Int. J. Open Problems Complex Analysis, Vol. 10, No. 1, March 2018
ISSN 2074-2827; Copyright ©ICSRS Publication, 2018
www.i-csrs.org

A Generalized Criterion for Starlikeness of Meromorphic Functions

Kuldeep Kaur, Sukhwinder Singh Billing

Department of Mathematics,
Sri Guru Granth Sahib World University
Fatehgarh Sahib-140407(Punjab), INDIA
e-mail: kkshergill16@gmail.com
e-mail: ssbilling@gmail.com

Received 1 February 2018; Accepted 11 March 2018

Abstract

In the present paper, we obtain a generalized criterion for meromorphic starlike functions. We claim that the result obtained here, unifies some known criteria of starlikeness of meromorphic functions.

Keywords: *Analytic function, Meromorphic function, Meromorphic Starlike function.*
Mathematics Subject Classification: *Primary 30C45, Secondary 30C80.*

1 Introduction

Let Σ be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{0}^{\infty} a_n z^n,$$

which are analytic in the punctured unit disc $\mathbb{E}_0 = \mathbb{E} \setminus \{0\}$, where $\mathbb{E} = \{z : |z| < 1\}$. A function $f \in \Sigma$ is said to be meromorphic starlike of order α if $f(z) \neq 0$ for $z \in \mathbb{E}_0$ and

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}).$$

The class of such functions is denoted by $\mathcal{MS}^*(\alpha)$ and write $\mathcal{MS}^* = \mathcal{MS}^*(0)$ - the class of meromorphic starlike functions.
A function $f \in \Sigma$ is called meromorphic convex of order α if $f'(z) \neq 0$ and

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}).$$

The class of functions defined above is denoted by $\mathcal{MC}(\alpha)$. The class $\sum_{\gamma}^{*}(\alpha)$ of $\gamma-$ meromorphic convex functions of order α consists of functions $f(z)$ with $f(z)f'(z) \neq 0$ satisfying

$$-\Re\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \alpha, \quad (z \in \mathbb{E}),$$

where α is real number and $\alpha < 1$.
Nunokawa and Ahuja [2] have proved the following results.

Theorem 1.1. *Let* $\alpha < 0$. *If*

$$f \in \mathcal{M}C\left(\frac{\alpha(3-2\alpha)}{2(1-\alpha)}\right),$$

then $f \in \mathcal{M}S^*(\alpha)$

Theorem 1.2. *Let* $\alpha < 0$ *and* $\gamma \geq 0$. *If*

$$f \in \Sigma_\gamma^*\left(\frac{2\alpha - 2\alpha^2 + \gamma\alpha}{2(1-\alpha)}\right),$$

then $f \in \mathcal{M}S^*(\alpha)$

Goyal and Prajapat [3] proved the following results:

Theorem 1.3. *If* $f(z) \in \Sigma$ *satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)}\right| < \frac{(1-\alpha)(3-\alpha)}{2-\alpha} \qquad (0 \leq \alpha < 1),$$

then $f(z) \in \mathcal{M}S^*(\alpha)$

Theorem 1.4. *If* $f(z) \in \Sigma$ *satisfies inequality*

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\frac{2zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} - 1\right)\right] > -\frac{1}{2},$$

then $f(z) \in \mathcal{M}S^*$.

The main objective of the present paper is to obtain a more general criterion for meromorphic starlike functions of certain order λ, $0 \leq \lambda < 1$, $\lambda \neq \frac{1}{2}$. We claim that our main result unifies certain known results implying meromorphic starlikeness.

2 Preliminaries

We shall need the following lemma of Miller and Mocanu [1] to prove our main result.

Lemma 2.1. *Let Ω be a subset of $\mathbb{C}^2 \times \mathbb{E}$ ($\mathbb{C}$ is the complex plane) and let $\psi : \Omega \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2,\ v = v_1 + iv_2$ (u_1, u_2, v_1, v_2 are reals), let ψ satisfy the following conditions:*
(i) $\psi(u, v)$ is continuous in Ω;
(ii) $(1, 0) \in \Omega$ and $\Re\psi(1, 0) > 0$; and
(iii) $\Re\ \{\psi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in \Omega$ such that $v_1 \leq -(1 + u_2^2)/2$.
Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots,$ be regular in the unit disc $\mathbb{E}$, such that $(p(z), zp'(z); z) \in \Omega$, for all $z \in \mathbb{E}$. If

$$\Re[\psi(p(z), zp'(z))] > 0, z \in \mathbb{E},$$

then $\Re\ p(z) > 0,\ z \in \mathbb{E}$.

3 Main Theorem

Theorem 3.1. *Let $\alpha,\ \alpha \geq 0,\ \lambda, 0 \leq \lambda < 1,\ \lambda \neq \frac{1}{2}$ and $\beta, 0 \leq \beta \leq 1,$ be given real numbers.*

(i) *For $0 \leq \lambda < \dfrac{1}{2}$, if a function $f \in \Sigma$, $\dfrac{zf'(z)}{f(z)} \neq 0$ in $\mathbb{E}$, satisfies*

$$-\Re\left[\frac{zf'(z)}{f(z)}\left(1 + \frac{\alpha z f''(z)}{f'(z)}\right) + \alpha\beta\left(1 - \frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > M(\alpha, \beta, \lambda), \tag{1}$$

then $f \in \mathcal{MS}^*(\lambda)$.

(ii) *For $\dfrac{1}{2} < \lambda < 1$, let $f \in \Sigma$, $\dfrac{zf'(z)}{f(z)} \neq 0$ in $\mathbb{E}$, satisfy*

$$-\Re\left[\frac{zf'(z)}{f(z)}\left(1 + \frac{\alpha z f''(z)}{f'(z)}\right) + \alpha\beta\left(1 - \frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > N(\alpha, \beta, \lambda), \tag{2}$$

then $f \in \mathcal{MS}^*(\lambda)$. Here

$$M(\alpha, \beta, \lambda) = [1 - \alpha + \alpha\beta]\lambda - \frac{\alpha(1-\beta)(1-\lambda)}{2} - \frac{\alpha(1-\beta)(\frac{3}{2} - 2\lambda)\lambda^2}{1-\lambda}$$

$$-\frac{2\alpha(\frac{1}{2} - \lambda)}{1-\lambda}[\sqrt{\beta\lambda(1-\beta)}] + \frac{\alpha\beta\lambda}{2(1-\lambda)} \tag{3}$$

and

$$N(\alpha,\beta,\lambda)=[1-\alpha+\alpha\beta]\lambda-\frac{\alpha(1-\beta)(1-\lambda)}{2}-\frac{\alpha(1-\beta)(\frac{3}{2}-2\lambda)\lambda^2}{1-\lambda}$$

$$+\frac{2\alpha(\frac{1}{2}-\lambda)}{1-\lambda}[\sqrt{\beta\lambda(1-\beta)}]+\frac{\alpha\beta\lambda}{2(1-\lambda)} \tag{4}$$

Proof. Define a function p by

$$-\frac{zf'(z)}{f(z)}=\lambda+(1-\lambda)p(z). \tag{5}$$

Then p is analytic in $\mathbb{E}$ and $p(0)=1$. A simple calculation yields

$$-\left[\frac{zf'(z)}{f(z)}\left(1+\frac{\alpha zf''(z)}{f'(z)}\right)+\alpha\beta\left(1-\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)\right]$$

$$=(1-\alpha+\alpha\beta)[\lambda+(1-\lambda)p(z)]-\alpha(1-\beta)[\lambda+(1-\lambda)p(z)]^2$$

$$+\alpha(1-\beta)(1-\lambda)zp'(z)-\alpha\beta\frac{(1-\lambda)zp'(z)}{\lambda+(1-\lambda)p(z)}$$

$$=\psi(p(z),zp'(z);z) \tag{6}$$

where,

$$\psi(u,v;z)=(1-\alpha+\alpha\beta)[\lambda+(1-\lambda)u]-\alpha(1-\beta)[\lambda+(1-\lambda)u]^2$$

$$+\alpha(1-\beta)(1-\lambda)v-\alpha\beta\frac{(1-\lambda)v}{\lambda+(1-\lambda)u}$$

Let $u=u_1+iu_2, v=v_1+iv_2$, where u_1,u_2,v_1,v_2 are all real with $v_1\leq -(1+u_2^2)/2$. Then, we have

$\Re\ \psi(iu_2,v_1;z)$

$$
\begin{aligned}
&= (1-\alpha+\alpha\beta)\lambda-\alpha(1-\beta)[\lambda^2-(1-\lambda)^2u_2^2] \\
&\qquad +\alpha(1-\beta)(1-\lambda)v_1-\alpha\beta\frac{\lambda(1-\lambda)v_1}{\lambda^2+(1-\lambda)^2u_2^2} \\
&\le (1-\alpha+\alpha\beta)\lambda-\alpha(1-\beta)[\lambda^2-(1-\lambda)^2u_2^2] \\
&\qquad -\frac{\alpha(1-\beta)(1-\lambda)(1+u_2^2)}{2}+\alpha\beta\frac{\lambda(1-\lambda)(1+u_2^2)}{2(\lambda^2+(1-\lambda)^2u_2^2)} \\
&= (1-\alpha+\alpha\beta)\lambda-\alpha(1-\beta)\lambda^2-\frac{\alpha(1-\beta)(1-\lambda)}{2} \\
&\qquad +\alpha(1-\beta)(1-\lambda)\left(\frac{1}{2}-\lambda\right)u_2^2+\alpha\beta\frac{\lambda(1-\lambda)(1+u_2^2)}{2(\lambda^2+(1-\lambda)^2u_2^2)} \\
&= (1-\alpha+\alpha\beta)\lambda-\alpha(1-\beta)\lambda^2-\frac{\alpha(1-\beta)(1-\lambda)}{2} \\
&\qquad +\alpha(1-\beta)(1-\lambda)\left(\frac{1}{2}-\lambda\right)t+\alpha\beta\frac{\lambda(1-\lambda)(1+t)}{2(\lambda^2+(1-\lambda)^2t)} \\
&= \phi(t) \quad \text{(say)}, \quad \text{where } u_2^2=t \\
&\le \max\phi(t). \qquad (7)
\end{aligned}
$$

Writing

$$(1-\alpha+\alpha\beta)\lambda-\alpha(1-\beta)\lambda^2-\frac{\alpha(1-\beta)(1-\lambda)}{2}=a,$$

$$(1-\beta)(1-\lambda)\left(\frac{1}{2}-\lambda\right)=b$$

and

$$\frac{\lambda}{1-\lambda}=c,$$

we have

$$\phi(t)=a+\alpha bt+\frac{\alpha\beta c}{2}\left(\frac{1+t}{c^2+t}\right).$$

Clearly, $\phi(t)$ is continuous at $t=0$. A simple calculation gives

$$\phi'(t)=\alpha b+\frac{\alpha\beta c}{2}\left(\frac{c^2-1}{(c^2+t)^2}\right).$$

and

$$\phi''(t)=\frac{\alpha\beta c(1-c^2)}{(c^2+t)^3}$$

Now, $\phi'(t)=0$ implies

$$\alpha b+\frac{\alpha\beta c}{2}\left(\frac{c^2-1}{(c^2+t)^2}\right)=0$$

which gives

$$t = -c^2 \pm \left(\sqrt{\frac{\beta c(1-c^2)}{2b}}\right).$$

Writing $-c^2 - \sqrt{\frac{\beta c(1-c^2)}{2b}} = t_1$ and $-c^2 + \sqrt{\frac{\beta c(1-c^2)}{2b}} = t_2$, we observe that $t_1 < 0$ and also, $t_1 < t_2$.

Case (i). When $0 \leq \lambda < \frac{1}{2}$, then $c = \frac{\lambda}{1-\lambda} < 1$ Since $\alpha \geq 0,\ 0 \leq \beta \leq 1$, therefore, $b \geq 0$. Hence,

$$\phi''(t_1) = -\alpha\beta c(1-c^2)\left(\frac{2b}{\beta c(1-c^2)}\right)^{\frac{3}{2}} \leq 0$$

Thus

$$\begin{aligned} \max \phi(t) &= \phi(t_1) \\ &= M(\alpha, \beta, \lambda). \end{aligned} \tag{8}$$

Let

$$\Omega = \{w : \Re\, w > M(\alpha, \beta, \lambda)\}.$$

Then from (1) and (6), we have $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{E}$, but in view of (7) and (8), $\psi(iu_2, v_1; z) \notin \Omega$. In the light of Lemma 2.1, from (5), we conclude that $f \in \mathcal{MS}^*(\lambda)$.

Case (ii). When $\frac{1}{2} < \lambda < 1$, we get $c = \frac{\lambda}{1-\lambda} > 1$.

$$\phi''(t_2) = \alpha\beta c(1-c^2)\left(\frac{2b}{\beta c(1-c^2)}\right)^{\frac{3}{2}} \leq 0$$

$$\begin{aligned} \max \phi(t) &= \phi(t_2) \\ &= N(\alpha, \beta, \lambda). \end{aligned} \tag{9}$$

Let

$$\Omega_1 = \{w : \Re\, w > N(\alpha, \beta, \lambda)\}.$$

Then from (2) and (6), we have $\psi(p(z), zp'(z); z) \in \Omega_1$ for all $z \in \mathbb{E}$, but in view of (7) and (9), $\psi(iu_2, v_1; z) \notin \Omega_1$. By the use of Lemma 2.1, from (5), we obtain that $f \in \mathcal{MS}^*(\lambda)$.

4 Deductions

Selecting $\beta = 1$ in Theorem 3.1, we obtain the following result:

Corollary 4.1. *For a real number* α, $\alpha \geq 0$, *let* $f \in \Sigma$, $\frac{zf'(z)}{f(z)} \neq 0$ *in* $\mathbb{E}$ *satisfy the differential inequality*

$$-\Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \lambda + \frac{\alpha\lambda}{2(1-\lambda)}, \quad z \in \mathbb{E},$$

then $f \in \mathcal{M}S^*(\lambda)$, $0 \leq \lambda < 1, \lambda \neq \frac{1}{2}$.

Taking $\beta = 0$ in Theorem 3.1, we get the following result:

Corollary 4.2. *Suppose that* α, $\alpha \geq 0$, $0 \leq \lambda < 1$, $\lambda \neq \frac{1}{2}$ *are real numbers and if* $f \in \Sigma$, $\frac{zf'(z)}{f(z)} \neq 0$ *in* $\mathbb{E}$ *satisfies the condition*

$$-\Re\left[\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right)\right] > (1-\alpha)\lambda - \alpha\frac{(1-\lambda)}{2} - \frac{\alpha(\frac{3}{2} - 2\lambda)\lambda^2}{1-\lambda}, \quad z \in \mathbb{E},$$

then $f \in \mathcal{M}S^*(\lambda)$.

Selecting $\alpha = \beta = 1$ in Theorem 3.1, we obtain the following result:

Corollary 4.3. *If* $f \in \Sigma$, $\frac{zf'(z)}{f(z)} \neq 0$ *in* $\mathbb{E}$, *satisfies*

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \lambda + \frac{\lambda}{2(1-\lambda)}, \quad z \in \mathbb{E},$$

then $f \in \mathcal{M}S^*(\lambda)$, $0 \leq \lambda < 1$, $\lambda \neq \frac{1}{2}$.

5 Open Problem

In the present paper, we obtain the sufficient conditions for starlikeness of order $\lambda, (0 \leq \lambda < 1, \lambda \neq \frac{1}{2})$ of meromorphic functions. The problem is yet open for starlikeness of order $\lambda = \frac{1}{2}$ of meromorphic functions.

References

[1] S. S. Miller and P. T. Mocanu , "Differential subordinations and inequalities in the complex plane", *J. Diff. Eqns.*, 67, (1987), pp.199-211.

[2] M.Nunokawa and O.P.Ahuja, "On meromorphic starlike and convex functions", *Indian J. Pure Appl. Math.*, 32(7), (2001), pp.1027-1032.

[3] S.P. Goyal and J.K. Prajapat, "A new class of meromorphic multivalent functions involving certain linear operator", *Tamsui Oxford Journal of Mathematical Sciences*, 25(2), (2009) pp.167-176.