laplace transform to the $\left[\frac{2}{2}\right]$ Pade approximate, we obtain the modified approximate solution $u(x) = \sin(x)$.

4. Results and Dissections

Numerical results that are obtained using the ADM procedure are formulated in tables and graphically plotted in figures. From that, we observed that the ADM is a powerful and effective procedure for solving this kind of differential equation in the form of a series that converges to the exact solutions in the limit of infinity terms. In this regard, instead of using many terms of the ADM approximate solution, we improve the accuracy of the ADM by using an alternative technique based on the ADM approximate solution, employing the Laplace transform on the truncated ADM series solution, then using the Pade approximants, and finally applying the invers of the Laplace transform to get an accurate solution that is close to the exact form.

5. Conclusion

In this research paper, The ADM procedure, the Laplace transform, and the Pade approximants are used and employed successfully to obtain the exact solution for several examples of linear and nonlinear integral equations of Volterra type. The capability of this algorithm was verified via several examples carried out in this manuscript. The graphs for each example lead us to conclude that the proposed algorithm was quite close to the exact solution. It was noticed that the presented algorithm was very simple, attractive, and straightforward.

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