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Wavelets and generalized windowed transform associated to partial differential operators

Chirine Chettaoui

Faculty of Sciences of Tunis,
Department of Mathematics,
CAMPUS, 2092 Tunis, Tunisia
e-mail: Chirine.Chettaoui@insat.rnu.tn

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Abstract

We consider the partial operators on $\mathcal{K} = [0, +\infty[\times\mathbb{R}$

$$\begin{cases} D_1 = \dfrac{\partial}{\partial\theta} \\ D_2 = \dfrac{\partial^2}{\partial y^2} + ((2\alpha+1)\coth y + \tanh y)\dfrac{\partial}{\partial y} - \dfrac{1}{\cosh^2 y}\dfrac{\partial^2}{\partial\theta^2} + (\alpha+1)^2, \end{cases}$$

where $\alpha \in \mathbb{R}$ **,**$\alpha \geq 0$.
For $\alpha = n-2$, $n \in \mathbb{N}$, $n \geq 2$, **the operators** D_1 **and** $D_2 - (\alpha+1)^2$ **are used to study a harmonic analysis associated to Harish-Chandra's spherical functions ou Riemannian symmetric spaces.(see [4]p.72)**

In this paper we give first harmonic analysis associated with the operators D_1, D_2 **(see [5]), next we define the wavelets and the generalized windowed transform and we prove for this transform Plancherel and inversion formulas.**

Keywords: *Partial differential operators; Wavelets; Generalized windowed transform.*

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Introduction

Let g be a non negligeable square integrable function on $\mathbb{R}^2$ with respect to the Lebesgue measure. The classical windowed transform Ψ_g is a transform which replace the usual Fourier transform on $\mathbb{R}^2$ of a function f is given by:

$$\Psi_g(f)(\lambda, y) = \int_{\mathbb{R}^2} f(x) g_{\lambda,y}(x) dx, \quad \lambda, y \in \mathbb{R}^2.$$

This transform is the product of the analyzed function f by the function $g_{\lambda,y}$ called the classical wavelet defined by

$$g_{\lambda,y}(x) = e^{-i<\lambda,x>} \frac{\tau_x g(y)}{||\tau_x g||_2},$$

with τ_x the classical translation operator defined for $x \in \mathbb{R}^2$, by

$$\tau_x g(y) = g(x-y), \quad y \in \mathbb{R}^2$$

The function g is called windowed function.
We prove for the transform Ψ_g Plancherel and inversion formulas.

In this paper, we introduce first the harmonic analysis associated with the operators D_1, D_2 (generalized Fourier transform, generalized Paley-Wiener transform, generalized Plancherel theorem, generalized translation operator $T_{(y,\theta)}$, $(y, \theta) \in \mathcal{K}$, and generalized convolution product)(see[4]).

Next, we consider a non negligeable function g on $\mathcal{K}$, and its translate $T_{(y,\theta)}g$ and we study first the properties of its L^2-norm $||T_{(y,\theta)}g||_{\alpha,2}$ with respect to the measure

$$\mathcal{A}_\alpha(y) dy d\theta = 2^{2(\alpha+1)} (\sinh y)^{2\alpha+1} \cosh y dy d\theta, (y, \theta) \in \mathcal{K}.$$

and we prove for all $(y, \theta) \in \mathcal{K}$ that the function $||T_{(y,\theta)}g||_{\alpha,2}$ is different from zero.
We define the wavelet $g^s_{(\lambda,\mu),(y,\theta)}$ associated with the operators D_1, D_2 given by

$$g^s_{(\lambda,\mu),(y,\theta)}(x, \tau) = \varphi_{\lambda,\mu}(x, \tau) \frac{T_{(y,\theta)} g(x, \tau)}{||T_{(x,\tau)} g||^s_{\alpha,2}}.$$

By using these wavelet we define the family of generalized windowed transform, $\Phi^s_g(f)$,
$s \in \mathbb{R}$, associated, with the operators D_1, D_2 given for regular functions f on $\mathcal{K}$ by

$$\Phi_g^s(f)((\lambda,\mu),(y,\theta)) = \int_{\mathcal{K}} f(x,\tau)(g^s_{(\lambda,\mu),(y,\theta)})^*(x,\tau)\mathcal{A}_\alpha(x)dxd\tau,\ \ (\lambda,\mu),(y,\theta)\in\mathcal{K},$$

where

$$\forall (x,\tau)\in\mathcal{K},\ \ (g^s_{(\lambda,\mu),(y,\theta)})^*(x,\tau) = \overline{g^s_{(\lambda,\mu),(y,\theta)}}(x,-\tau),$$

and we prove for this transform Plancherel and inversion formulas.
The contents of the paper is as follows:

In the first section we give the main results concerning the harmonic analysis associated with the operators D_1, D_2.
We study in the second section the generalized translation operator associated with the operators D_1, D_2.
The third section we define and study the Wavelets associated with the operators D_1, D_2
In the fourth section is devoted to an example of wavelets associated with the operators D_1, D_2 .
In the last section we give the generalized windowed transform associated with the operators D_1, D_2.

As example we give the Gaussian wavelets and the Gaussian windowed transform associated with the operators D_1, D_2.

1 Harmonic analysis associated with the operators D_1, D_2

Notations. We denote by
- $\mathcal{E}_*(\mathbb{R}^2)$ (resp. $\mathcal{D}_*(\mathbb{R}^2)$) the space of C^∞-functions on $\mathbb{R}^2$ even with respect to the first variable (resp. with compact support even with respect to the first variable).
- $S_*(\mathbb{R}^2)$ the Schwartz space of functions on $\mathbb{R}^2$ even with respect to the first variable.
- $\Gamma = \{(\lambda,\mu)\in\mathbb{R}\times\mathbb{C}/|Im\mu|\le\alpha+1\}\cup\{(\lambda,\mu)\in\mathbb{R}\times\mathbb{C}/\mu=i\eta,$ $\eta\ge-(\alpha+1),\lambda=\pm(\alpha+2m+1+\eta), m\in\mathbb{N}\}$.

We provide these spaces with the classical topologies.

We consider the following system of partial differential operators defined by

$$\begin{cases} D_1 = \dfrac{\partial}{\partial \theta}. \\ D_2 = \dfrac{\partial^2}{\partial y^2} + ((2\alpha+1)\coth y + \tanh y)\dfrac{\partial}{\partial y} - \dfrac{1}{\cosh^2 y}\dfrac{\partial^2}{\partial \theta^2} + (\alpha+1)^2. \end{cases} \tag{1.1}$$

where $(y,\theta) \in [0,+\infty[\times\mathbb{R}$ and $\alpha \in \mathbb{R}$,$\alpha \geq 0$
We denote by $\varphi_{\lambda,\mu}(y,\theta)$ the is the unique solution of the system

$$\begin{cases} D_1 U = i\lambda U, & \lambda \in \mathbb{C}; \\ D_2 U = -\mu^2 U, & \mu \in \mathbb{C}; \\ U(0,0) = 1, \dfrac{\partial U}{\partial y}(0,\theta) = 0 & \forall \theta \in]0,+\infty[. \end{cases} \tag{1.2}$$

Proposition 1.1 *For every $(\lambda,\mu) \in \mathbb{C}\times\mathbb{C}$ the unique solution of the system (1.2) is defined by*

$$\varphi_{\lambda,\mu}(y,\theta) = e^{i\lambda\theta}(\cosh y)^{\lambda}\varphi_{\mu}^{(\alpha,\lambda)}(y) = e^{i\lambda\theta}(\cosh y)^{-\lambda}\varphi_{\mu}^{(\alpha,-\lambda)}(y) \tag{1.3}$$

where $\varphi_{\mu}^{(\alpha,\lambda)}$ is the Jacobi function defined by

$$\varphi_{\mu}^{(\alpha,\lambda)}(y) = {}_2F_1\left(\frac{\alpha+\lambda+1+i\mu}{2}, \frac{\alpha+\lambda+1-i\mu}{2}, \alpha+1; -\sinh^2 y\right).$$

$_2F_1$ denotes the hypergeometric function (See [6])

Corollary 1.1 *1. For all $(y,\theta) \in \mathcal{K}$, we have*

$$\forall(\lambda,\mu) \in \Gamma, |\varphi_{\lambda,\mu}(y,\theta)| \leq 1. \tag{1.4}$$

2. *For all $(y,\theta) \in \mathcal{K}$, the function $(\lambda,\mu) \to \varphi_{\lambda,\mu}(y,\theta)$ is analytic function on $\mathbb{C}^2$.*

3. *For all $(y,\theta) \in \mathcal{K}, \lambda \in \mathbb{C}$, the function $\mu \to \varphi_{\lambda,\mu}(y,\theta)$ is even satisfies the relation*
$$\overline{\varphi_{\lambda,\mu}(y,\theta)} = \varphi_{-\lambda,\mu}(y,\theta). \tag{1.5}$$

4. *For all $(\lambda,\mu) \in \mathbb{C}\times\mathbb{C}$, the function $(y,\theta) \to \varphi_{\lambda,\mu}(y,\theta)$ is a C^{∞}- function on $\mathcal{K}$.*

Proposition 1.2 *The function $\varphi_{\lambda,\mu}$, $(\lambda,\mu)\in\ \mathbb{C}\times\mathbb{C}$, satisfies the following product formula*

1. *If $\alpha>0$ then for all $(y,\theta),(x,\tau)\in\mathcal{K}$,*

$$\varphi_{\lambda,\mu}(y,\theta)\varphi_{\lambda,\mu}(x,\tau)=\frac{\alpha}{\pi}\int_D\varphi_{\lambda,\mu}[\cosh y\cosh xe^{i(\theta+\tau)}+\sinh y\sinh x\xi]\times(1-|\xi|^2)^{\alpha-1}dm(\xi) \tag{1.6}$$

where D is the unit disk of $\mathbb{C}$ of center o and $dm(\xi_1+i\xi_2)=d\xi_1d\xi_2$

2. *If $\alpha=0$ then for all $(y,\theta),(x,\tau)\in\mathcal{K}$*

$$\varphi_{\lambda,\mu}(y,\theta)\varphi_{\lambda,\mu}(x,\tau)=\frac{1}{2\pi}\int_0^{2\pi}\varphi_{\lambda,\mu}[\cosh y\cosh xe^{i(\theta+\tau)}+\sinh y\sinh xe^{i\psi}]d\psi. \tag{1.7}$$

1.1 The Fourier transform associated with the operators D_1, D_2

Notations. We denote by:
- $C_*(\mathbb{R}^2)$ the space of continuous functions on $\mathbb{R}^2$ even with respect to the first variable .
- $L^p_{A_\alpha}(\mathcal{K}),\ 1\le p\le+\infty$, the space of measurable functions on $\mathcal{K}$ such that

$$||f||_{\alpha,p}=(\int_{\mathcal{K}}|f(y,\theta)|^pA_\alpha(y)dyd\theta)^{\frac{1}{p}}<+\infty,\qquad if\ 1\le p<+\infty,$$

where A_α is the function defined by:

$$\forall y\in[0,+\infty[, A_\alpha(y)=2^{2(\alpha+1)}(\sinh y)^{2\alpha+1}\cosh y. \tag{1.8}$$

and

$$||f||_{\alpha,\infty}=ess\ sup_{(y,\theta)\in\mathcal{K}}|f(y,\theta)|<+\infty,\qquad if\ p=+\infty.$$

- $\widetilde{C}=\{(\lambda,\mu)\in\mathbb{C}\times\mathbb{C}/\lambda\in\mathbb{R},\mu\ge0\}$
- $\widetilde{D}=\{(\lambda,\mu)\in\mathbb{C}\times\mathbb{C}/\lambda\in\mathbb{R},-i\mu=\eta>0,C_1(\lambda,-\mu)=0\}=\{(\lambda,\mu)\in\mathbb{R}\times\mathbb{C}/\mu=i\eta,$
$\eta\ge-(\alpha+1),\lambda=\pm(\alpha+2m+1+\eta),m\in\mathbb{N}\}$
and

$$C_1(\lambda,\mu)=\frac{2^{\alpha-i\mu+1}\Gamma(i\mu)\Gamma(\alpha+1)}{\Gamma(\frac{\alpha+\lambda+1+i\mu}{2})\Gamma(\frac{\alpha-\lambda+1+i\mu}{2})}. \tag{1.9}$$

$$d\gamma(\lambda,\mu)=\frac{1}{(2\pi)^2}|C_1(\lambda,\mu)|^{-2}\chi_{\widetilde{C}}(\lambda,\mu)d\lambda d\mu+\frac{1}{(2\pi)^2}C_2(\lambda,\mu)\chi_{\widetilde{D}}(\lambda,\mu)d\lambda d\mu. \tag{1.10}$$

where for $(\lambda_0,\mu_0)\in\widetilde{D}$ we denote by

$$C_2(\lambda_0,\mu_0)=Res_{\mu=\mu_0}[C_1(\lambda_0,\mu).C_1(\lambda_0,-\mu)]^{-1}. \tag{1.11}$$

- $L^p(\widetilde{C}\cup\widetilde{D},d\gamma)$ the space of measurable functions on $\widetilde{C}\cup\widetilde{D}$ such that

$$||f||_{\gamma,p}=(\int_{\widetilde{C}\cup\widetilde{D}}|f(\lambda,\mu)|^p d\gamma(\lambda,\mu))^{\frac{1}{p}}<+\infty,\qquad if\ 1\le p<+\infty,$$
$$||f||_{\gamma,\infty}=ess\ sup_{(\lambda,\mu)\in\widetilde{C}\cup\widetilde{D}}|f(\lambda,\mu)|<+\infty,\qquad if\ p=+\infty.$$

- $H_*(\mathbb{C}^2)$ the space of entire functions on $\mathbb{C}^2$, even with respect to the first variable, rapidly decreasing and of exponential type.
- $H_*^0(\mathbb{C}^2)$ the space of entire functions ψ in $H_*(\mathbb{C}^2)$, rapidly decreasing on $\widetilde{D}$

$$\forall k\in\mathbb{N},\ \sup_{(\lambda,\mu)\in\widetilde{D}}(1+|\lambda|^2+|\mu|^2)^k|\psi(\lambda,\mu)|<+\infty$$

We provide these spaces with the classical topologies.

Definition 1.1 *The Fourier transform associated with the operators D_1,D_2 of a function f in $\mathcal{D}_*(\mathbb{R}^2)$ is defined by*

$$\forall(\lambda,\mu)\in\mathbb{C}^2,\quad \mathcal{F}(f)(\lambda,\mu)=\int_{\mathcal{K}}f(y,\theta)\varphi_{-\lambda,\mu}(y,\theta)\mathcal{A}_\alpha(y)dyd\theta. \tag{1.12}$$

The following Proposition gives some properties of the transform $\mathcal{F}$.

Proposition 1.3 *For $f\in L^1_{A_\alpha}(\mathcal{K})$ we have*

$$||\mathcal{F}(f)||_{\gamma,\infty}\le||f||_{\alpha,1}. \tag{1.13}$$

Theorem 1.1 *The Fourier transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{D}_*(\mathbb{R}^2)$ onto $H_*^0(\mathbb{C}^2)$.*

Theorem 1.2 *For every $f\in L^2_{A_\alpha}(\mathcal{K})$ such that $\mathcal{F}(f)\in L^1(\widetilde{C}\cup\widetilde{D},d\gamma)$, we have the following inversion formula*

$$f(y,\theta)=\int_{\widetilde{C}\cup\widetilde{D}}\varphi_{\lambda,\mu}(y,\theta)\mathcal{F}(f)(\lambda,\mu)d\gamma(\lambda,\mu),\ \text{a.e on } \mathcal{K} \tag{1.14}$$

Theorem 1.3 *i) Plancherel formula: For all f in $\mathcal{D}_*(\mathbb{R}^2)$ we have*

$$\int_{\mathcal{K}}|f(y,\theta)|^2\mathcal{A}_\alpha(y)dyd\theta=\int_{\widetilde{C}\cup\widetilde{D}}|\mathcal{F}(f)(\lambda,\mu)|^2d\gamma(\lambda,\mu). \tag{1.15}$$

ii) Plancherel theorem: The Fourier transform can be extended to an isometric isomorphism from $L^2_{A_\alpha}(\mathcal{K})$ onto $L^2(\widetilde{C}\cup\widetilde{D},d\gamma(\lambda,\mu))$.
(see [5-6]).

2 The generalized translation operators associated with the operators D_1, D_2

Definition 2.1 *The generalized translation operators $T_{(y,\theta)}$, $(y,\theta) \in \mathcal{K}$, associated with the operators D_1, D_2 are defined for $f \in C_*(\mathbb{R}^2)$, by*
i) If $\alpha > 0$, for all $(y,\theta), (x,\tau) \in \mathcal{K}$

$$T_{(y,\theta)} f(x,\tau) = \frac{\alpha}{\pi} \int_D f[\cosh y \cosh x e^{i(\theta+\tau)} + \sinh y \sinh x \xi](1 - |\xi|)^{\alpha-1} dm(\xi). \tag{2.1}$$

where D is the unit disk of $\mathbb{C}$ of center o and $dm(\xi_1 + i\xi_2) = d\xi_1 d\xi_2$
ii) If $\alpha = 0$, for all $(y,\theta), (x,\tau) \in \mathcal{K}$

$$T_{(y,\theta)} f(x,\tau) = \frac{1}{2\pi} \int_0^{2\pi} f[\cosh y \cosh x e^{i(\theta+\tau)} + \sinh y \sinh x e^{i\Psi}] d\Psi \tag{2.2}$$

Proposition 2.1 *For $f \in C_*(\mathbb{R}^2)$ we have*
i) For all $\theta \in \mathbb{R}$,

$$T_{(0,\theta)} f(x,\tau) = f(x, \theta + \tau)$$

ii)For all $(y,\theta), (x,\tau) \in \mathcal{K}$,

$$T_{(y,\theta)} f(x,\tau) = T_{(x,\tau)} f(y,\theta)$$

$$T_{(y,\theta)} \circ T_{(x,\tau)} = T_{(x,\tau)} \circ T_{(y,\theta)}$$

$$T_{(0,0)} = Id$$

Proposition 2.2 *The generalized translation operators $T_{(y,\theta)}, (y,\theta) \in \mathcal{K}$, satisfy:*
i) For every bounded function f in $\mathbb{C}_*(\mathbb{R}^2)$ and for all $(y,\theta) \in \mathcal{K}$, the function $T_{(y,\theta)} f$ belongs to $\mathbb{C}_*(\mathbb{R}^2)$.
ii) (Product formula) For all $(y,\theta), (x,\tau) \in \mathcal{K}$ and $(\lambda,\mu) \in \mathbb{C}^2$ we have,

$$T_{(y,\theta)} \varphi_{\lambda,\mu}(x,\tau) = \varphi_{\lambda,\mu}(y,\theta) \varphi_{\lambda,\mu}(x,\tau). \tag{2.3}$$

Definition 2.2 *The translation operators $T_{(y,\theta)}$, $(y,\theta) \in \mathcal{K}$, associated with the operators D_1, D_2 are defined for f in $L^2_{A_\alpha}(\mathcal{K})$, by*

$$\forall (\lambda,\mu) \in \Gamma, \mathcal{F}(T_{(y,\theta)} f)(\lambda,\mu) = \varphi_{\lambda,\mu}(y,\theta) \mathcal{F}(f)(\lambda,\mu). \tag{2.4}$$

2.1 The convolution product associated with the operators D_1, D_2

Definition 2.3 *The convolution product associated with the operators D_1, D_2 of two functions f and g in $\mathcal{D}_*(\mathbb{R}^2)$ is defined by*

$$f * g(y,\theta) = \int_{\mathcal{K}} f(x,\tau) T_{(y,\theta)} g(x,-\tau) \mathcal{A}_\alpha(x) dx d\tau, \tag{2.5}$$

Proposition 2.3 *i) Let f,g be in $L^2_{A_\alpha}(\mathcal{K})$. Then the function $f * g$ given for $(y,\theta) \in \mathcal{K}$, by*

$$f * g(y,\theta) = \int_{\mathcal{K}} f(x,\tau) T_{(y,\theta)} g(x,-\tau) \mathcal{A}_\alpha(x) dx d\tau, \tag{2.6}$$

is continuous on $\mathcal{K}$, tends to zero at infinity, and we have

$$\sup_{(y,\theta)\in\mathcal{K}} |f * g(y,\theta)| \leq ||f||_{\alpha,2} ||g||_{\alpha,2}. \tag{2.7}$$

ii) Let f be in $L^2_{A_\alpha}(\mathcal{K})$ and g in $L^1_{A_\alpha}(\mathcal{K})$ then,
- the function $f * g$ defined almost everywhere on $\mathcal{K}$, by

$$f * g(y,\theta) = \int_{\mathcal{K}} f(x,\tau) T_{(y,\theta)} g(x,-\tau) \mathcal{A}_\alpha(x) dx d\tau, \tag{2.8}$$

belongs to $L^2_{A_\alpha}(\mathcal{K})$ and we have

$$||f * g||_{\alpha,2} \leq ||f||_{\alpha,2} ||g||_{\alpha,1}. \tag{2.9}$$

and

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g). \tag{2.10}$$

2.2 Properties of the L^2-norm of the generalized translation operators of functions of $L^1_{A_\alpha}(\mathcal{K}) \cap L^2_{A_\alpha}(\mathcal{K})$

Proposition 2.4 *For $(y,\theta) \in \mathcal{K}$ and $f \in L^2_{A_\alpha}(\mathcal{K})$, the function $T_{(y,\theta)} f$ belongs to $L^2_{A_\alpha}(\mathcal{K})$ and we have*

$$||T_{(y,\theta)} f||_{\alpha,2} \leq ||f||_{\alpha,2}. \tag{2.11}$$

Proposition 2.5 *Let g be a function in $L^1_{A_\alpha}(\mathcal{K}) \cap L^2_{A_\alpha}(\mathcal{K})$.*
i) We have for all $(y,\theta) \in \mathcal{K}$,

$$||T_{(y,\theta)} g||^2_{\alpha,2} = \int_{\widetilde{C}\cup\widetilde{D}} |\varphi_{\lambda,\mu}(y,\theta)|^2 |\mathcal{F}(g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu). \tag{2.12}$$

ii) We have

$$||T_{(y,\theta)}g||^2_{\alpha,2} = T_{(y,\theta)}(g*g^*)(y,\theta), \quad a.e \ \ on \ \ \mathcal{K}, \tag{2.13}$$

with

$$g^*(y,\theta) = \overline{g}(y,-\theta). \tag{2.14}$$

Proof.

i) From Theorem 1.3 and (2.4), we have for $(y,\theta)\in\mathcal{K}$,

$$\begin{aligned} ||T_{(y,\theta)}g||^2_{\alpha,2} &= \int_{\mathcal{K}} |T_{(y,\theta)}(g)(t,\tau)|^2\mathcal{A}_\alpha(t)dtd\tau \\ &= \int_{\widetilde{C}\cup\widetilde{D}} |\mathcal{F}(T_{(y,\theta)}g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu) \\ &= \int_{\widetilde{C}\cup\widetilde{D}} |\varphi_{\lambda,\mu}(y,\theta)|^2|\mathcal{F}(g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu). \end{aligned}$$

ii) As the function g is in $L^1_{A_\alpha}(\mathcal{K})\cap L^2_{A_\alpha}(\mathcal{K})$,then from (2.8) , the function $g*g^*$ belongs to $L^2_{A_\alpha}(\mathcal{K})$ and from (2.4),(2.10),(2.9),(1.5),we have for $(y,\theta)\in\mathcal{K},\forall(\lambda,\mu)\in\widetilde{C}\cup\widetilde{D}$:

$$\begin{aligned} \mathcal{F}(T_{(y,\theta)}(g*g^*))(\lambda,\mu) &= \varphi_{\lambda,\mu}(y,\theta)\mathcal{F}(g*g^*)(\lambda,\mu) \\ &= \varphi_{\lambda,\mu}(y,\theta)\mathcal{F}(g)(\lambda,\mu)\mathcal{F}(g^*)(\lambda,\mu) \\ &= \varphi_{\lambda,\mu}(y,\theta)\mathcal{F}(g)(\lambda,\mu)\overline{\mathcal{F}(g)(\lambda,\mu)}. \end{aligned}$$

Thus,

$$\mathcal{F}(T_{(y,\theta)}(g*g^*))(\lambda,\mu) = \varphi_{\lambda,\mu}(y,\theta)|\mathcal{F}(g)(\lambda,\mu)|^2.$$

On the other hand, from Theorem 1.3 and (1.4), we deduce that $\mathcal{F}(T_{(y,\theta)}(g*g^*))$ belongs to $L^1(\widetilde{C}\cup\widetilde{D},d\gamma)$. Thus from Theorem 1.2 we deduce that for almost all
$(x,\tau)\in\mathcal{K}$, we have

$$T_{(y,\theta)}(g*g^*)(x,\tau) = \int_{\widetilde{C}\cup\widetilde{D}} \varphi_{\lambda,\mu}(x,\tau)\varphi_{\lambda,\mu}(y,\theta)|\mathcal{F}(g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu). \tag{2.15}$$

We deduce (2.13) by taking $(x,\tau)=(y,\theta)$ in this relation and (2.12).

Proposition 2.6 *Let g be a non negligible function in $L^1_{A_\alpha}(\mathcal{K})\cap L^2_{A_\alpha}(\mathcal{K})$. Then,*
i) The function

$$(y,\theta)\longrightarrow ||T_{(y,\theta)}g||_{\alpha,2} \ \textit{is continuous on } \mathcal{K}. \qquad (2.16).$$

ii) For all $(y,\theta)\in\mathcal{K}$,

$$||T_{(y,\theta)}g||_{\alpha,2} \neq 0. \tag{2.17}$$

To prove this proposition we need the following Lemma.

Lemma 2.1 *We consider an entire function f on $\mathbb{C}^2$, and $N = \{\lambda \in \mathbb{R}^2,\ \ f(\lambda) = 0\}$ it's set of real zero. Then the Lebesgue measure of the set N is equal to zero.*

Proof
We write the function $f(\lambda)$ in the following form

$$f(\lambda) = \sum_{\alpha \in \mathbb{N}^2} a_\alpha \lambda^\alpha, \quad (\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2)$$

where a_α are complex constants and $\lambda^\alpha = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}, \alpha = (\alpha_1, \alpha_2)$.
We write $f_{/\mathbb{R}^2}$ the restriction of f on $\mathbb{R}^2$ by

$$\forall \lambda \in \mathbb{R}^2, \quad f_{/\mathbb{R}^2}(\lambda) = \phi_1(\lambda) + i\phi_2(\lambda),$$

where ϕ_1 and ϕ_2 are real analytic functions.
More precisely for all $\lambda \in \mathbb{R}^2$, we have

$$\phi_1(\lambda) = \sum_{\alpha \in \mathbb{N}^2} Re(a_\alpha) \lambda^\alpha,$$

and

$$\phi_2(\lambda) = \sum_{\alpha \in \mathbb{N}^2} Im(a_\alpha) \lambda^\alpha.$$

Then

$$N = N_{\phi_1} \cap N_{\phi_2},$$

where N_{ϕ_1} and N_{ϕ_2} are respectively the set of real zero of the functions ϕ_1 and ϕ_2.
On the other hand, the set of zero of a real analytic function is of the form $N = S_1 \cup S_2$ (disjoint union) where S_j is a sub-variety (real analytic) of dimension j. The set S_j can be empty.
But it is well known that the Lebesgue measure of any sub-variety of $\mathbb{R}^2$ of dimension 1 is equal to zero. Then the Lebesgue measures of N_{ϕ_1} and N_{ϕ_2} are equal to zero and thus the Lebesgue measure of N is equal to zero.
Proof of Proposition 2.6
i) From Proposition 2.5, we have

$$\forall (y, \theta) \in \mathcal{K}, \ \ ||T_{(y,\theta)} g||^2_{\alpha,2} = \int_{\widetilde{C} \cup \widetilde{D}} |\varphi_{\lambda,\mu}(y, \theta)|^2 |\mathcal{F}(g)(\lambda, \mu)|^2 d\gamma(\lambda, \mu).$$

For all $(\lambda, \mu) \in \widetilde{C} \cup \widetilde{D}$, the function $(y, \theta) \longrightarrow |\varphi_{\lambda,\mu}(y, \theta)|^2 |\mathcal{F}(g)(\lambda, \mu)|^2$ is continuous on $\mathcal{K}$ and bounded by $|\mathcal{F}(g)(\lambda, \mu)|^2$ which is in $L^1(\widetilde{C} \cup \widetilde{D}, d\gamma)$, then from the dominated convergence theorem, the function $(y,\theta) \longrightarrow ||T_{(y,\theta)}g||_{\alpha,2}$ is continuous on $\mathcal{K}$.

ii) - If $(y,\theta) = (0,0)$, we have

$$||T_{(0,0)}g||_{\alpha,2} = ||g||_{\alpha,2} \neq 0.$$

- If $(y,\theta) \in \mathcal{K}\backslash\{(0,0)\}$. Suppose that there exists $(y_0,\theta_0) \in \mathcal{K}\backslash\{(0,0)\}$ such that

$$||T_{(y_0,\theta_0)}g||_{\alpha,2} = 0.$$

From Proposition 2.5, we have

$$\int_{\widetilde{C}\cup\widetilde{D}} |\varphi_{\lambda,\mu}(y_0,\theta_0)|^2 |\mathcal{F}(g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu) = 0.$$

The function $(\lambda,\mu) \longrightarrow \varphi_{\lambda,\mu}(y_0,\theta_0)$ is even with respect to the variable μ and entire on $\mathbb{C}^2$. We denote by $N_\alpha(y_0,\theta_0) = \{(\lambda,\mu) \in \widetilde{C}, \quad \varphi_{\lambda,\mu}(y_0,\theta_0) = 0\}$. We have

$$\begin{aligned}\int_{\widetilde{C}\cup\widetilde{D}} |\varphi_{\lambda,\mu}(y_0,\theta_0)|^2 |\mathcal{F}(g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu) &= \int_{N_\alpha(y_0,\theta_0)} |\varphi_{\lambda,\mu}(y_0,\theta_0)|^2 |\mathcal{F}(g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu) \\ &+ \int_{N_\alpha^c(y_0,\theta_0)} |\varphi_{\lambda,\mu}(y_0,\theta_0)|^2 |\mathcal{F}(g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu) = 0,\end{aligned}$$

where $N_\alpha^c(y_0,\theta_0)$ is the complementary of $N_\alpha(y_0,\theta_0)$.

From Lemma 2.1 the Lebesgue measure of $N_\alpha(y_0,\theta_0)$ is equal to zero. Then

$$\int_{N_\alpha^c(y_0,\theta_0)} |\varphi_{\lambda,\mu}(y_0,\theta_0)|^2 |\mathcal{F}(g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu) = 0.$$

Thus for all $(\lambda,\mu) \in N_\alpha^c(y_0,\theta_0)$, we have

$$|\mathcal{F}(g)(\lambda,\mu)|^2 = 0.$$

On the other hand from the relation (1.15) we have

$$\begin{aligned}||g||_{\alpha,2}^2 &= \int_{\widetilde{C}\cup\widetilde{D}} |\mathcal{F}(g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu) \\ &= \int_{N_\alpha(y_0,\theta_0)} |\mathcal{F}(g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu) + \int_{N_\alpha^c(y_0,\theta_0)} |\mathcal{F}(g)(\lambda,\mu)|^2 d\gamma(\lambda,\mu). \quad (2.18)\end{aligned}$$

By applying to this relation the fact that the Lebesgue measure of $N_\alpha(y_0,\theta_0)$ is equal to zero and the relation (2.18), we deduce that

$$||g||_{\alpha,2} = 0.$$

This contradicts the fact that $||g||_{\alpha,2} \neq 0$.

2.3 The Gauss kernel associated with the operators D_1, D_2

2.3.1 Definition and properties of the heat kernel E_t

Definition 2.4 *The heat kernel $E_t, t > 0$, associated with the operators D_1, D_2 is given by*

$$\forall (y,\theta) \in \mathcal{K}, E_t(y,\theta) = \int_{\widetilde{C}\cup\widetilde{D}} e^{-t(\lambda^2+\mu^2+\frac{9}{4})}\varphi_{\lambda,\mu}(y,\theta)d\gamma(\lambda,\mu). \tag{2.19}$$

The function $E_t, t > 0$, possesses the following proprieties
The function $E_t, t > 0$ is of class C^∞ on $\mathcal{K}$
i) We have

$$||E_t||_{\alpha,1} = 1. \tag{2.20}$$

ii) For all $(\lambda,\mu) \in \Gamma \cup \{(0, i\frac{3}{2})\}$, we have

$$\mathcal{F}(E_t)(\lambda,\mu) = e^{-t(\lambda^2+\mu^2+\frac{9}{4})}. \tag{2.21}$$

iii) For all $t > 0,\ s > 0$, we have

$$\forall (y,\theta) \in K, \quad E_t * E_s(y,\theta) = E_{t+s}(y,\theta). \tag{2.22}$$

2.3.2 Properties of the L^2-norm of the Gauss kernel

The Gauss kernel $E(t,(y,\theta),(x,\tau))$ associated with the operators D_1, D_2 is defined by

$$E(t,(y,\theta),(x,\tau)) = T_{(y,\theta)}(E_t)(x,\tau), \quad (y,\theta),(x,\tau) \in \mathcal{K}, \tag{2.23}$$

Remark 2.1 *By using the relation (2.19) and (2.4), the relation (2.23) can also written in the form*

$$\forall (y,\theta),(x,\tau) \in \mathcal{K}, E(t,(y,\theta),(x,\tau)) = \int_{\widetilde{C}\cup\widetilde{D}} e^{-t(\lambda^2+\mu^2+\frac{9}{4})}\varphi_{\lambda,\mu}(y,\theta)\varphi_{\lambda,\mu}(x,-\tau)d\gamma(\lambda,\mu). \tag{2.24}$$

Proposition 2.7 *i)For all $t > 0$ we have*

$$\forall (x,\tau) \in \mathcal{K}, \quad ||E(t,(x,\tau),(.,.))||^2_{\alpha,2} = E(2t,(x,\tau),(x,\tau)). \tag{2.25}$$

ii)For all $t > 0$ we have

$$E(2t,(y,\theta),(y,\theta)) \le ||E_t||^2_{\alpha,2}. \tag{2.26}$$

Proof.

i) From Proposition 2.5 ii), the fact that the function E_t belongs to $S_*(\mathbb{R}^2)$ and the relations (2.22),(2.23), we have

$$\begin{aligned}\forall (x,\tau)\in\mathcal{K},\quad ||E(t,(x,\tau),(.,.))||^2_{\alpha,2} &= T_{(x,\tau)}(E_t*(E_t)^*)(x,\tau)\\ &= T_{(x,\tau)}(E_{2t})(x,\tau)\\ &= E(2t,(x,\tau),(x,\tau)).\end{aligned}$$

ii)From the relations (2.23) ,(2.25) and (2.11) we deduce that for all $t>0$ we have

$$E(2t,(y,\theta),(y,\theta))\leq ||E_t||^2_{\alpha,2}.$$

Remark 2.2 *From Theorem (1.1) and (2.23) we deduce that the function $E(t,(y,\theta),(x,\tau))$ is bounded.*

3 Wavelets associated with the operators D_1, D_2

We consider in this section a non negligible function g in $L^1_{A_\alpha}(\mathcal{K})\cap L^2_{A_\alpha}(\mathcal{K})$.
Notation. We denote by $\mathcal{M}^p_{g,s}(\mathcal{K})$, $s\in\mathbb{R}$, $p=1,2$, the space of measurable functions on $\mathcal{K}$, such that

$$||f||^p_{\mathcal{M}^p_{g,s}}=\int_{\mathcal{K}}|f((y,\theta))|^p\frac{\mathcal{A}_\alpha(y)dyd\theta}{||T_{(y,\theta)}g||^{2(s-1)}_{\alpha,2}}<+\infty.$$

Remark 3.1 *From the relation (2.11), we deduce that*
- If $s<1$. $L^p_{A_\alpha}(\mathcal{K})\subset\mathcal{M}^p_{g,s}(\mathcal{K})$.
- If $s=1$. $\mathcal{M}^p_{g,s}(\mathcal{K})=L^p_{A_\alpha}(\mathcal{K})$.
- If $s>1$. $\mathcal{M}^p_{g,s}(\mathcal{K})\subset L^p_{A_\alpha}(\mathcal{K})$.

Definition 3.1 *Let $(\lambda,\mu)\in\widetilde{C}\cup\widetilde{D},(y,\theta)\in\mathcal{K}$ and $s\in\mathbb{R}$. The family of wavelets $\{g^s_{(\lambda,\mu),(y,\theta)}\}_{s\in\mathbb{R}}$ associated with the operators D_1,D_2 is defined on $\mathcal{K}$ by*

$$g^s_{(\lambda,\mu),(y,\theta)}(x,\tau)=\varphi_{\lambda,\mu}(x,\tau)\frac{T_{(y,\theta)}g(x,\tau)}{||T_{(x,\tau)}g||^s_{\alpha,2}}. \tag{3.1}$$

Proposition 3.1 *We suppose that the function g is such that, for all $(y,\theta)\in\mathcal{K}$ and $s\in\mathbb{R}$, the function $(x,\tau)\longrightarrow\dfrac{T_{(y,\theta)}g(x,\tau)}{||T_{(x,\tau)}g||^s_{\alpha,2}}$ belongs to $L^\infty_{A_\alpha}(\mathcal{K})\cap L^2_{A_\alpha}(\mathcal{K})$ (resp. $L^\infty_{A_\alpha}(\mathcal{K})\cap\mathcal{M}^2_{g,s}(\mathcal{K})$).*
Then the function $g^s_{(\lambda,\mu),(y,\theta)}$ belongs to $L^\infty_{A_\alpha}(\mathcal{K})\cap L^2_{A_\alpha}(\mathcal{K})$ (resp. $L^\infty_{A_\alpha}(\mathcal{K})\cap\mathcal{M}^2_{g,s}(\mathcal{K})$).

Proof.
We deduce the results from the relations (3.1),(1.4).

Proposition 3.2 *Under the hypothesis of Proposition 3.1 and if moreover*
i) for $s \leq 1$. *For* $(x,\tau) \in \mathcal{K}$ *the function* $(y,\theta) \longrightarrow \dfrac{T_{(y,\theta)}g(x,\tau)}{||T_{(x,\tau)}g||^s_{\alpha,2}}$ *is continuous from* $\mathcal{K}$ *into* $L^2_{A_{\alpha,2}}(\mathcal{K})$.
ii) For $s > 1$. *For* $(x,\tau) \in \mathcal{K}$ *the function* $(y,\theta) \longrightarrow \dfrac{T_{(y,\theta)}g(x,\tau)}{||T_{(x,\tau)}g||^s_{\alpha,2}}$ *is continuous from* $\mathcal{K}$ *into* $\mathcal{M}^2_{g,s}(K)$.
Then,
i) For $s \leq 1$. *The function* $((\lambda,\mu),(y,\theta)) \longrightarrow g^s_{(\lambda,\mu),(y,\theta)}$ *is continuous from* $\widetilde{C} \cup \widetilde{D} \times \mathcal{K}$ *into* $L^2_{A_\alpha}(\mathcal{K})$.
ii) For $s > 1$. *The function* $((\lambda,\mu),(y,\theta)) \longrightarrow g^s_{(\lambda,\mu),(y,\theta)}$ *is continuous from* $\widetilde{C} \cup \widetilde{D} \times \mathcal{K}$ *into* $\mathcal{M}^2_{g,s}(\mathcal{K})$.

Proof.
i) If $s \leq 1$. Let $((\lambda_0,\mu_0),(y_0,\theta_0)) \in \widetilde{C} \cup \widetilde{D} \times \mathcal{K}$. Using (3.1) and the fact that $T_{(y,\theta)}g(x,\tau) = T_{(x,\tau)}g(y,\theta)$ we obtain

$$\begin{aligned}
&||g^s_{(\lambda,\mu),(y,\theta)} - g^s_{(\lambda_0,\mu_0),(y_0,\theta_0)}||_{\alpha,2} \\
&\qquad \leq ||\varphi_{\lambda_0,\mu_0}(x,\tau)\Big(\tfrac{T_{(x,\tau)}g(y,\theta)}{||T_{(x,\tau)}g||^s_{\alpha,2}} - \tfrac{T_{(x,\tau)}g(y_0,\theta_0)}{||T_{(x,\tau)}g||^s_{\alpha,2}}\Big)||_{\alpha,2} \\
&\qquad + ||(\varphi_{\lambda,\mu}(x,\tau) - \varphi_{\lambda_0,\mu_0}(x,\tau)).\tfrac{T_{(x,\tau)}g(y_0,\theta_0)}{||T_{(x,\tau)}g||^s_{\alpha,2}}||_{\alpha,2} \\
&\qquad + ||(\varphi_{\lambda,\mu}(x,\tau) - \varphi_{\lambda_0,\mu_0}(x,\tau))\Big(\tfrac{T_{(x,\tau)}g(y,\theta)}{||T_{(x,\tau)}g||^s_{\alpha,2}} - \tfrac{T_{(x,\tau)}g(y_0,\theta_0)}{||T_{(x,\tau)}g||^s_{\alpha,2}}\Big)||_{\alpha,2}
\end{aligned}$$

Using (1.4), we get

$$\begin{aligned}
||g^s_{(\lambda,\mu),(y,\theta)} - g^s_{(\lambda_0,\mu_0),(y_0,\theta_0)}||_{\alpha,2} &\leq 3||\frac{T_{(x,\tau)}g(y,\theta)}{||T_{(x,\tau)}g||^s_{\alpha,2}} - \frac{T_{(x,\tau)}g(y_0,\theta_0)}{||T_{(x,\tau)}g||^s_{\alpha,2}}||_{\alpha,2} \\
&+ ||(\varphi_{\lambda,\mu}(x,\tau) - \varphi_{\lambda_0,\mu_0}(x,\tau)).\frac{T_{(x,\tau)}g(y_0,\theta_0)}{||T_{(x,\tau)}g||^s_{\alpha,2}}||_{\alpha,2}
\end{aligned}$$

From hypothesis i) we obtain

$$\lim_{(y,\theta)\to(y_0,\theta_0)} ||\frac{T_{(x,\tau)}g(y,\theta)}{||T_{(x,\tau)}g||^s_{\alpha,2}} - \frac{T_{(x,\tau)}g(y_0,\theta_0)}{||T_{(x,\tau)}g||^s_{\alpha,2}}||_{\alpha,2} = 0, \qquad (3.2)$$

and from Proposition 3.1, the relation (1.6) and the dominated convergence theorem, we get

$$\lim_{(\lambda,\mu)\to(\lambda_0,\mu_0)} ||(\varphi_{\lambda,\mu}(x,\tau) - \varphi_{\lambda_0,\mu_0}(x,\tau)).\frac{T_{(x,\tau)}g(y_0,\theta_0)}{||T_{(x,\tau)}g||^s_{\alpha,2}}||_{\alpha,2} = 0. \qquad (3.3)$$

Using (3.2),(3.3) we deduce that

$$\lim_{((\lambda,\mu),(y,\theta))\to((\lambda_0,\mu_0),(y_0,\theta_0))} ||g^s_{(\lambda,\mu),(y,\theta)} - g^s_{(\lambda_0,\mu_0),(y_0,\theta_0)}||_{\alpha,2} = 0.$$

ii) If $s > 1$. Using the same proof as for case $s \leq 1$, by changing the measure $\mathcal{A}_\alpha(x)dx$ by the measure $\dfrac{\mathcal{A}_\alpha(x)dx}{||T_{(x,\tau)}g||_{\alpha,2}^{2(s-1)}}$ we deduce that the function $(\lambda, y) \longrightarrow g^s_{(\lambda,\mu),(y,\theta)}$ is continuous from $\widetilde{C} \cup \widetilde{D} \times \mathcal{K}$ into $\mathcal{M}^2_{g,s}(\mathcal{K})$.

4 Example

As an example of the function g considered in the previous section, we take the heat kernel $E_t,\ t > 0$, associated with the operators D_1, D_2. We obtain the Gaussian wavelets associated with the operators D_1, D_2.

Definition 4.1 *Let* $((\lambda,\mu),(y,\theta)) \in \widetilde{C} \cup \widetilde{D} \times \mathcal{K}$ *. The family of Gaussian wavelets* $G^s_{(\lambda,\mu),(y,\theta)}$ *given by*

$$G^s_{(\lambda,\mu),(y,\theta)}(x,\tau) = \varphi_{\lambda,\mu}(x,\tau)\frac{T_{(x,\tau)}E_t(y,\theta)}{\|T_{(x,\tau)}E_t(y,\theta)\|^s}. \tag{4.1}$$

Remark 4.1 *The family of Gaussian wavelets* $G^s_{(\lambda,\mu),(y,\theta)}$ *is also given by*

$$G^s_{(\lambda,\mu),(y,\theta)}(x,\tau) = \varphi_{\lambda,\mu}(x,\tau)\frac{E(t,(y,\theta),(x,\tau))}{(E(2t,(x,\tau),(x,\tau)))^{s/2}}. \tag{4.2}$$

Proposition 4.1 *For all* $(y,\theta), \in \mathcal{K}$ *and* $s \leq 0$*, the function* $\dfrac{E(t,(y,\theta),(x,\tau))}{(E(2t,(x,\tau),(x,\tau)))^{s/2}}$ *belongs to* $L^\infty_{A_\alpha}(\mathcal{K}) \cap L^2_{A_\alpha}(\mathcal{K})$ *(resp.* $L^\infty_{A_\alpha}(\mathcal{K}) \cap \mathcal{M}^2_{g,s}(\mathcal{K})$*), then from the Proposition 3.1 the function* $G^s_{(\lambda,\mu),(y,\theta)}$ *belongs to* $L^\infty_{A_\alpha}(\mathcal{K}) \cap L^2_{A_\alpha}(\mathcal{K})$ *(resp.* $L^\infty_{A_\alpha}(\mathcal{K}) \cap \mathcal{M}^2_{g,s}(\mathcal{K})$*).*

Proof.
From the relations (2.23),(2.25) and (2.26), we deduce that there exists a positive constant $M_0(t)$ such that for all $(y,\theta) \in \mathcal{K}$, we have

$$\forall (x,\tau) \in \mathcal{K}, \quad \frac{E(t,(y,\theta),(x,\tau))}{(E(2t,(x,\tau),(x,\tau)))^{s/2}} \leq M_0(t). \tag{4.3}$$

We obtain the result asked from (1.4), the continuity of the function $(x,\tau) \longmapsto \dfrac{E(t,(y,\theta),(x,\tau))}{(E(2t,(x,\tau),(x,\tau)))^{s/2}}$ on $\mathcal{K}$, and the relation (4.3).

5 The generalized windowed transform associated with the operators D_1, D_2

In this section, we take a non negligible function g in $L^1_{A_\alpha}(\mathcal{K}) \cap L^2_{A_\alpha}(\mathcal{K})$ satisfying the hypothesis of Propositions 3.1, 3.2.

Definition 5.1 *Let $s \in \mathbb{R}$. The generalized windowed transform Φ_g^s is defined for regular function f on $\mathcal{K}$ by*

$$\Phi_g^s(f)((\lambda,\mu),(y,\theta)) = \int_{\mathcal{K}} f(x,\tau)(g^s_{(\lambda,\mu),(y,\theta)})^*(x,\tau)\mathcal{A}_\alpha(x)dxd\tau, \quad ((\lambda,\mu),(y,\theta)) \in \widetilde{C}\cup\widetilde{D}\times\mathcal{K}, \tag{5.1}$$

where

$$\forall (x,\tau) \in \mathcal{K}, \quad (g^s_{(\lambda,\mu),(y,\theta)})^*(x,\tau) = \overline{g^s_{(\lambda,\mu),(y,\theta)}}(x,-\tau).$$

Remark 5.1 *The relation (5.1) can also be written in the following two forms.*

$$i) \qquad \Phi_g^s(f)((\lambda,\mu),(y,\theta)) = \left(\frac{\varphi_{\lambda,\mu} f}{||T_{(.,.)}g||^s_{\alpha,2}}\right) * g^*(y,-\theta), \tag{5.2}$$

where $*$ is the convolution product defined by (2.6).

$$ii) \qquad \Phi_g^s(f)((\lambda,\mu),(y,\theta)) = \mathcal{F}\left(f.\frac{T_{(y,\theta)}(g^*)}{||T_{(.,.)}g||^s_{\alpha,2}}\right)(-\lambda,\mu), \tag{5.3}$$

where $\mathcal{F}$ is Fourier transform associated with the operators D_1, D_2 given by (1.12).

5.1 Plancherel formula for the generalized windowed transform

Theorem 5.1 *For all $s \in \mathbb{R}$, we have for the transform Φ_g^s the following Plancherel formula*

$$\int_{\widetilde{C}\cup\widetilde{D}} \int_{\mathcal{K}} |\Phi_g^s(f)((\lambda,\mu),(y,\theta))|^2 d\gamma(\lambda,\mu)\mathcal{A}_\alpha(y)dyd\theta = ||f||^2_{\mathcal{M}^2_{g,s}}.$$

This formula is true for the functions of the following spaces.

i) If $s \leq 1$. $f \in L^2_{A_\alpha}(\mathcal{K})$.

ii) If $s > 1$. $f \in \mathcal{M}^2_{g,s}(\mathcal{K})$.

Proof.
i) If $s \leq 1$. For all $(y,\theta) \in \mathcal{K}$, the function $\dfrac{T_{(y,\theta)}(g^*)(x,\tau)}{||T_{(x,\tau)}g||^s_{\alpha,2}}$ is in $L^\infty_{A_\alpha}(\mathcal{K})$ and as f is in $L^2_{A_\alpha}(\mathcal{K})$, then the function $(x,\tau) \longrightarrow f(x,\tau)\dfrac{T_{(y,\theta)}(g^*)(x,\tau)}{||T_{(x,\tau)}g||^s_{\alpha,2}}$ belongs to $L^2_{A_\alpha}(\mathcal{K})$. Thus, from (5.3) we deduce that

$$\int_{\widetilde{C}\cup\widetilde{D}}\int_{\mathcal{K}}|\Phi_g^s(f)((\lambda,\mu),(y,\theta))|^2\mathcal{A}_\alpha(y)dyd\theta d\gamma(\lambda,\mu)$$
$$=\int_{\widetilde{C}\cup\widetilde{D}}\int_{\mathcal{K}}|\mathcal{F}(f.\frac{T_{(y,\theta)}(g^*)}{||T_{(.,.)}g||^s_{\alpha,2}})(\lambda,-\mu)|^2\mathcal{A}_\alpha(y)dyd\theta d\gamma(\lambda,\mu).$$

From Theorem 1.3, the fact that

$$||T_{(x,\tau)}(g^*)||_{\alpha,2}=||T_{(x,\tau)}(g)||_{\alpha,2}$$

and Fubini-Tonnelli's theorem we obtain

$$\int_{\mathcal{K}}\int_{\widetilde{C}\cup\widetilde{D}}\Phi_g^s(f)((\lambda,\mu),(y,\theta))|^2d\gamma(\lambda,\mu)\mathcal{A}_\alpha(y)dyd\theta$$

$$\begin{aligned}
&= \int_{\mathcal{K}}\frac{|f(x,\tau)|^2}{||T_{(x,\tau)}(g)||^{2s}_{\alpha,2}}(\int_{\mathcal{K}}|T_{(y,\theta)}g(x,\tau)|^2\mathcal{A}_\alpha(y)dyd\theta)\mathcal{A}_\alpha(x)dxd\tau\\
&= \int_{\mathcal{K}}\frac{|f(x,\tau)|^2}{||T_{(x,\tau)}(g)||^{2(s-1)}_{\alpha,2}}\mathcal{A}_\alpha(x)dxd\tau\\
&= ||f||^2_{\mathcal{M}^2_{g,s}}.
\end{aligned}$$

ii) If $s>1$. We obtain the result in this case by using the same proof as for the case $s\leq 1$.

5.2 Inversion formula for the generalized windowed transform

Theorem 5.2 *For all $s\in\mathbb{R}$, the transform Φ_g^s admits the following inversion formula. Let $S_{p,q}$ be the subset of $\widetilde{C}\cup\widetilde{D}$ and $\lim_{(p,q)\to+\infty}S_{p,q}=\widetilde{C}\cup\widetilde{D}$*
Then we have for $(x,\tau)\in\mathcal{K}$

$$f(x,\tau)=\lim_{(p,q)\to+\infty}\int_{S_{p,q}}\int_{\mathcal{K}}\Phi_g^s(f)((\lambda,\mu),(y,\theta))g^{2-s}_{(\lambda,\mu),(y,\theta)}(x,-\tau)\mathcal{A}_\alpha(y)dyd\theta d\gamma(\lambda,\mu). \tag{5.4}$$

the limit is in $L^2_\alpha(\mathcal{K})$. This formula is true for the functions f of the following spaces.

i) If $s\leq 1$. $f\in L^1_{A_\alpha}(\mathcal{K})\cap L^2_{A_\alpha}(\mathcal{K})$.

ii) If $s>1$. $f\in\mathcal{M}^1_{g,s}(\mathcal{K})\cap\mathcal{M}^2_{g,s}(\mathcal{K})$.

To prove this theorem, we need the following lemma.

Lemma 5.1 *For all $(\lambda,\mu)\in\widetilde{C}\cup\widetilde{D}$, the integral*

$$\int_{\mathcal{K}}\Phi_g^s(f)((\lambda,\mu),(y,\theta))g^{2-s}_{(\lambda,\mu),(y,\theta)}(x,-\tau)\mathcal{A}_\alpha(y)dyd\theta, \tag{5.5}$$

is absolutely convergent and satisfies for all $(\lambda,\mu),(y,\theta)\in\widetilde{C}\cup\widetilde{D}\times\mathcal{K}$, the following relation

$$\int_{\mathcal{K}}\Phi_g^s(f)((\lambda,\mu),(y,\theta))g^{2-s}_{(\lambda,\mu),(y,\theta)}(x,-\tau)\mathcal{A}_\alpha(y)dyd\theta=\frac{\varphi_{\lambda,\mu}(x,-\tau)}{||T_{(x,-\tau)}g||^2_{\alpha,2}}\mathcal{F}(f.T_{(x,-\tau)}(g*g^*))(-\lambda,\mu). \tag{5.6}$$

These results are true for the functions f of the following spaces.

i) If $s\leq 1$. $f\in L^1_{A_\alpha}(\mathcal{K})\cap L^2_{A_\alpha}(\mathcal{K})$.

ii) If $s>1$. $f\in\mathcal{M}^1_{g,s}(\mathcal{K})\cap\mathcal{M}^2_{g,s}(\mathcal{K})$.

Proof.
i) If $s\leq 1$. Using (1.4), we have for all $((\lambda,\mu),(x,\tau))\in\widetilde{C}\cup\widetilde{D}\times\mathcal{K}$,

$$\int_{\mathcal{K}}|\Phi_g^s(f)((\lambda,\mu),(y,\theta))g^{2-s}_{(\lambda,\mu),(y,\theta)}(x,-\tau)|\mathcal{A}_\alpha(y)dyd\theta$$
$$\leq\frac{1}{||T_{(x,-\tau)}g||^{2-s}_{\alpha,2}}\int_{\mathcal{K}}|\Phi_g^s(f)((\lambda,\mu),(y,\theta))||T_{(y,\theta)}g(x,-\tau)|\mathcal{A}_\alpha(y)dyd\theta.$$

Using Hölder's inequality and the relation (5.2) we obtain,

$$\int_{\mathcal{K}}|\Phi_g^s(f)((\lambda,\mu),(y,\theta))g^{2-s}_{(\lambda,\mu),(y,\theta)}(x,-\tau)|\mathcal{A}_\alpha(y)dyd\theta$$
$$\leq\frac{1}{||T_{(x,-\tau)}g||^{2-s}_{\alpha,2}}||\Phi_g^s(f)((\lambda,\mu),(.,.))||_{\alpha,2}||g||_{\alpha,2}$$
$$\leq\frac{||f||_{\alpha,1}||g||^2_{\alpha,2}}{||T_{(x,-\tau)}g||^2_{\alpha,2}}<+\infty.$$

Thus the integral (5.5) is absolutely convergent.
By using (5.2), we obtain for all $((\lambda,\mu),(x,\tau))\in\widetilde{C}\cup\widetilde{D}\times\mathcal{K}$,

$$\int_{\mathcal{K}}\Phi_g^s(f)((\lambda,\mu),(y,\theta))g^{2-s}_{(\lambda,\mu),(y,\theta)}(x,-\tau)\mathcal{A}_\alpha(y)dyd\theta$$
$$=\frac{\varphi_{\lambda,\mu}(x,-\tau)}{||T_{(x,-\tau)}g||^2_{\alpha,2}}\int_{\mathcal{K}}[(\varphi_{\lambda,\mu}(.,.)f)*g^*(y,-\theta)].T_{(y,\theta)}g(x,-\tau)\mathcal{A}_\alpha(y)dyd\theta. \tag{5.7}$$

But from the associativity of the convolution product associated with the operators D_1,D_2, we get

$$\int_{\mathcal{K}}[(\varphi_{\lambda,\mu}(.,.)f)*g^*(y,-\theta))].T_{(y,\theta)}g(x,-\tau)\mathcal{A}_\alpha(y)dyd\theta = (\varphi_{\lambda,\mu}(.,.)f)*(g*g^*)(x,-\tau)$$

$$= \mathcal{F}(f.T_{(x,-\tau)}(g*g^*))(-\lambda,\mu). \quad (5.8)$$

Thus, we deduce (5.6) from the relation (5.7), (5.8).

ii) If $s > 1$. The same arguments used in i) imply the results of the Lemma 5.1 for the function f of the space $\mathcal{M}^1_{g,s}(\mathcal{K}) \cap \mathcal{M}^2_{g,s}(\mathcal{K})$.

Proof of Theorem 5.2.

i) If $s \leq 1$. For all f in $L^1_{A_\alpha}(\mathcal{K}) \cap L^2_{A_\alpha}(\mathcal{K})$ and $(x,\tau) \in \mathcal{K}$, we have from Lemma 5.1

$$\int_{S_{p,q}}(\int_{\mathcal{K}} \Phi^s_g(f)((\lambda,\mu),(y,\theta))g^{2-s}_{(\lambda,\mu),(y,\theta)}(x,-\tau)\mathcal{A}_\alpha(y)dyd\theta)d\gamma(\lambda,\mu)$$

$$= \frac{1}{||T_{(x,\tau)}g||^2_{\alpha,2}} \int_{S_{p,q}} \mathcal{F}(f.T_{(x,-\tau)}(g*g^*))(\lambda,-\mu)\varphi_{\lambda,\mu}(x,-\tau)d\gamma(\lambda,\mu).$$

As the functions f and g are in $L^1_{A_\alpha}(\mathcal{K}) \cap L^2_{A_\alpha}(\mathcal{K})$, then from Proposition 2.3 , the function $(t,\rho) \longrightarrow f(t,\rho).T_{(x,-\tau)}(g*g^*)(t,\rho)$ belongs to $L^1_{A_\alpha}(\mathcal{K}) \cap L^2_{A_\alpha}(\mathcal{K})$. Then, from Theorem 1.2 and Proposition 2.5, we deduce that

$$\lim_{(p,q)\to+\infty} \int_{S_{p,q}}(\int_{\mathcal{K}} \Phi^s_g(f)((\lambda,\mu),(y,\theta))g^{2-s}_{(\lambda,\mu),(y,\theta)}(x,-\tau)\mathcal{A}_\alpha(y)dyd\theta)d\gamma(\lambda,\mu)$$

$$\begin{aligned} &= \frac{1}{||T_{(x,\tau)}g||^2_{\alpha,2}} f(x,\tau).T_{(x,-\tau)}(g*g^*)(x,\tau) \\ &= f(x,\tau). \end{aligned}$$

ii) If $s > 1$. Let f be in $\in \mathcal{M}^1_{g,s}(\mathcal{K}) \cap \mathcal{M}^2_{g,s}(\mathcal{K})$. We obtain the result of this case by using the same proof as for the previous case.

Theorem 5.3 *We consider the function g in $S_*(\mathbb{R}^2)$. Then for all f in $S_*(\mathbb{R}^2)$and $s \in \mathbb{R}$, we have the following inversion formula, $\forall (x,\tau) \in \mathcal{K}$,*

$$f(x,\tau) = \int_{\widetilde{C}\cup\widetilde{D}} \int_{\mathcal{K}} \Phi^s_g(f)((\lambda,\mu),(y,\theta))g^{2-s}_{(\lambda,\mu),(y,\theta)}(x,-\tau)\mathcal{A}_\alpha(y)dyd\theta d\gamma(\lambda,\mu). \quad (5.9)$$

Proof.

We deduce the relation (5.9) from (5.6), Proposition 2.5 and Theorem 1.2.

6 Example

The Gaussian windowed transform $\Phi_G^s,\ s \leq 0$, associated with the operators D_1, D_2 is defined for regular function f by

$$\Phi_G^s(f)((\lambda,\mu),(y,\theta)) = \int_{\mathcal{K}} f(x,\tau)(G^s_{(\lambda,\mu),(y,\theta)})^*(x,\tau)\mathcal{A}_\alpha(x)dxd\tau, \tag{6.1}$$

$(\lambda,\mu),(y,\theta) \in \widetilde{C} \cup \widetilde{D} \times \mathcal{K}$ where $G^s_{(\lambda,\mu),(y,\theta)}$ is the Gaussian wavelet given by (4.1).

By applying to this transform the results of the previous sections we obtain for the transform $\Phi_G^s, s \in \mathbb{R}$, analogous Plancherel and inversion formulas.

7 Open Problem

In the future work I will to study the wavelet and the generalized windowed transform on the generalized Sobolev spaces.

References

[1] L.C. Andrews, *Special functions of Mathematics for engineers*, second edition. Oxford University Press-Oxford-Tokyo-Melbourn, 1998.

[2] A. Hassini and K. Trimèche, *Wavelets and generalized windowed transforms associated with the Dunkl-Bessel-Laplace operator on* $\mathbb{R}^d \times \mathbb{R}_+$, Mediterr. J. Math. 12,(2015), p. 1323-1344.

[3] T.H. Koornwinder, *The continuous wavelet transform. Series in Approximations and Decompositions*, Vol.1. Wavelets: An elementary treatment of theory and applications. Edited by T.H.Koornwinder, World Scientific, (1993), p.27-48.

[4] M. Flensted-Jensen, *Spherical Functions on a Simply Connected Semigroupe wavelet Lie Group II.The Paley-Wiener Theorem for the Rank one Case*, Math.Ann.(1977),228, p.65-92.

[5] K.Trimèche, *Opérateurs de permutation et analyse harmonique associés a des opérateurs aux dérives partielles*, Math. Pures et Appl.(1991),70,p.1-73.

[6] K.Trimèche, *Generalized Harmonic Analysis and Wavelet Packets*, Gordon and Breach Sci. Publishers, 2001.

[7] K.Trimèche, *Generalized Wavelets and Hypergroups*, Gordon and Breach Science Publishers, 1997.