Int. J. Open Problems Complex Analysis, Vol. 10, No. 2, July 2018
ISSN 2074-2827; Copyright ©ICSRS Publication, 2018
www.i-csrs.org

Some Classes of Analytic Functions Associated with the Wright Generalized Hypergeometric Function

E. A. Adwan

Common First Year, Saudi Electronic University,
Riyadh, Saudi Arabia
e-mail: eman.a2009@yahoo.com

Received 15 April 2018; Accepted 1 July 2018
Communicated by Iqbal H. Jebril

Abstract

In this paper, we introduce some classes of multivalent functions associated with the Wright generalized hypergeometric function and derive several interesting results of these classes.

Keywords: *MAnalytic function, Hadamard product, starlike function, convex function, Wright generalized hypergeometric function.*

2010 Mathematical Subject Classification: 30C45.

1 Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1+p}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1}$$

which are analytic and p-valent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $A(1) = A$.

Let $P_k(\gamma, p)$ $(k \geq 2, 0 \leq \gamma < p, p \in \mathbb{N})$ denote the class of functions

$$g(z) = p + \sum_{k=1}^{\infty} c_k z^k \tag{2}$$

which are analytic in U and satisfy for every $r < 1 \left(z = re^{i\theta} \in U\right)$ the conditions

(1) $g(0) = p$,

$$(2) \int_0^{2\pi} \frac{|\mathrm{Re}\{g(z)\} - \gamma|}{(p-\gamma)} d\theta \leq k\pi. \tag{3}$$

The class $P_k(\gamma, p)$ was introduced and studied by Aouf [2].

We note that:

(1) $P_k(\alpha, 1) = P_k(\gamma)$ $(k \geq 2, 0 \leq \gamma < 1)$ (see Padmanabhan and Parvatham [17]);

(2) $P_k(0, 1) = P_k$ $(k \geq 2)$ (see Pinchuk [20] and Robertson [23]);

(3) $P_2(\gamma, p) = P(\gamma, p)$, $(0 \leq \gamma < p,\ p \in \mathbb{N})$, where $P(\gamma, p)$ is the class of functions g of the form (2) and satisfy the conditions $g(0) = p$ and $\mathrm{Re}\{g(z)\} > \gamma$, $(0 \leq \gamma < p)$ in U;

(4) $P_2(0, 1) = P$, where P is the class of functions with positive real part in U;

(5) $P_2(\gamma, 1) = P(\gamma)$ $(0 \leq \gamma < 1)$, where $h(\gamma) = (1-\gamma)p(z) + \gamma$, $h(z) \in P(\gamma)$ and $p(z) \in P$.

From (1.2), we have $g(z) \in P_k(\gamma, p)$ if and only if there exists $g_i \in P(\gamma, p)$, $i = 1, 2$ such that (see [2])

$$g(z) = \left(\frac{k}{4} + \frac{1}{2}\right) g_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) g_2(z) \quad (z \in U). \tag{4}$$

For analytic functions $f(z) \in A(p)$, given by (1) and $\phi(z) \in A(p)$ given by $\phi(z) = z^p + \sum\limits_{k=1+p}^{\infty} b_k z^k$ $(p \in \mathbb{N})$, the Hadamard product (or convolution) of $f(z)$ and $\phi(z)$, is defined by

$$(f * \phi)(z) = z^p + \sum_{k=1+p}^{\infty} a_k b_k z^k = (\phi * f)(z). \tag{5}$$

Let $\alpha_1, A_1, ..., \alpha_q, A_q$ and $\beta_1, B_1, ..., \beta_s, B_s$ $(q, s \in \mathbb{N})$ be positive real parameters such that

$$1+\sum_{j=1}^{s}B_j-\sum_{j=1}^{q}A_j\geq 0.$$

The Wright generalized hypergeometric function [28] (see also [27])

$${}_q\Psi_s\left[(\alpha_1,A_1),...,(\alpha_q,A_q);(\beta_1,B_1),...,(\beta_s,B_s);z\right]={}_q\Psi_s\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]$$

is defined by

$${}_q\Psi_s\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{q}\Gamma(\alpha_i+kA_i)}{\prod_{i=1}^{s}\Gamma(\beta_i+kB_i)}.\frac{z^n}{k!}\quad(z\in U).$$

If $A_i=1(i=1,...,q)$ and $B_i=1\,(i=1,...,s)$, we have the relationship:

$$\Omega_q\Psi_s\left[(\alpha_i,1)_{1,q};(\beta_i,1)_{1,s};z\right]={}_qF_s\left(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z\right),$$

where ${}_qF_s\left(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z\right)$ is the generalized hypergeometric function (see [27]) and

$$\Omega=\frac{\prod_{i=1}^{s}\Gamma(\beta_i)}{\prod_{i=1}^{q}\Gamma(\alpha_i)}.\tag{6}$$

The Wright generalized hypergeometric functions were invoked in the geometric function theory (see [21] and [22]).
By using the generalized hypergeometric function Dziok and Srivastava [10] introduced a linear operator. In [9] Dziok and Raina and in [4] Aouf and Dziok extended this linear operator by using Wright generalized hypergeometric function.
Aouf et al. [6] considered the following linear operator

$$\theta_{p,q,s}\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s}\right]:A(p)\to A(p),$$

defined by the following Hadamard product:

$$\theta_{p,q,s}\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s}\right]f(z)={}_q\Phi_s^p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]*f(z),$$

where ${}_q\Phi_s^p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]$ is given by

$$ {}_q\Phi_s^p\left[(\alpha_i, A_i)_{1,q}\,;(\beta_i, B_i)_{1,s}\,;z\right] = \Omega\ z^p\ {}_q\Psi_s\left[(\alpha_i, A_i)_{1,q}\,;(\beta_i, B_i)_{1,s}\,;z\right]. $$

We observe that, for a function $f(z)$ of the form (1), we have

$$ \theta_{p,q,s}\left[(\alpha_i, A_i)_{1,q}\,;(\beta_i, B_i)_{1,s}\right] f(z) = z^p + \sum_{k=1+p}^{\infty} \Omega\sigma_{k,p}(\alpha_1)\, a_k z^k, \tag{7} $$

where Ω is given by (6) and $\sigma_{k,p}(\alpha_1)$ is defined by

$$ \sigma_{k,p}(\alpha_1) = \frac{\Gamma(\alpha_1 + A_1(k-p))\ldots\Gamma(\alpha_q + A_q(k-p))}{\Gamma(\beta_1 + B_1(k-p))\ldots\Gamma(\beta_s + B_s(k-p))(k-p)!}. \tag{8} $$

If, for convenience, we write

$$ \theta_{p,q,s}[\alpha_1, A_1, B_1] f(z) = \theta_{p,q,s}[(\alpha_1, A_1), ..., (\alpha_q, A_q); (\beta_1, B_1), ..., (\beta_s, B_s)] f(z), $$

then one can easily verify from (7) that

$$ zA_1(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))' = \alpha_1\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f(z) $$

$$ -(\alpha_1 - pA_1)\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z),\ (A_1 > 0). \tag{9} $$

For $p = 1$, $\theta_{1,q,s}[\alpha_1, A_1, B_1] = \theta[\alpha_1]$ which was introduced by Dziok and Raina [9] and studied by Aouf and Dziok [4]. We note that, for $f(z) \in A(p)$, $A_i = 1\,(i = 1, 2, ..., q)$ and $B_i = 1\,(i = 1, 2, ..., s)$, we obtain $\theta_{p,q,s}[\alpha_1, 1, 1] f(z) = H_{p,q,s}[\alpha_1]f(z)$, where $H_{p,q,s}[\alpha_1]$ is the Dziok-Srivastava operator (see [10]).

We note also that, for $f(z) \in A(p), q = 2,\ s = 1$ and $A_1 = A_2 = B_1 = 1$, we have:

(1) $\theta_{p,2,1}[a, 1; c] f(z) = L_p(a, c) f(z)\ \ (a > 0, c > 0, p \in \mathbb{N})$ (see [25]);

(2) $\theta_{p,2,1}[\mu + p, 1; 1] f(z) = D^{\mu+p-1} f(z)\ \ (\mu > -p, p \in \mathbb{N})$, where $D^{\mu+p-1} f(z)$ is the $(\mu + p - 1)$ −the order Ruscheweyh derivative (see [11]);

(3) $\theta_{p,2,1}[\nu + p, 1; \nu + p + 1] f(z) = F_{\nu,p}(f)(z)\ \ (\nu > -p, p \in \mathbb{N})$, where $F_{\nu,p}(f)(z)$ is the generalized Bernardi-Libera-Livingston-integral operator (see [8]);

(4) $\theta_{p,2,1}[c, 1; a] f(z) = I^a_{c,p} f(z)\ \left(a \in \mathbb{R}, c \in \mathbb{C}\backslash\mathbb{Z}_0^-, p \in \mathbb{N}\right)$,where the operator $I^a_{c,p}$ was introduced and studied by by AL-Kharasani and Al-Hajiry (see [1]);

(5) $\theta_{p,2,1}[p+1,1;n+p]\,f(z)=I_{n,p}f(z)\ (n\in\mathbb{Z};n>-p,p\in\mathbb{N})$,where the operator $I_{n,p}$ was introduced and studied by by Liu and Noor (see [14]);

(6) $\theta_{p,2,1}[\lambda+p,c;a]\,f(z)=I_p^{\lambda}(a,c)\,f(z)\ \ (a,c\in\mathbb{R}\backslash\mathbb{Z}_o^-;\lambda>-p,p\in\mathbb{N})$, where $I_p^{\lambda}(a,c)$ is the Cho-Kwon-Srivastava operator (see [7]);

(7) $\theta_{p,2,1}[1+p,1;1+p-\mu]\,f(z)=\Omega_z^{(\mu,p)}f(z)\ \ (-\infty<\mu<1+p,p\in\mathbb{N})$, where the operator $\Omega_z^{(\mu,p)}$ was introduced and studied by Patel and Mishra (see [18]) and studied by Srivastava and Aouf [26] when $(0\leq\mu<1)$.

Now we define the following classes of the class $A(p)$ for $0\leq\gamma,\lambda<p$ and $k\geq 2$:

$$S_k(p,\gamma)=\left\{f(z)\in A(p):\frac{zf'(z)}{f(z)}\in P_k(p,\gamma)\,,z\in U\right\}, \tag{10}$$

$$C_k(p,\gamma)=\left\{f(z)\in A(p):\frac{\left(zf'(z)\right)'}{f'(z)}\in P_k(p,\gamma)\,,z\in U\right\}, \tag{11}$$

and

$$V_k(p,\gamma,\lambda)=\left\{f(z)\in A(p)\,,g(z)\in S_2(p,\gamma):\frac{zf'(z)}{g(z)}\in P_k(p,\lambda)\,,z\in U\right\}. \tag{12}$$

We can easily see that:

$$f(z)\in C_{k,p}(\gamma)\iff\frac{zf'(z)}{p}\in S_{k,p}(\gamma)\,. \tag{13}$$

We note that, for special choices for the parameters k and γ involved in the above classes, we can obtain well-known subclasses of $A(p)$. For example, we have

$$S_{2,p}(\gamma)=S_p^*(\gamma),C_{2,p}(\gamma)=K_p(\gamma)\text{ and }V_{2,p}(\gamma,\lambda)=K_p(\gamma,\lambda).$$

The classes $S_p^*(\gamma)$, $K_p(\gamma)$ and $K_p(\gamma,\lambda)$ denoted by p-valently starlike, convex and close-to-convex of order γ and type $\lambda\,(0\leq\gamma,\lambda<p,p\in\mathbb{N})$. The classes $S_p^*(\gamma)$ and $K_p(\gamma)$ were studied by Patil and Thakare [19] and Owa [16] and the class $K_p(\gamma,\lambda)$ was studied by Aouf [3]. Note that $S_1^*(\gamma)=S^*(\gamma)\ \ (0\leq\gamma<1)$ is the class of starlike function of order γ.

Next, by using the Wright generalized hypergeometric functions, we introduce the following classes of analytic functions for $0 \le \gamma, \lambda < p$, and $k \ge 2$

$$S_{k,p}\left(\alpha_1, A_1, B_1; \gamma\right) = \left\{f\left(z\right) \in A\left(p\right) : \theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f\left(z\right) \in S_k\left(p, \gamma\right), z \in U\right\}, \tag{14}$$

$$C_{k,p}\left(\alpha_1, A_1, B_1; \gamma\right) = \left\{f\left(z\right) \in A\left(p\right) : \theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f\left(z\right) \in C_k\left(p, \gamma\right), z \in U\right\}, \tag{15}$$

and

$$V_{k,p}\left(\alpha_1, A_1, B_1; \gamma, \lambda\right) = \left\{f\left(z\right) \in A\left(p\right) : \theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f\left(z\right) \in V_k\left(p, \lambda\right), z \in U\right\}. \tag{16}$$

We also note that

$$f\left(z\right) \in C_{k,p}\left(\alpha_1, A_1, B_1; \gamma\right) \Leftrightarrow \frac{zf'\left(z\right)}{p} \in S_{k,p}\left(\alpha_1, A_1, B_1; \gamma\right). \tag{17}$$

Putting $q = 2, s = 1,\ \alpha_1 = c, \alpha_2 = 1,\ \beta_1 = a\ (a, c > 0, p \in \mathbb{N})$ and $A_1 = A_2 = B_1 = 1$, in the above classes we obtain, respectively, the following classes:

$$S_{k,p}\left(a, c; \gamma\right) = \left\{f\left(z\right) \in A\left(p\right) : L_p^*\left(a, c\right) f\left(z\right) \in S_k\left(p, \gamma\right), z \in U\right\}, \tag{18}$$

$$C_{k,p}\left(a, c; \gamma\right) = \left\{f\left(z\right) \in A\left(p\right) : L_p^*\left(a, c\right) f\left(z\right) \in C_k\left(p, \gamma\right), z \in U\right\}, \tag{19}$$

and

$$V_{k,p}\left(a, c; \gamma, \lambda\right) = \left\{f\left(z\right) \in A\left(p\right) : L_p^*\left(a, c\right) f\left(z\right) \in V_k\left(p, \lambda\right), z \in U\right\}. \tag{20}$$

We also note that

$$f\left(z\right) \in C_{k,p}\left(a, c; \gamma\right) \Leftrightarrow \frac{zf'\left(z\right)}{p} \in S_{k,p}\left(a, c; \gamma\right). \tag{21}$$

Remark 1. (1) *The classes* $S_{k,p}\left(a, c; \gamma\right), C_{k,p}\left(a, c; \gamma\right)$ *and* $V_{k,p}\left(a, c; \gamma, \lambda\right)$ *given by (18),(19) and (20), respectively, correct the definitions of the classes introduced by Hussain et al. [13, Definations 1.1, 1.2 and 1.3, respectively];*

(2) *Putting* $k = 2$, *in (18), (19) and (20), respectively, we correct the classes introduced by Hussain [12, Definations 1.1, 1.2 and 1.3, respectively].*

2 Preliminary results

In order to prove our results, we need the following lemmas.
Lemma 1 [15]. *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and $\Phi(u, v)$ be a complex-valued function satisfying the conditions:*
(1) $\Phi(u, v)$ *is continuous in a domain* $D \in \mathbb{C}^2$.
(2) $(0, 1) \in D$ *and* $Re\Phi(1, 0) > 0$.
(3) $\Re e\{\Phi(iu_2, v_1)\} > 0$ *where* $(iu_2, v_1) \in D$ *and* $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.
If $h(z) = 1 + c_1z + c_2z^2 + ...$ *is analytic in* U *such that* $(h(z), zh^{'}(z)) \in D$ *and* $Re\{\Phi(h(z), zh^{'}(z))\} > 0$ *for* $z \in U$, *then* $Re\{h(z)\} > 0$ *in* U.

Lemma 2 [24]. *Let* $p(z)$ *be analytic in* U *with* $p(0) = 1$ *and* $\Re\{p(z)\} > 0, z \in U$. *Then for* $s > 0$ *and* $\sigma \neq -1$ *(complex),*

$$\mathrm{Re}\left\{p(z) + \frac{szp^{'}(z)}{p(z) + \sigma}\right\} > 0 \quad (|z| < r_0),$$

where r_0 *is given by*

$$r_0 = \frac{|\sigma + 1|}{\sqrt{A + (A^2 - |\sigma^2 - 1|)^{\frac{1}{2}}}}, A = 2(s + 1)^2 + |\sigma|^2 - 1,$$

and this radius is best possible.

Lemma 3 [24]. Let ϕ be convex and f be starlike in U. Then, for F analytic in U with $F(0) = 1$, $\frac{\phi * Ff}{\phi * f}$ is contained in the convex hull of $F(U)$.

3 Main Results

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters $\alpha_1, A_1, ..., \alpha_q, A_q$ and $\beta_1, B_1, ..., \beta_s, B_s$ $(q, s \in \mathbb{N})$ are positive real numbers, $0 \leq \gamma, \lambda < p$, $k \geq 2$ and $z \in U$.
Theorem 1. *Let* $0 \leq \eta \leq \gamma < p$, $\frac{\alpha_1}{A_1} > p$ *and* $k \geq 2$, *then*

$$S_{k,p}(\alpha_1 + 1, A_1, B_1; \gamma) \subset S_{k,p}(\alpha_1, A_1, B_1; \eta), \tag{22}$$

where

$$\eta = \frac{2[p - 2\gamma(p - \frac{\alpha_1}{A_1})]}{2\frac{\alpha_1}{A_1} - 2p - 2\gamma + 1 + \sqrt{\left(2\frac{\alpha_1}{A_1} - 2p - 2\gamma + 1\right)^2 + 8[p - 2\gamma(p - \frac{\alpha_1}{A_1})}}. \tag{23}$$

Proof. Let $f(z)\in S_{k,p}(\alpha_1+1,A_1,B_1;\gamma)$ and

$$\frac{z\left(\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)\right)'}{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)}=H(z)=(p-\eta)h(z)+\eta, \tag{24}$$

where

$$h(z)=\left(\frac{k}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)h_2(z), \tag{25}$$

where $h_i(z)$ $(i=1,2)$ are analytic in U and $h_i(0)=1$ $(i=1,2)$. Using *(9)* in *(24)* and differentiating the resulting equation with respect to z, we have

$$\frac{z\left(\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]f(z)\right)'}{\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]f(z)}-\gamma=\eta-\gamma+(p-\eta)h(z)+\frac{(p-\eta)zh'(z)}{(p-\eta)h(z)+\eta+\frac{\alpha_1}{A_1}-p}. \tag{26}$$

Now we will show that $H(z)\in P_k(\gamma,p)$ or $h_i(z)\in P$. From *(25)* and *(26)* we have

$$\frac{z\left(\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]f(z)\right)'}{\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]f(z)}-\gamma=\left(\frac{k}{4}+\frac{1}{2}\right)\{\eta-\gamma+(p-\eta)h_1(z)$$

$$+\frac{(p-\eta)zh_1'(z)}{(p-\eta)h_1(z)+\eta+\frac{\alpha_1}{A_1}-p}\Bigg\}-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{\eta-\gamma+(p-\eta)h_2(z)+\frac{(p-\eta)zh_2'(z)}{(p-\eta)h_2(z)+\eta+\frac{\alpha_1}{A_1}-p}\right\},$$

this implies that

$$\operatorname{Re}\left\{\eta-\gamma+(p-\eta)h_i(z)+\frac{(p-\eta)zh_i'(z)}{(p-\eta)h_i(z)+\eta+\frac{\alpha_1}{A_1}-p}\right\}>0 \qquad (i=1,2).$$

We form the functional $\Phi(u,v)$ by taking $u=h_i(z)$, $v=zh_i'(z)$,

$$\Phi(u,v)=\eta-\gamma+(p-\eta)u+\frac{(p-\eta)v}{(p-\eta)u+\eta+\frac{\alpha_1}{A_1}-p}.$$

Clearly, the first two conditions of Lemma 1 are satisfied in the domain $D\subseteq\mathbb{C}\times\mathbb{C}\backslash\frac{\eta+\frac{\alpha_1}{A_1}-p}{\eta-p}$. Now, we verify the condition (iii) as follows:

$$\begin{aligned}\operatorname{Re}\{\Phi(iu_2,v_1)\} &= (\eta-\gamma)+\operatorname{Re}\left\{\frac{(p-\eta)v_1}{(p-\beta)iu_2+\eta+\frac{\alpha_1}{A_1}-p}\right\}\\ &\le (\eta-\gamma)-\frac{(p-\eta)\left(\eta+\frac{\alpha_1}{A_1}-p\right)(1+u_2^2)}{2\left[(p-\eta)^2u_2^2+\left(\eta+\frac{\alpha_1}{A_1}-p\right)^2\right]}\\ &= \frac{A+Bu_2^2}{2C},\end{aligned}$$

where

$$A = 2(\eta-\gamma)\left(\eta+\frac{\alpha_1}{A_1}-p\right)^2-(p-\eta)\left(\eta+\frac{\alpha_1}{A_1}-p\right),$$
$$B = 2(\eta-\gamma)(p-\eta)^2-(p-\eta)\left(\eta+\frac{\alpha_1}{A_1}-p\right),$$
$$C = (p-\eta)^2 u_2^2+\left(\eta+\frac{\alpha_1}{A_1}-p\right)^2.$$

We note that $\operatorname{Re}\{\Phi(iu_2,v_1)\}<0$ if and only if $A\le 0$ and $B<0$. From $A\le 0$, we obtain η as given by *(23)* and from $0\le\eta\le\gamma<p$ we have $B<0$. Therefore applying Lemma 1, $h_i(z)\in P\,(i=1,2)$ and consequently $f\in S_{k,p}(\alpha_1,A_1,B_1;\eta)$. This completes the proof of Theorem 1.

Putting $q=2, s=1, \alpha_1=c, \alpha_2=1, \beta_1=a$ and $A_1=A_2=B_1=1$ $(c\in\mathbb{R}, a\in\mathbb{C}\backslash\mathbb{Z}_0^-, p\in\mathbb{N})$ in Theorem 1, we obtain the following corollary .

Corollary 1. *Let $0\le\eta\le\gamma<p, c>p$ and $k\ge 2$, then*

$$S_{k,p}(a,c+1,\gamma)\subset S_{k,p}(a,c,\eta),$$

where

$$\eta=\frac{2\left[p-2\gamma(p-c)\right]}{2c-2p-2\gamma+1+\sqrt{(2c-2p-2\gamma+1)^2+8\left[p-2\gamma(p-c)\right]}}. \tag{27}$$

Remark 2. *The result in Corollary 1 corrects the result obtained by Hussain et al. [13, Theorem 1].*

Putting $k=2$, in Corollary 1, we obtain the following corollary which corrects the result obtained by Hussain [12, Theorem 1].

.

Corollary 2. *Let $0\le\eta\le\gamma<p$ and $c>p$, then*

$$S_p(a,c+1,\gamma)\subset S_p(a,c,\eta),$$

where η given by (27).

Theorem 2. *Let $0\le\eta\le\gamma<p$, $\frac{\alpha_1}{A_1}>p$ and $k\ge 2$, then*

$$C_{k,p}(\alpha_1+1,A_1,B_1;\gamma)\subset C_{k,p}(\alpha_1,A_1,B_1;\eta). \tag{28}$$

Proof. Applying *(17)* and Theorem 1, we observe that

$$f(z)\in C_{k,p}(\alpha_1+1,A_1,B_1;\gamma) \iff \frac{zf'(z)}{p}\in S_{k,p}(\alpha_1+1,A_1,B_1;\gamma)$$

$$\Longrightarrow \frac{zf'(z)}{p} \in S_{k,p}\left(\alpha_1, A_1, B_1; \eta\right) \Longleftrightarrow f(z) \in C_{k,p}\left(\alpha_1, A_1, B_1; \eta\right),$$

which evidently proves Theorem 2.

Theorem 3. *Let* $0 \leq \gamma, \lambda < p$, $\frac{\alpha_1}{A_1} > p$ *and* $k \geq 2$, *then*

$$V_{k,p}\left(\alpha_1 + 1, A_1, B_1; \gamma, \lambda\right) \subset V_{k,p}\left(\alpha_1, A_1, B_1; \gamma, \lambda\right). \tag{29}$$

Proof. Let $f(z) \in V_{k,p}\left(\alpha_1 + 1, A_1, B_1; \gamma, \lambda\right)$. Then, in view of the definition of the class $V_{k,p}\left(\alpha_1 + 1, A_1, B_1; \gamma, \lambda\right)$, there exists a function $g(z) \in S_{2,p}\left(\alpha_1 + 1, A_1, B_1; \gamma\right)$ such that

$$\frac{z\left(\theta_{p,q,s}\left[\alpha_1 + 1, A_1, B_1\right] f(z)\right)'}{\theta_{p,q,s}\left[\alpha_1 + 1, A_1, B_1\right] g(z)} \in P_k\left(\lambda, p\right) \quad (z \in U).$$

Now let

$$\frac{z\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)\right)'}{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] g(z)} = G(z) = (p - \lambda)\, h(z) + \lambda, \tag{30}$$

where $h(z)$ is given by *(25)*. Using *(9)* in *(30)*, we have

$$\frac{\alpha_1}{A_1}\theta_{p,q,s}\left[\alpha_1 + 1, A_1, B_1\right] f(z) - \left(\frac{\alpha_1}{A_{1\,1}} - p\right)\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)$$

$$= \left[(p - \lambda)\, h(z) + \lambda\right]\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] g(z). \tag{31}$$

Differentiating *(31)* with respect to z and multiplying by z, we obtain

$$\frac{\alpha_1}{A_1} z\left(\theta_{p,q,s}\left[\alpha_1 + 1, A_1, B_1\right] f(z)\right)' - \left(\frac{\alpha_1}{A_1} - p\right) z\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)\right)'$$

$$= (p - \lambda)\, h'(z)\, \theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] g(z) + \left[(p - \lambda)\, h(z) + \lambda\right] z\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] g(z)\right)'. \tag{32}$$

Since $g(z) \in S_{2,p}\left(\alpha_1 + 1, A_1, B_1; \gamma\right)$, by Theorem 1, $g(z) \in S_{2,p}\left(\alpha_1, A_1, B_1; \gamma\right)$, then we have

$$\frac{z\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] g(z)\right)'}{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] g(z)} = (p - \gamma)\, q(z) + \gamma,$$

where $q(z) = 1 + c_1 z + c_2 z^2 + ...$ is analytic in U with $q(0) = 1$. Then by using *(9)*, we have

$$\frac{\alpha_1}{A_1}\frac{\theta_{p,q,s}\left[\alpha_1 + 1, A_1, B_1\right] g(z)}{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] g(z)} = (p - \gamma)\, q(z) + \frac{\alpha_1}{A_1} - p + \gamma. \tag{33}$$

From *(32)* and *(33)*, we obtain

$$\frac{z\left(\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]f\left(z\right)\right)'}{\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]g(z)}-\lambda=\left(p-\lambda\right)h\left(z\right)+\frac{\left(p-\lambda\right)zh'(z)}{\left(p-\gamma\right)q(z)+\frac{\alpha_1}{A_1}-p+\gamma}. \tag{34}$$

Now we will show that $G\left(z\right)\in P_k\left(\lambda,p\right)$ or $h_i\left(z\right)\in P$, $i=1,2$. From *(25)* and *(34)* we have

$$\begin{aligned}&\frac{z\left(\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]f(z)\right)'}{\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]g(z)}-\lambda\\ &=\left(\frac{k}{4}+\frac{1}{2}\right)\left\{\left(p-\lambda\right)h_1\left(z\right)+\frac{\left(p-\lambda\right)zh_1'(z)}{\left(p-\gamma\right)q(z)+\frac{\alpha_1}{A_1}-p+\gamma}\right\}\\ &-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{\left(p-\lambda\right)h_2\left(z\right)+\frac{\left(p-\lambda\right)zh_2'(z)}{\left(p-\gamma\right)q(z)+\frac{\alpha_1}{A_1}-p+\gamma}\right\},\end{aligned}$$

this implies that

$$\operatorname{Re}\left\{\left(p-\lambda\right)h_i\left(z\right)+\frac{\left(p-\lambda\right)zh_i'(z)}{\left(p-\gamma\right)q(z)+\frac{\alpha_1}{A_1}-p+\gamma}\right\}>0\quad\left(z\in U;i=1,2\right).$$

We form the functional $\Phi\left(u,v\right)$ by choosing $u=h_i(z)$, $v=zh_i'\left(z\right)$,

$$\Phi\left(u,v\right)=\left(p-\lambda\right)u+\frac{\left(p-\lambda\right)v}{\left(p-\gamma\right)q(z)+\frac{\alpha_1}{A_1}-p+\gamma}.$$

Clearly, the first two conditions of Lemma 1 are satisfied in the domain $D\subseteq\mathbb{C}\times\mathbb{C}\backslash\frac{\frac{\alpha_1}{A_1}-p+\gamma}{\gamma-p}$ and $q(z)=q_1+iq_2$. Now, we verify the condition (iii) as follows:

$$\begin{aligned}\operatorname{Re}\left\{\Phi\left(iu_2,v_1\right)\right\}&=\operatorname{Re}\left\{\frac{\left(p-\lambda\right)v_1}{\left(p-\gamma\right)\left(q_1+iq_2\right)+\frac{\alpha_1}{A_1}-p+\gamma}\right\}\\ &\le-\frac{\left[\left(p-\gamma\right)q_1+\frac{\alpha_1}{A_1}-p+\gamma\right]\left(p-\lambda\right)\left(1+u_2^2\right)}{2\left\{\left[\left(p-\gamma\right)q_1+\frac{\alpha_1}{A_1}-p+\gamma\right]^2+\left[\left(p-\gamma\right)q_2\right]^2\right\}}.\\ &<0.\end{aligned}$$

By applying Lemma 1, $h_i\left(z\right)\in P$ $(i=1,2)$ and consequently $f\left(z\right)\in V_{k,p}\left(\alpha_1,A_1,B_1;\gamma,\lambda\right)$. This completes the proof of Theorem 3.

Theorem 4. *If* $0\le\gamma<p$, $k\ge2$ *and* $f\in S_{k,p}\left(\alpha_1,A_1,B_1;\gamma\right)$ *for* $z\in U$, *then* $f\in S_{k,p}\left(\alpha_1+1,A_1,B_1;\gamma\right)$ *for*

$$|z|<r_0=\frac{|\sigma+1|}{\sqrt{A+\left(A^2-|\sigma^2-1|\right)^{\frac{1}{2}}}}, \tag{35}$$

where $A = 2(s+1)^2 + |\sigma|^2 - 1$, *with* $\sigma = \frac{\gamma + \frac{\alpha_1}{A_1} - p}{p-\gamma} \neq -1$ *and* $s = \dfrac{1}{p-\gamma}$. *This radius is best possible.*

Proof. Let $f \in S_{k,p}(\alpha_1, A_1, B_1, \gamma)$ for $z \in U$ and

$$\frac{z\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)\right)'}{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)} = (p-\gamma) h(z) + \gamma, \tag{36}$$

where $h(z)$ is given by *(25)*. Using *(9)* in *(36)*, and differentiating the resulting equation with respect to z, we obtain

$$\begin{aligned}
\frac{1}{p-\gamma}\left\{\frac{z\left(\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)\right)'}{\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)} - \gamma\right\} &= h(z) + \frac{\left(\frac{1}{p-\gamma}\right) z h'(z)}{h(z) + \left(\frac{\gamma+\alpha_1 - A_1 p}{p-\gamma}\right)} \\
&= \left(\frac{k}{4} + \frac{1}{2}\right)\left\{h_1(z) + \frac{\left(\frac{1}{p-\gamma}\right) z h_1'(z)}{h_1(z) + \left(\frac{\gamma + \frac{\alpha_1}{A_1} - p}{p-\gamma}\right)}\right\} \\
&\quad - \left(\frac{k}{4} - \frac{1}{2}\right)\left\{h_2(z) + \frac{\left(\frac{1}{p-\gamma}\right) z h_2'(z)}{h_2(z) + \left(\frac{\gamma + \frac{\alpha_1}{A_1} - p}{p-\gamma}\right)}\right\}.
\end{aligned}$$

Applying Lemma 2 with $s = \left(\frac{1}{p-\gamma}\right)$ and $\sigma = \frac{\gamma + \frac{\alpha_1}{A_1} - p}{p-\gamma} \neq -1$, we get

$$\operatorname{Re}\left\{h_i(z) + \frac{\left(\frac{1}{p-\gamma}\right) z h_i'(z)}{h_i(z) + \left(\frac{\gamma + \frac{\alpha_1}{A_1} - p}{p-\gamma}\right)}\right\} > 0 \; for \;\; |z| < r_0, \tag{37}$$

where r_0 is given by *(35)* and this radius is the best possible. This completes the proof of Theorem 4.

Theorem 5. *Let* ϕ *be convex and* $f \in S_{2,p}\left(\alpha_1, A_1, B_1; p\gamma'\right)$. *Then* $G \in S_{2,p}\left(\alpha_1, A_1, B_1; p\gamma'\right)$, *where* $G = \phi * f$ and $\left(0 \leq \gamma' < 1\right)$.

Proof. To show that $G = \phi * f \in S_{2,p}\left(\alpha_1, A_1, B_1; p\gamma'\right)\left(0 \leq \gamma' < 1\right)$, it sufficient to show that $\frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] G)'}{p\theta_{p,q,s}[\alpha_1, A_1, B_1] G}$ contained in the convex hull of $F(U)$. Now

$$\frac{z\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] G\right)'}{p\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] G} = \frac{\phi * F\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f}{\phi * \theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f}, \tag{38}$$

where $F = \frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{p\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}$ is analytic in U and $F(0) = 1$. From Lemma 3, we can see that $\frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] G)'}{p\theta_{p,q,s}[\alpha_1, A_1, B_1] G}$ is contained in the convex hull of $F(U)$.

Since $\frac{z(\theta_{p,q,s}[\alpha_1,A_1,B_1]G)'}{p\theta_{p,q,s}[\alpha_1,A_1,B_1]G}$ is analytic in U and

$$F(U) \subseteq \Omega = \left\{ w : \frac{z\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] w\left(z\right)\right)'}{p\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] w\left(z\right)} \in P\left(\gamma^{'}\right) \right\},$$

then $\frac{z(\theta_{p,q,s}[\alpha_1,A_1,B_1]G)'}{p\theta_{p,q,s}[\alpha_1,A_1,B_1]G}$ lies in Ω, this implies that $G = \phi * f \in S_{2,p}\left(\alpha_1, A_1, B_1; p\gamma^{'}\right)$.

In [8] Choi et al. defined the familiar integral operator $F_{\nu,p}(f)(z)$ as follows:

$$\begin{aligned} F_{\nu,p}(f)(z) &= \frac{\nu+p}{z^{\nu}} \int_0^z t^{\nu-1} f(t)dt \qquad (\nu > -p, p \in \mathbb{N}) \\ &= z^p + \sum_{k=1+p}^{\infty} \left(\frac{\nu+p}{\nu+k}\right) a_k z^k. \end{aligned} \tag{39}$$

It follows that:

$$\begin{aligned} z\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] F_{\nu,p}(f)(z)\right)^{'} &= (\nu+p)\,\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f\left(z\right) \\ &\quad -\nu\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] F_{\nu,p}(f)(z). \end{aligned} \tag{40}$$

Theorem 6. *If* $0 \leq \gamma < p$, $k \geq 2$ *and* $f \in S_{k,p}\left(\alpha_1, A_1, B_1; \gamma\right)$, *then* $F_{\nu,p}(f)(z) \in S_{k,p}\left(\alpha_1, A_1, B_1; \gamma\right)$ $(\nu \geq 0)$.

Proof. Let $f \in S_{k,p}\left(\alpha_1, A_1, B_1; \gamma\right)$ and set

$$\frac{z\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] F_{\nu,p}(f)(z)\right)^{'}}{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] F_{\nu,p}(f)(z)} = M(z) = (p-\gamma)\,h\left(z\right) + \gamma, \tag{41}$$

where $h(z)$ is given by *(25)*. Using *(40)* and *(41)*, we have

$$(\nu+p)\,\frac{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f\left(z\right)}{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] F_{\nu,p}(f)(z)} = (p-\gamma)\,h\left(z\right) + \gamma + \nu. \tag{42}$$

Taking the logarithmic differentiation on both sides of *(42)* with respect to z and multiplying by z, we have

$$\frac{z\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f\left(z\right)\right)^{'}}{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f\left(z\right)} - \gamma = (p-\gamma)\,h\left(z\right) + \frac{(p-\gamma)\,zh^{'}\left(z\right)}{(p-\gamma)\,h\left(z\right) + \gamma + \nu}. \tag{43}$$

Now we will show that $M\left(z\right) \in P_k\left(\gamma, p\right)$ or $h_i(z) \in P$. From *(25)* and *(43)* we have

$$\frac{z(\theta_{p,q,s}[\alpha_1,A_1,B_1]f(z))'}{\theta_{p,q,s}[\alpha_1,A_1,B_1]f(z)}-\gamma = \left(\frac{k}{4}+\frac{1}{2}\right)\left\{(p-\gamma)h_1(z)+\frac{(p-\gamma)zh_1'(z)}{(p-\gamma)h_1(z)+\gamma+\nu}\right\} -\left(\frac{k}{4}-\frac{1}{2}\right)\left\{(p-\gamma)h_2(z)+\frac{(p-\gamma)zh_2'(z)}{(p-\gamma)h_2(z)+\gamma+\nu}\right\},$$

this implies that

$$\operatorname{Re}\left\{(p-\gamma)h_i(z)+\frac{(p-\gamma)zh_i'(z)}{(p-\gamma)h_i(z)+\gamma+\nu}\right\}>0 \quad (z\in U; i=1,2). \qquad (44)$$

We form the functional $\Phi(u,v)$ by choosing $u=h_i(z)$, $v=zh_i'(z)$,

$$\Phi(u,v)=(p-\gamma)u+\frac{(p-\gamma)v}{(p-\gamma)u+\gamma+\nu}.$$

Then clearly $\Phi(u,v)$ satisfies all the conditions of Lemma 1. Hence $h_i(z)\in P\,(i=1,2)$ and consequently $h(z)\in P_k$ for $z\in U$, which implies that $F_{\nu,p}(f)(z)\in S_{n,p}(\alpha_1,A_1,B_1,\gamma)$. This completes the proof of Theorem 6.

Next, we derive an inclusion property for the subclass $C_{k,p}(\alpha_1,A_1,B_1,\gamma)$ involving $F_{\nu,p}(f)(z)$, which is given by the following theorem.

Theorem 7. *If $0\le\gamma<p$, $k\ge 2$, $\nu\ge 0$ and $f\in C_{k,p}(\alpha_1,A_1,B_1,\gamma)$, then $F_{\nu,p}(f)(z)\in C_{k,p}(\alpha_1,A_1,B_1,\gamma)$.*

Proof. By applying Theorem 5, it follows that

$$\begin{aligned} f \quad &\in \quad C_{k,p}(\alpha_1,A_1,B_1,\gamma)\Longleftrightarrow\frac{zf'}{p}\in S_{k,p}(\alpha_1,A_1,B_1,\gamma)\\ &\Longrightarrow \quad F_{\nu,p}(f)(z)\left(\frac{zf'}{p}\right)\in S_{k,p}(\alpha_1,A_1,B_1,\gamma)\\ &\Longleftrightarrow \quad \frac{z(F_{\nu,p}(f)(z))'}{p}\in S_{k,p}(\alpha_1,A_1,B_1,\gamma)\Longleftrightarrow F_{\nu,p}(f)(z)\in C_{k,p}(\alpha_1,A_1,B_1,\gamma).\end{aligned}$$

This completes the proof of Theorem 7.

Using *(40)* instead of *(9)* and the techniques of the proof of Theorem 3, we can prove the following theorem.

Theorem 8. *If $0\le\gamma,\lambda<p$, $k\ge 2$, $\nu\ge 0$ and $f\in V_{k,p}(\alpha_1,A_1,B_1,\gamma,\lambda)$, then $F_{\nu,p}(f)(z)\in V_{k,p}(\alpha_1,A_1,B_1,\gamma,\lambda)$.*

Remark 3. *Putting $q=2, s=1$, $\alpha_1=c,\alpha_2=1$, $\beta_1=a$ and $A_1=A_2=B_1=1\,(a,c>0,p\in\mathbb{N})$, our results in this paper correct the results of Hussain et al. [13].*

Remark 4. *Specializing* $q,\ s, \alpha_1, A_1, ..., \alpha_q, A_q$ *and* $\beta_1, B_1, ...\beta_s, B_s$, *in the above results, we obtain the corresponding results for different classes associated with the operators (1-7) defined in the introduction.*

4 Open Problem

The authors suggest to study these classes defined by the Aouf et al. [5] operator:

$$D^m_{\lambda,p}(f * g)(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty}[1 + \lambda(k + p)]^m a_k b_k z^k$$

Acknowledgements. The author would like to thank the Prof. Dr. M.K. Aouf and Dr. A.O. Mostafa, for yuor valiable suggestions.

References

[1] H. A. Al-Kharasani and S.S. Al-Hajiry, A linear operator and its applications on p-valent functions, Internat. J. Math. Analysis, (2007), 627-634.

[2] M. K. Aouf, A generalized of functions with real part bounded in the mean on the unit disc, Math. Japon., 33 (1988), no. 2, 175-182.

[3] M. K. Aouf, On a class of p-valent close-to-convex functions, Internat. J. Math. Math. Sci., 11 (1988), no. 2, 259–266.

[4] M. K. Aouf and J. Dziok, Certain class of analytic functions associated with the Wright generalized hypergeometric function, J. Math. Appl. 30(2008), 23-32.

[5] M. K. Aouf, A. Shamandy, A.O. Mostafa and S. M. Madian, Properties of some families of meromorphic p-valent functions involving certain differential operator, Acta Univ. Apulensis, 20 (2009), 7-16.

[6] M. K. Aouf, A. Shamandy, A. O. Mostafa and S. M. Madian, Certain class of p-valent functions associated with the Wright generalized hypergeometric function, Demonstratio Math., (2010), no. 1, 40-54.

[7] N. E. Cho, O.S. Kwon and H.M. Srivastava, Inclusion and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl., 292 (2004), 470–483.

[8] J. H. Choi, M. Saigo and H.M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl., 276 (2002), no.1, 432–445.

[9] J. Dziok and R. K. Raina, Families of analytic functions associated with the Wright generalized hypergeometric function, Demonstratio Math., 37(2004), no.3, 533-542.

[10] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput,. 103 (1999), 1–13.

[11] R.M. Goel and N.S. Sohi, A new criterion for p-valent functions, Proc. Amer. Math. Soc., 78 (1980), 353–357.

[12] S. Hussain, Some new classes of analytic functions defined by mean of a convolution operator, Acta Univ. Apulensis, 20 (2009), 203-209.

[13] S. Hussain, N. Anjum and I. Z. Cheema, Certain classes of multivalent functions related with a linear operator, Acta Univ. Apulensis, 28 (2011), 367-377.

[14] J.-L. Liu and K.I. Noor, Some properties of Noor integral operator, J. Natur. Geom., 21 (2002), 81–90.

[15] S. S. Miller and P. T. Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl., 65 (1978), no. 2, 289-305.

[16] S. Owa, On certain classes of p-valent functions with negative coefficients, Simon Stevin, 59 (1985), no. 4, 385-402.

[17] K.S. Padmanabhan and R. Parvatham, Properties of a class of functions with bounded boundary rotation, Ann. Polon. Math., 31 (1975), 311–323

[18] J. Patel and A.K. Mishra, On certain subclasses of multivalent functions associated with an extended fractional differintegral operator, J. Math. Anal. Appl., 332 (2007), 109–122.

[19] D.A. Patil and N.K. Thakare, On convex hulls and extreme points of p-valent starlike and convex classes with applications, Bull. Math. Soc. Sci. Math. R.S. Roumaie (N.S), 27 (1983), no.75, 145-160.

[20] B. Pinchuk, Functions with bounded boundary rotation, Isr. J. Math., 10 (1971), 7–16.

[21] R. K. Raina, On certain classes of analytic functions and application to fractional calculas operator, Integral Transform. Spec. Funct. 5(1997), 247-260.

[22] R. K. Raina and T. S. Nahar, On univalent and starlike Wright generalized hypergeometric functions, Rend. Sen. Mat. Univ. Padova, 95(1996), 11-22.

[23] M. S. Robertson, Variational formulas for several classes of analytic functions, Math. Z, 118 (1976), 311-319.

[24] St. Ruscheweyh and T. Shiel-small, Hadmard prouduct of schlicht functions and Polya-Schoenberg conjecture, Comment. Math. Helv., 48(1973), 119-135.

[25] H. Saitoh, A linear operator and its applications of first order differential subordinations, Math. Japon., 44 (1996), 31–38.

[26] H. M. Srivastava and M.K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I, J. Math. Anal. Appl., 171(1992), 1-13; II, J. Math. Anal. Appl., 192 (1995), 673-688.

[27] H. M. Srivastava and P. W. Karlsson, Multiple Gaussian Hypergeometric Series, Ellis Horwood Ltd., Chichester, Halsted Press (John Wiley & Sons, Inc.), New York, 1985.

[28] E. M. Wright, The asymptotic expansion of the generalized hypergeometric functions, Proc. London Math. Soc. 46(1946), 389-408.