Int. J. Open Problems Complex Analysis, Vol. 10, No. 2, July 2018
ISSN 2074-2827; Copyright ©ICSRS Publication, 2018
www.i-csrs.org

Non-Bazilevic Results for Classes of Multivalent Functions Defined by Integral Operator

M. K. Aouf and A. O. Mostafa

Department of Mathematics,Faculty of Science, Mansoura University
Mansoura 35516, Egypt
e-mail: mkaouf127@yahoo.com
e-mail: adelaeg254@yahoo.com

G. M. El-Hawsh

Department of Mathematics,Faculty of Science, Fayoum University
Fayoum 63514, Egypt
e-mail: gma05@fayoum.edu.eg

Received 15 January 2018; Accepted 1 April 2018

Communicated by Imran Faisal

Abstract

By making use of the principle of subordination between analytic functions, we introduce non-Bazlevic classes of multivalent functions defined by integral operator. Various results as subordination, superordination, sandwich type result and distortion theorems are obtained.

Keywords: *Analytic function, $p-$valent function, Hadamarh product, integral opeator.*
2010 Mathematical Subject Classification: 30C45.

1 Introduction

Let $H[a, k]$ be the class of analytic functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} + ... \qquad (z \in \mathbb{U}),$$

and $\mathbb{A}(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and $p-$valent in $\mathbb{U} = \{z : |z| < 1\}$.

For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists a Schwarz function $\omega(z)$, which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $f(z) = g(\omega(z))$ $(z \in \mathbb{U})$.

We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see for details [1], [3] and [6]; see also [9]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let M be the class of functions $\Phi(z)$ which are analytic and univalent in $\mathbb{U}$ and for which $\Phi(\mathbb{U})$ is convex with $\Phi(0) = 1$ and $Re\{\ \Phi(z)\} > 0$.

Tang et al. [10] (see also Seoudy and Aouf [8], Aouf et al. [2]), defined the operator $H_{p,\eta,\mu}^{\lambda,\delta} : \mathbb{A}(p) \to \mathbb{A}(p)$ by

$$\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\delta+p)_n (1+p-\mu)_n (1+p+\eta-\lambda)_n}{(1)_n (1+p)_n (1+p+\eta-\mu)_n} a_{p+n} z^{p+n}$$

$$(\mu, \eta \in R; \mu < p+1; -\infty < \lambda < \eta + p + 1 \textit{ and } \delta > -p). \tag{1.2}$$

Specializing the parameters p, η, μ, λ and δ, we obtain various new operators, for example,

$$\begin{aligned} \mathbb{H}_{1,\eta,\mu}^{\lambda,\delta} f(z) &= z + \sum_{n=2}^{\infty} \frac{(\delta+1)_{n-1} (2-\mu)_{n-1} (2+\eta-\lambda)_{n-1}}{(1)_{n-1} (2)_{n-1} (2+\eta-\mu)_{n-1}} a_n z^n \\ (\delta &> -1;\ \eta, \mu \in \mathbb{R};\ \mu < 2;\ -\infty < \lambda < \eta + 2); \end{aligned}$$

$$\begin{aligned} \mathbb{H}_{p,\eta,\mu}^{\lambda,1} f(z) &= z^p + \sum_{n=1}^{\infty} \frac{(1+p-\mu)_n (1+p+\eta-\lambda)_n}{(1)_n (1+p+\eta-\mu)_n} a_{p+n} z^{p+n} \\ (\eta, \mu &\in \mathbb{R};\ \mu < p+1;\ -\infty < \lambda < \eta + p + 1); \end{aligned}$$

$$\begin{aligned} \mathbb{H}_{p,\eta,\lambda}^{\lambda,\delta} f(z) &= \mathbb{D}_p^{\lambda,\delta} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\delta+p)_n (1+p-\lambda)_n}{(1)_n (1+p)_n} a_{p+n} z^{p+n} \\ (\ \delta &> -p;\ -\infty < \lambda < \eta + p + 1); \end{aligned}$$

and

$$\mathbb{H}^{\mu,\delta}_{p,\eta,\mu} f(z) \;=\; \mathbb{D}^{\mu,\delta}_{p} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\delta+p)_n(1+p-\mu)_n}{(1)_n(1+p)_n} a_{p+n} z^{p+n}$$
$$(\mu \;\in\; \mathbb{R};\ \mu < p+1;\ \delta > -p).$$

From (1.2), we can easily obtain the following identities:

$$z(\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu} f(z))' = (\delta+p)\mathbb{H}^{\lambda,\delta+1}_{p,\eta,\mu} f(z) - \delta \mathbb{H}^{\lambda,\delta}_{p,\eta,\mu} f(z) \tag{1.3}$$

and

$$z(\mathbb{H}^{\lambda+1,\delta}_{p,\eta,\mu} f(z))' = (p+\eta-\lambda)\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu} f(z) - (\eta-\lambda)\mathbb{H}^{\lambda+1,\delta}_{p,\eta,\mu} f(z). \tag{1.4}$$

Using the operator $\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu} f(z)$ and for $\rho \in \mathbb{C}$, $-1 \le B < A \le 1$, let:

(i) $R^{\lambda,\delta}_{p,\eta,\mu}(\alpha,\rho;A,B) =$

$$\left\{ f \in \mathbb{A}(p) : \begin{array}{l} \chi(z) = (1+\rho)\left(\frac{z^p}{H^{\lambda,\delta}_{p,\eta,\mu} f(z)}\right)^{\alpha} - \rho\left(\frac{H^{\lambda,\delta+1}_{p,\eta,\mu} f(z)}{H^{\lambda,\delta}_{p,\eta,\mu} f(z)}\right)\left(\frac{z^p}{H^{\lambda,\delta}_{p,\eta,\mu} f(z)}\right)^{\alpha} \\ \prec \frac{1+Az}{1+Bz}, \end{array} \right\}, \tag{1.5}$$

(ii) $T^{\lambda,\delta}_{p,\eta,\mu}(\alpha,\rho;A,B) =$

$$\left\{ f \in \mathbb{A}(p) : \begin{array}{l} (1+\rho)\left(\frac{z^p}{H^{\lambda+1,\delta}_{p,\eta,\mu} f(z)}\right)^{\alpha} - \rho\left(\frac{H^{\lambda,\delta}_{p,\eta,\mu} f(z)}{H^{\lambda+1,\delta}_{p,\eta,\mu} f(z)}\right)\left(\frac{z^p}{H^{\lambda+1,\delta}_{p,\eta,\mu} f(z)}\right)^{\alpha} \\ \prec \frac{1+Az}{1+Bz}. \end{array} \right\} \tag{1.6}$$

Throughout this paper unless otherwise stated the parameters $\eta, \mu, \lambda, \delta, \rho, \alpha, A$ and B satisfy the constraints:

$$\eta,\mu \;\in\; \mathbb{R},\ \mu < p+1,\ \ -\infty < \lambda < \eta+p+1,\ \delta > -p,$$
$$0 \;<\; \alpha < 1 \text{ and } p \in \mathbb{N}.$$

and all powers are understood as being principle values.

2 Preliminary results

In order to establish our main results, we need the following definition and Lemmas.

Definition 3 [7]**.** Denote by $\mathcal{L}$ the set of all functions f that are analytic and injective on $\bar{U}\backslash E(f)$, where

$$E(f) = \left\{ \xi \in \partial\mathbb{U} : \lim_{z\to\xi} f(z) = \infty \right\},$$

and such that $f'(\xi) \neq 0$ for $\xi \in \bar{U} \backslash E(f)$.

Lemma 1[6]**.** Let $h(z)$ be analytic and convex (univalent) in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function $g(z)$ given by

$$g(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + ... \tag{2.1}$$

is analytic in $\mathbb{U}$. If

$$g(z) + \frac{zg'(z)}{\gamma} \prec h(z) \qquad (Re(\gamma) > 0), \tag{2.2}$$

then

$$g(z) \prec q(z) = \frac{\gamma}{k} z^{-\frac{\gamma}{k}} \int h(t) t^{\frac{\gamma}{k}-1} dt \prec h(z),$$

and $q(z)$ is the best dominant of (2.2).

Lemma 2 [9]**.** Let $q(z)$ be a convex univalent function in $\mathbb{U}$ and let $\sigma \in \mathbb{C}$, $\tau \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ with

$$\Re \left(1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -Re\left(\frac{\sigma}{\tau} \right) \right\}.$$

If the function $g(z)$ is analytic in $\mathbb{U}$ and

$$\sigma g(z) + \tau z g'(z) \prec \sigma q(z) + \tau z q'(z),$$

then $g(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 3 [7]**.** Let $q(z)$ be convex univalent in $\mathbb{U}$ and $m \in \mathbb{C}$. Further assume that $Re(m) > 0$. If $g(z) \in H[q(0), 1] \cap \mathcal{L}$, and $g(z) + mzg'(z)$ is univalent in $\mathbb{U}$, then

$$q(z) + mzq'(z) \prec g(z) + mzg'(z),$$

implies $q(z) \prec g(z)$ and $q(z)$ is the best subordinant.

Lemma 4 [4]**.** let F be analytic and convex in $\mathbb{U}$. If $f, g \in \mathbb{A} = \mathbb{A}(1)$ and $f,\, g \prec F$ then

$$\lambda f(z) + (1-\lambda) g(z) \prec F(z) \quad (0 \leq \lambda \leq 1).$$

3 Main results

In the remender of this paper, $\chi(z)$ is given by (1.5).

Theorem 1. Let $f(z) \in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho; A, B)$ with $Re\,(\rho) > 0$. Then

$$\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu} f(z)} \right)^{\alpha} \prec q(z) = \frac{\alpha(p+\delta)}{\rho} \int\limits_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{\alpha(p+\delta)}{\rho} - 1} du$$

$$\prec \frac{1+Az}{1+Bz} \tag{3.1}$$

and $q(z)$ is the best dominant.

Proof. Let

$$g(z) = \left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta} f(z)}\right)^{\alpha}. \tag{3.2}$$

Then $g(z)$ is of the form (2.1) and is analytic in $\mathbb{U}$. Differentiating (3.2) and using (1.3), we get

$$\chi(z) = g(z) + \frac{\rho z g'(z)}{\alpha(p+\delta)}. \tag{3.3}$$

As $f(z) \in R_{p,\eta,\mu}^{\lambda,\delta}(\alpha, \rho; A, B)$, we have

$$g(z) + \frac{\rho z g'(z)}{\alpha(p+\delta)} \prec \frac{1+Az}{1+Bz}.$$

Applying Lemma 1 with $\gamma = \frac{\alpha(p+\delta)}{\rho}$, we get

$$\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta} f(z)}\right)^{\alpha} \prec q(z) = \frac{\alpha\,(p+\delta)}{\rho} z^{\frac{-\alpha(p+\delta)}{\rho}} \int_0^z \frac{1+At}{1+Bt} t^{\frac{\alpha(p+\delta)}{\rho}-1} dt$$

$$= \frac{\alpha(p+\delta)}{\rho} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\alpha(p+\delta)}{\rho}-1} du \prec \frac{1+Az}{1+Bz}, \tag{3.4}$$

and $q(z)$ is the best dominant, which ends the proof of Theorem 1.

Theorem 2. Let $q(z)$ be univalent in $\mathbb{U}, \rho \in \mathbb{C}^*$, satisfies

$$Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -Re\left(\frac{\alpha(p+\delta)}{\rho}\right)\right\}. \tag{3.5}$$

If $f(z) \in \mathbb{A}(p)$ satisfies

$$\chi(z) \prec q(z) + \frac{\rho z q'(z)}{\alpha(p+\delta)}, \tag{3.6}$$

then

$$\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta} f(z)}\right)^{\alpha} \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let $g(z)$ be defined by (3.2), then (3.3) holds. Combining (3.3) and (3.6), we fined that

$$g(z) + \frac{\rho z g'(z)}{\alpha(p+\delta)} \prec q(z) + \frac{\rho z q'(z)}{\alpha(p+\delta)}. \tag{3.7}$$

By using Lemma 2 and (3.7), we easily get the assertion of Theorem 2.

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2, we get the following result.

Corollary 1. Let $\rho \in \mathbb{C}^*$ and $-1 \leq B < A \leq 1$, such that

$$Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -Re\left(\frac{\alpha(p+\delta)}{\rho}\right)\right\}.$$

If $f(z) \in \mathbb{A}(p)$ satisfies

$$\chi(z) \prec \frac{1+Az}{1+Bz} + \frac{\rho\,(A-B)\,z}{\alpha(p+\delta)(1+Bz)^2},$$

then

$$\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta} f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz},$$

and $\dfrac{1+Az}{1+Bz}$ is the best dominant.

Theorem 3. Let $q(z)$ be convex univalent in $\mathbb{U}$ with $Re\,(\rho) > 0$. Also let

$$\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta} f(z)}\right)^{\alpha} \in H\,[q(0),1] \cap \mathcal{L}$$

and $\chi(z)$ be univalent in $\mathbb{U}$. If

$$q(z) + \frac{\rho z q'(z)}{\alpha(p+\delta)} \prec \chi(z),$$

then

$$q(z) \prec \left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta} f(z)}\right)^{\alpha},$$

and the function $q(z)$ is the best subdominant.

Proof. Let $g(z)$ be defined by (3.2). Then

$$q(z) + \frac{\rho z q'(z)}{\alpha(p+\delta)} \prec \chi(z) = g(z) + \frac{\rho z g'(z)}{\alpha(p+\delta)}.$$

Applying Lemma 3 yields the assertion of Theorem 3.

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 3, we get the following result.

Corollary 2. Let $q(z)$ be convex univalent in $\mathbb{U}$ and $-1 \leq B < A \leq 1$ with $Re(\rho) > 0$. Also let

$$\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta} f(z)}\right)^{\alpha} \in H\,[q(0),1] \cap \mathcal{L},$$

and $\chi(z)$ be univalent in $\mathbb{U}$. If

$$\frac{1+Az}{1+Bz}+\frac{\rho(A-B)z}{\alpha(p+\delta)(1+Bz)^2}\prec\chi(z),$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha},$$

and the function $\dfrac{1+Az}{1+Bz}$ is the best subdominant.

Combining Theorem 2 and Theorem 3, we easily get the following "Sandwich type result".

Theorem 4. Let $q_1(z)$ be convex univalent, $q_2(z)$ be univalent in $\mathbb{U}$ and satisfies (3.5) with $\rho\in\mathbb{C}^*$. If

$$\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}\in H\left[q(0),1\right]\cap\mathcal{L},$$

and $\chi(z)$ is univalent in $\mathbb{U}$, and if also

$$q_1(z)+\frac{\rho zq_1'(z)}{\alpha(p+\delta)}\prec\chi(z)=q_2(z)+\frac{\rho zq_2'(z)}{\alpha(p+\delta)},$$

then

$$q_1(z)\prec\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}\prec q_2(z),$$

and $q_1(z)$ and $q_2(z)$ are the best subordinant and dominant respectively.

Theorem 5. If ρ, $\alpha>0$ and $f(z)\in R_{p,\eta,\mu}^{\lambda,\delta}(\alpha,0;1-2\psi,-1)$ $(0\le\psi<1)$, then $f(z)\in R_{p,\eta,\mu}^{\lambda,\delta}(\alpha,\rho;1-2\psi,-1)$ for $|z|<R$, where

$$R=\left(\sqrt{\left(\frac{\rho}{\alpha\left(p+\delta\right)}\right)^{2}+1}-\frac{\rho}{\alpha\left(p+\delta\right)}\right). \tag{3.8}$$

The bound R is the best possible.

Proof. Write

$$\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}=\psi+(1-\psi)p(z). \tag{3.9}$$

Then, clearly, $p(z)$ is of the form (2.1), analytic and has positive real part in $\mathbb{U}$. Differentiating (3.9) and using (1.3), we obtain

$$\frac{1}{1-\psi}\left(\chi(z)-\psi\right)=p(z)+\frac{\rho zp'(z)}{\alpha(p+\delta)}. \tag{3.10}$$

By making use of the following well-known estimate (see [5]):

$$\frac{|zp'(z)|}{Re\{p(z)\}} \leq \frac{2r}{1-r^2} \quad (|z| = r < 1)$$

(3.10) leads to

$$Re\left(\frac{1}{1-\psi}\{\chi(z) - \psi\}\right) \geq Re\{p(z)\}\left(1 - \frac{2\rho r}{\alpha(p+\delta)(1-r^2)}\right). \tag{3.11}$$

It is seen that the right-hand side of (3.11) is positive, provided that $r < R$, where R is given by (3.8).

In order to show that the bound R is the best possible, we consider the function $f(z) \in \mathbb{A}(p)$ defined by

$$\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha} = \psi + (1-\psi)\left(\frac{1+z}{1-z}\right).$$

Noting that

$$\frac{1}{1-\psi}\left\{(1+\rho)\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha} - \rho\left(\frac{\mathbb{H}^{\lambda,\delta+1}_{p,\eta,\mu}f(z)}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha} - \psi\right\}$$

$$= \frac{1+z}{1-z} + \frac{2\rho z}{\alpha(p+\delta)(1-z)^2} = 0, \tag{3.12}$$

for $|z| = R$, we conclude that the bound is the best possible, which ends the proof of Theorem 5.

Theorem 6. Let $f(z) \in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho; A, B)$ with $Re(\rho) > 0$. Then

$$f(z) = \left(z^p\left(\frac{1+B\omega(z)}{1+A\omega(z)}\right)^{\frac{1}{\alpha}}\right) * \left(z^p + \sum_{n=1}^{\infty}\frac{(1)_n(1+p)_n(1+p+\eta-\mu)_n}{(\delta+p)_n(p+1-\mu)_n(1+p-\lambda+\eta)_n}z^{p+n}\right), \tag{3.13}$$

where $\omega(z)$ is analytic function with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Proof. Suppose that $f(z) \in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho; A, B)$ with $Re(\rho) > 0$. It follows from (3.1) that

$$\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha} = \frac{1+A\omega(z)}{1+B\omega(z)}, \tag{3.14}$$

that is,

$$\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z) = z^p\left(\frac{1+B\omega(z)}{1+A\omega(z)}\right)^{\frac{1}{\alpha}}. \tag{3.15}$$

Combining (1.2) and (3.15), we have

$$\left(z^p + \sum_{n=1}^{\infty}\frac{(\delta+p)_n(p+1-\mu)_n(1+p-\lambda+\eta)_n}{(1)_n(1+p)_n(1+p+\eta-\mu)_n}z^{p+n}\right) * f(z) = z^p\left(\frac{1+B\omega(z)}{1+A\omega(z)}\right)^{\frac{1}{\alpha}}. \tag{3.16}$$

The assertion (3.13) of Theorem 6 can now easily be derived from (3.16).

Theorem 7. Let $f(z) \in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho; A, B)$ with $Re(\rho) > 0$. Then

$$\frac{1}{z^p}\left[\begin{array}{c}\left(1+Ae^{i\theta}\right)^{\frac{1}{\alpha}}\left(z^p+\sum\limits_{n=1}^{\infty}\frac{(\delta+p)_n\,(p+1-\mu)_n\,(1+p-\lambda+\eta)_n}{(1)_n(1+p)_n(1+p+\eta-\mu)_n}z^{p+n}\right)\\ *f(z)-z^p\left(1+Be^{i\theta}\right)^{\frac{1}{\alpha}}\end{array}\right]$$

$$\neq 0 \quad (\, 0 < \theta < 2\pi) \,. \tag{3.17}$$

Proof. Suppose that $f(z) \in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho; A, B)$ with $Re(\rho) > 0$. We know that (3.1) holds, implying that

$$\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha} \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}} \quad (\, 0 < \theta < 2\pi) \,. \tag{3.18}$$

It is easy to see that the condition (3.18) can be written as follows:

$$\frac{1}{z^p}\left[\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)\left(1+Ae^{i\theta}\right)^{\frac{1}{\alpha}} - z^p\left(1+Be^{i\theta}\right)^{\frac{1}{\alpha}}\right] \neq 0 \quad (\, 0 < \theta < 2\pi) \,. \tag{3.19}$$

Combining (1.2) and (3.19), we easily get the convolution property (3.17).

Theorem 8. Let $\rho_2 \geq \rho_1 \geq 0$ and $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$. Then

$$R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho_2; A_2, B_2) \subset R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho_1; A_1, B_1). \tag{3.20}$$

Proof. Suppose that $f(z) \in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho_2; A_2, B_2)$. We have

$$(1+\rho_2)\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha} - \rho_2\left(\frac{\mathbb{H}^{\lambda,\delta+1}_{p,\eta,\mu}f(z)}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha} \prec \frac{1+A_2z}{1+B_2z}.$$

As $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, we easily find that

$$(1+\rho_2)\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha} - \rho_2\left(\frac{\mathbb{H}^{\lambda,\delta+1}_{p,\eta,\mu}f(z)}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha}$$

$$\prec \frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z}, \tag{3.21}$$

which means that $f(z) \in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho_2; A_1, B_1)$. Thus the assertion (3.20) holds for $\rho_2 = \rho_1 \geq 0$. If $\rho_2 > \rho_1 \geq 0$, by Theorem 1 and (3.21), we know that $f(z) \in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, 0; A_1, B_1)$, that is,

$$\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha} \prec \frac{1+A_1z}{1+B_1z}. \tag{3.22}$$

At the same time, we have

$$(1+\rho_1)\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}-\rho_1\left(\frac{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}$$

$$=(1-\frac{\rho_1}{\rho_2})\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}+\frac{\rho_1}{\rho_2}\left[\begin{array}{c}(1+\rho_2)\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}\\ -\rho_2\left(\frac{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}\end{array}\right]. \quad (3.23)$$

Moreover

$$0\le\frac{\rho_1}{\rho_2}<1,$$

and the function $\frac{1+A_1z}{1+B_1z}$ $(-1\le B_1<A_1\le 1;\ z\in\mathbb{U})$ is analytic and convex in $\mathbb{U}$. Combining (3.21) – (3.23) and Lemma 4, we find that

$$(1+\rho_1)\left(\frac{z^p}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}-\rho_1\left(\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)\left(\frac{z^p}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}\prec\frac{1+A_1z}{1+B_1z},$$

which means that $f(z)\in R_{p,\eta,\mu}^{\lambda,\delta}(\alpha,\rho_1;A_1,B_1)$, which implies that the assertion (3.20) of Theorem 8 holds.

Theorem 9. Let $f(z)\in R_{p,\eta,\mu}^{\lambda,\delta}(\alpha,\rho;A,B)$ with $\rho>0$ and $-1\le B<A\le 1$. Then

$$\frac{\alpha(p+\delta)}{\rho}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\alpha(p+\delta)}{\rho}-1}du<Re\left(\frac{z^p}{\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}$$

$$<\frac{\alpha(p+\delta)}{\rho}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\alpha(p+\delta)}{\rho}-1}du. \quad (3.24)$$

The extremal function of (3.24) , is given by

$$\mathbb{H}_{p,\eta,\mu}^{\lambda,\delta}F(z)=z^p\left(\frac{\alpha(p+\delta)}{\rho}\int_0^1\frac{1+Az^nu}{1+Bz^nu}u^{\frac{\alpha(p+\delta)}{\rho}-1}du\right)^{\frac{-1}{\alpha}}. \quad (3.25)$$

Proof. Let $f(z)\in R_{p,\eta,\mu}^{\lambda,\delta}(\alpha,\rho;A,B)$ with $\rho>0$. From Theorem 1, we know that (3.1) holds, which implies that

$$Re\left(\frac{z^p}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\alpha}<\sup_{z\in U}Re\left\{\frac{\alpha(p+\delta)}{\rho}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\alpha(p+\delta)}{\rho}-1}du\right\}$$

$$\leq \frac{\alpha(p+\delta)}{\rho}\int_0^1 \sup_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\alpha(p+\delta)}{\rho}-1}du$$

$$< \frac{\alpha(p+\delta)}{\rho}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha(p+\delta)}{\rho}-1}du, \tag{3.26}$$

$$Re\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha} > \inf_{z\in U} Re\left\{\frac{\alpha(p+\delta)}{\rho}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha(p+\delta)}{\rho}-1}du\right\}$$

$$\geq \frac{\alpha(p+\delta)}{\rho}\int_0^1 \inf_{z\in U} Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\alpha(p+\delta)}{\rho}-1}du$$

$$> \frac{\alpha(p+\delta)}{\rho}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha(p+\delta)}{\rho}-1}du. \tag{3.27}$$

Combining (3.26) and (3.27), we get (3.24). By noting that the function $\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}F(z)$, defined by (3.25), belongs to the class $R^{\lambda,\delta}_{p,\eta,\mu}(\alpha,\rho;A,B)$, we obtain that equality (3.24) is sharp. This completes the proof of Theorem 9.

In a similar way, applying the method used in the proof of Theorem 9, we easily get the following result.

Corollary 3. Let $f(z)\in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha,\rho;A,B)$ with $\rho>0$ and $-1\leq A<B\leq 1$. Then

$$\frac{\alpha(p+\delta)}{\rho}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha(p+\delta)}{\rho}-1}du < \left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)}\right)^{\alpha}$$

$$< \frac{\alpha(p+\delta)}{\rho}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha(p+\delta)}{\rho}-1}du. \tag{3.28}$$

The extremal function of (3.28), is given by (3.25).

In view of Theorem 9 and Corollary 3, we easily derive the following distortion theorems for the class $R^{\lambda,\delta}_{p,\eta,\mu}(\alpha,\rho;A,B)$.

Corollary 4. Let $f(z)\in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha,\rho;A,B)$ with $\rho>0$ and $-1\leq B<A\leq 1$. Then for $|z|=r<1$, we have

$$r^p\left(\frac{\alpha(p+\delta)}{\rho}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\alpha(p+\delta)}{\rho}-1}du\right)^{\frac{1}{\alpha}} < \left|\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu}f(z)\right|$$

$$< r^p \left(\tfrac{\alpha(p+\delta)}{\rho} \int_0^1 \tfrac{1+Aur}{1+Bur} u^{\frac{\alpha(p+\delta)}{\rho}-1} du \right)^{\frac{1}{\alpha}}. \tag{3.29}$$

The extremal function of (3.29) is defined by (3.25).

Corollary 5. Let $f(z) \in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho; A, B)$ with $\rho > 0$ and $-1 \leq A < B \leq 1$. Then for $|z| = r < 1$, we have

$$r^p \left(\tfrac{\alpha(p+\delta)}{\rho} \int_0^1 \tfrac{1+Aur}{1+Bur} u^{\frac{\alpha(p+\delta)}{\rho}-1} du \right)^{\frac{1}{\alpha}} < \left| \mathbb{H}^{\lambda,\delta}_{p,\eta,\mu} f(z) \right|$$

$$< r^p \left(\tfrac{\alpha(p+\delta)}{\rho} \int_0^1 \tfrac{1-Aur}{1-Bur} u^{\frac{\alpha(p+\delta)}{\rho}-1} du \right)^{\frac{1}{\alpha}}. \tag{3.30}$$

The extremal function of (3.30) is defined by (3.25).

By noting that

$$(Re\,(v))^{\frac{1}{2}} \leq Re\left(v^{\frac{1}{2}}\right) \leq |v|^{\frac{1}{2}} \quad (v \in \mathbb{C}; Re\,(v) \geq 0). \tag{3.31}$$

We easily derive from Theorem 9 and Corollary 3 the following results.

Corollary 6. Let $f(z) \in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho; A, B)$ with $\rho > 0$ and $-1 \leq B < A \leq 1$. Then

$$\left(\frac{\alpha\,(p+\delta)}{\rho} \int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\alpha(p+\delta)}{\rho}-1} du \right)^{\frac{1}{2}} < Re\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu} f(z)} \right)^{\frac{\alpha}{2}}$$

$$< \left(\frac{\alpha\,(p+\delta)}{\rho} \int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\alpha(p+\delta)}{\rho}-1} du \right)^{\frac{1}{2}}.$$

The extremal function is defined by (3.25).

Corollary 7. Let $f(z) \in R^{\lambda,\delta}_{p,\eta,\mu}(\alpha, \rho; A, B)$ with $\rho > 0$ and $-1 \leq A < B \leq 1$. Then

$$\left(\frac{\alpha\,(p+\delta)}{\rho} \int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\alpha(p+\delta)}{\rho}-1} du \right)^{\frac{1}{2}} < Re\left(\frac{z^p}{\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu} f(z)} \right)^{\frac{\alpha}{2}}$$

$$< \left(\frac{\alpha\,(p+\delta)}{\rho} \int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\alpha(p+\delta)}{\rho}-1} du \right)^{\frac{1}{2}}.$$

The extremal function is defined by (3.25) .

Remarks

(i) Using (1.4) instead of (1.3) in the above results, we get the corresponding results for the class $T_{p,\eta,\mu}^{\lambda,\delta}(\alpha,\rho;A,B)$;

(ii) Taking $p=1,\delta=1,\mu=\lambda$ and $\lambda=\mu$, respectively, in the above results, we obtain results corresponding to the operators $\mathbb{H}_{1,\eta,\mu}^{\lambda,\delta}f(z)$, $\mathbb{H}_{p,\eta,\mu}^{\lambda,1}f(z)$, $\mathbb{D}_{p}^{\lambda,\delta}f(z)$ and $\mathbb{D}_{p}^{\mu,\delta}f(z)$ given in the introduction.

Open Problem

The authors suggest to study these classes defined by the operator

$$\begin{aligned}\mathbb{I}_p^{\alpha}f(z) &= \frac{(p+1)^{\alpha}}{z\Gamma(\alpha)}\int_0^z\left(\log\frac{z}{t}\right)^{\alpha-1}f(t)dt\\ &= z^p+\sum_{n=1}^{\infty}\left(\frac{p+1}{n+p+1}\right)^{\alpha}a_{n+p}z^{n+p}\ \ (p\in\mathbb{N};\ \alpha>0)\,.\end{aligned}$$

Acknowledgement. The authors are thankful to the referees for helpful suggestions.

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