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Certain Subclasses of Harmonic Multivalent Functions Involving an Integral Operator

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Abstract

For $p \geq 1$ $(p \in \mathbb{N} := \{1, 2, 3, \cdots\})$***, let*** $\mathcal{H}_p$ ***denote the set of all normalized multivalent harmonic functions*** f***, sense preserving in the open unit disk*** Δ***. Let*** $SH^{n,l}_{p,\lambda}(\alpha, \beta), (0 \leq \alpha < p; l, \lambda, \beta \geq 0; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ ***denote the family of multivalent harmonic functions satisfying:***

$$\Re\left\{(1-\beta)\frac{\widetilde{\Theta}^{n,l}_{p,\lambda} f(z)}{z^p} + \beta\frac{(\widetilde{\Theta}^{n,l}_{p,\lambda} f(z))'}{pz^{p-1}}\right\} > \frac{\alpha}{p};\ \ (z \in \Delta),$$

where the operator $\widetilde{\Theta}^{n,l}_{p,\lambda} f(z) : \mathcal{H}_p \to \mathcal{H}_p$ ***is introduced as a generalized Catas operator (cf.[12]). In the present paper various basic properties such as coefficient conditions, growth theorem, covering theorem, extreme points, inclusion relations, neighborhood property***

and radius of starlikeness are discussed for the subclass $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$ ***of*** $SH^{n,l}_{p,\lambda}(\alpha,\beta)$***. Apart from this, we also show that the newly defined subclasses are closed under convolution, convex combination and invariant under generalized Bernardi-Libera-Livingston operator.***

Keywords: *Harmonic multivalent functions, Coefficient estimates, Distortion inequality, convolution, convex combination, integral operator, neighborhood, radius of starlikeness.*

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1 Introduction and Preliminaries

Let S_H be the class of all complex valued, harmonic, orientation-preserving, univalent functions f in the open unit disk Δ normalized by $f(0) = f_z(0) - 1 = 0$. Clunie and Sheil-Small [14] studied the family S_H together with some geometric subclasses of S_H. They proved that although S_H is not compact, it is normal with respect to the topology of uniform convergence on compact subsets of Δ. Moreover, the subclass S^0_H of S_H consisting of the functions having the property $f_{\bar{z}}(0) = 0$ is compact [19]. This pioneering work of Clunie and Sheil-Small on harmonic univalent mappings gave rise to the birth of theory of harmonic univalent mappings. This theory has recently attracted the function theorists to look at harmonic analogues of the theory of analytic univalent or multivalent functions in the open unit disk. For systematic study we refer [1], [20], [21], [24], [25], and [28] (also see recent books [10], [11] and [19]).

For $p \geq 1$ ($p \in \mathbb{N} := \{1, 2, 3, \cdots\}$), we denote by $\mathcal{A}_p$ the class of all functions which are analytic multivalent in Δ and $\mathcal{H}_p$, the set of all multivalent harmonic functions $f = h + \overline{g}$, where h and g are of the form:

$$h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=p}^{\infty} b_k z^k, \qquad \text{and} \qquad |b_p| < 1, \tag{1}$$

which are analytic multivalent harmonic functions and sense preserving in the open unit disk Δ normalized by the conditions $h(0) = h'(0) = ... = h^{(p-1)}(0) = h^{(p)}(0) - p! = 0$ and $g(0) = g'(0) = \cdots = g^{p-1}(0) = 0$. The function h is called analytic part of f and g is called the co analytic part of f. The above defined class $\mathcal{H}_p$ has been introduced and studied extensively by Jahangiri and Ahuja [2](also see [22]). It is well known that, every analytic function is complex valued harmonic. Thus $\mathcal{A}_p$ is a subset of $\mathcal{H}_p$.

Following the work of Silverman [29], several subclasses of analytic functions with negative coefficients have been introduced and studied (cf.[7], [15],

[16], [17], [18], and [22]). In this work, we introduce following new subclasses of multivalent harmonic functions associated with generalized multiplier transformation.

Definition 1.1 *For $f = h + \bar{g}$ given by (1), let $SH^{n,l}_{p,\lambda}(\alpha, \beta)$ denote the family of multivalent harmonic functions satisfying the following:*

$$\Re\left\{(1-\beta)\frac{\widetilde{\Theta}^{n,l}_{p,\lambda}f(z)}{z^p} + \beta\frac{(\widetilde{\Theta}^{n,l}_{p,\lambda}f(z))'}{pz^{p-1}}\right\} > \frac{\alpha}{p};$$
$$(z \in \Delta;\ 0 \leq \alpha < p;\ n \in \mathbb{N}_0;\ l,\ \lambda,\ \beta\ \geq 0),$$

where the operator $\widetilde{\Theta}^{n,l}_{p,\lambda}f(z) : \mathcal{H}_p \to \mathcal{H}_p$ is defined by

$$\widetilde{\Theta}^{n,l}_{p,\lambda}f(z) = \Theta^{n,l}_{p,\lambda}h(z) + (-1)^n\overline{\Theta^{n,l}_{p,\lambda}g(z)}, \tag{2}$$

$$\Theta^{n,l}_{p,\lambda}h(z) = z^p + \sum_{k=p+1}^{\infty}\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^n a_k z^k \tag{3}$$

and

$$\Theta^{n,l}_{p,\lambda}g(z) = \sum_{k=p}^{\infty}\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^n b_k z^k.$$

Unless otherwise mentioned through out this paper we shall assume that $p,\ n \in \mathbb{N}_0$ are fixed, $0 \leq \alpha < p$ and $l, \beta, \lambda \geq 0$.

It is observed that the operator $\Theta^{n,l}_{p,\lambda}$ given in (3) unifies various classical and recently studied operators which have significant applications in Geometric function theory, for example:

For $p = 1$, $\Theta^{n,l}_{p,\lambda}$ reduces to the modified multiplier transform introduced and studied by Catas [12]. For $l = 0$, $\Theta^{n,l}_{p,\lambda}$ gives Modified El-Ashwah-Aouf operator [5]. The condition $l = 0$ and $\lambda = 1$, reduces $\Theta^{n,l}_{p,\lambda}$ to the operator defined by Aouf and Mostafa[4]. For $p = \lambda = 1$ and $l = 0$, we obtain the modified Sălăgean operator [27]. Again, for $p = \lambda = 1$, we get the modified operator due to Cho and Srivastava [13], and for $p = 1$ and $l = 0$, the operator $\Theta^{n,l}_{p,\lambda}$ generalizes to modified operator studied by Al-Oboudi [8]. It is worthy to mention here that, for $n = 0$, the newly defined class $SH^{n,l}_{p,\lambda}(\alpha, \beta)$ generalizes the classical subclass which is introduced and studied by Ahuja and Jahangiri [3]. For $\lambda = 1$, the operator $\widetilde{\Theta}^{n,l}_{p,\lambda}$ reduces to the operator $D^{n,l}_p$ which is introduced by EL-Ashwah and Aouf [6].

In the present investigation, we also define the following subclass of $SH^{n,l}_{p,\lambda}(\alpha, \beta)$, with the additional features as follows:

Definition 1.2 *The subclass* $\widehat{SH}_{p,\lambda}^{n,l}(\alpha,\beta)$ *of* $SH_{p,\lambda}^{n,l}(\alpha,\beta)$ *contains the harmonic mappings* $f_n(z) = h(z) + \overline{g_n(z)}$*, where*

$$h(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k, \quad g_n(z) = (-1)^n \sum_{k=p}^{\infty} |b_k| z^k; \quad |b_p| < 1; \quad (z \in \Delta). \quad (4)$$

The class $\widehat{SH}_{p,\lambda}^{n,l}(\alpha,\beta)$ reduces to the subfamily $\bar{H}^n(1,l,\beta,\alpha)$, studied by EL-Ashwah and Aouf [6], for $n = 0$ and $\lambda = 1$. When $n = 0$, $l = 0$ and $\lambda = 1$, the defined class $\widehat{SH}_{p,\lambda}^{n,l}(\alpha,\beta)$, becomes the class $\mathcal{H}_p\mathcal{R}(1,\beta,\alpha)$ introduced and studied by Ahuja and Jahangiri [3] (see also [30]). When $\beta = 0$ and $n = 0$, the class $\widehat{SH}_{p,\lambda}^{n,l}(\alpha,\beta)$, consists of multivalent functions f, where $\Re\left(\frac{pf(z)}{z^p}\right) > \alpha$. Therefore, in the present investigation we extend the results of [6], [3] and [30] to the class $\widehat{SH}_{p,\lambda}^{n,l}(\alpha,\beta)$ and also obtain some additional results in different prospective.

This paper is organized as follows: The main results are given in section 2 which is further subdivided into subsections in accordance with the content of the results therein. In the subsection 2.1, the necessary and sufficient conditions for functions to be in the newly defined classes are discussed. We investigated the growth theorem in subsection 2.2 and determined the extreme points of the defined class in subsection 2.3. The inclusion properties in the context of convolution and convex combination are discussed in subsection 2.4. Furthermore, the inclusion property based on neighborhood properties of the class is given in subsection 2.5. In subsection 2.6, we studied the radius of starlikeness of the introduced subclass. Lastly, the problems for further investigation, which may be studied as a sequel to our investigations are given in the section 3.

2 Main Results

2.1 Coefficient Estimates

We begin with the following sufficient condition for a function $f \in \mathcal{H}_p$ to be in the class $SH_{p,\lambda}^{n,l}(\alpha,\beta)$.

Theorem 2.1 *Let* $f = h + \overline{g}$ *in* $\mathcal{H}_p$ *be given by* (1)*. Then* $f \in SH_{p,\lambda}^{n,l}(\alpha,\beta)$

if the following inequality holds true :

$$\sum_{k=p+1}^{\infty}[(k-p)\beta+p]\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|a_k| \\ +\sum_{k=p}^{\infty}|(k+p)\beta-p|\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|b_k| \leq (p-\alpha). \quad (5)$$

Sharpness in (5) occurs for the function

$$f(z)=z^p+\sum_{k=p+1}^{\infty}\frac{x_k}{((k-p)\beta+p)\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}}z^k \\ +\sum_{k=p}^{\infty}\frac{\overline{y}_k}{|(k+p)\beta-p|\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}}\overline{z}^k,$$

where

$$\sum_{k=p+1}^{\infty}|x_k|+\sum_{k=p}^{\infty}|y_k|=p-\alpha.$$

Proof. Let $\Phi(z)=(1-\beta)\frac{\widetilde{\Theta}^{n,l}_{p,\lambda}f(z)}{z^p}+\beta\frac{(\widetilde{\Theta}^{n,l}_{p,\lambda}f(z))'}{pz^{p-1}}$. It is enough to show that for $0\leq\alpha<p$, we have

$$|p-\alpha+p\Phi(z)|\geq|p+\alpha-p\Phi(z)|. \quad (6)$$

Now upon using equations (2) to (6) and substituting in $\Phi(z)$, we have

$$|p-\alpha+p\Phi(z)|\geq 2p-\alpha-\sum_{k=p+1}^{\infty}[(k-p)\beta+p]\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|a_k||z|^{k-p} \\ -\sum_{k=p}^{\infty}|(k+p)\beta-p|\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|b_k||z|^{k-p}, \quad (7)$$

and

$$|p+\alpha-p\Phi(z)|\leq\alpha+\sum_{k=p+1}^{\infty}[(k-p)\beta+p]\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|a_k||z|^{k-p} \\ +\sum_{k=p}^{\infty}|(k+p)\beta-p|\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|b_k||z|^{k-p}. \quad (8)$$

Now (6), (7) and (8) yields

$$|p-\alpha+p\Phi(z)|-|p+\alpha-p\Phi(z)| \geq 2\Big[p-\alpha \\ -\sum_{k=p+1}^{\infty}[(k-p)\beta+p]\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|a_k||z|^{k-p} \\ -\sum_{k=p}^{\infty}|(k+p)\beta-p|\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|b_k||z|^{k-p}\Big],$$

which is greater than equal to zero only if (5) holds true. This completes the proof of Theorem 2.1.

In the following theorem, we show that (5) is both necessary and sufficient for a function f to be in $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$.

Theorem 2.2 *Let $f_n = h+\overline{g_n} \in \mathcal{H}_p$, where h and g_n be of the form (4). Then $f_n \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$ if and only if*

$$\sum_{k=p+1}^{\infty}((k-p)\beta+p)\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|a_k| \\ +\sum_{k=p}^{\infty}|(k+p)\beta-p|\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|b_k| \leq p-\alpha. \quad (9)$$

Proof. The sufficient condition is proved as in Theorem 2.1. Thus we prove the necessary part. We have

$$\Re\Big\{(1-\beta)\frac{\widetilde{\Theta}^{n,l}_{p,\lambda}f(z)}{z^p}+\beta\frac{(\widetilde{\Theta}^{n,l}_{p,\lambda}f(z))'}{pz^{p-1}}\Big\} > \frac{\alpha}{p}.$$

Therefore, upon suitable substitutions, we get

$$\Re\left\{(1-\beta)\frac{\Theta^{n,l}_{p,\lambda}h(z)+(-1)^n\overline{\Theta^{n,l}_{p,\lambda}g(z)}}{z^p}+\beta\frac{(\Theta^{n,l}_{p,\lambda}h(z)+(-1)^n\overline{\Theta^{n,l}_{p,\lambda}g(z))'}}{pz^{p-1}}\right\} \\ \geq 1-\frac{1}{p}\sum_{k=p+1}^{\infty}((k-p)\beta+p)\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|a_k||z|^{k-p} \\ -\frac{1}{p}\sum_{k=p}^{\infty}|(k+p)\beta-p|\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|b_k||z|^{k-p} \geq \frac{\alpha}{p}. \quad (10)$$

Taking z over positive real axis, $(0 \leq z = r < 1)$, equation (10) gives

$$1-\frac{1}{p}\sum_{k=p+1}^{\infty}((k-p)\beta+p)\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|a_k||r|^{k-p} \\ -\frac{1}{p}\sum_{k=p}^{\infty}|((k+p)\beta-p)|\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^{n}|b_k||r|^{k-p} \geq \frac{\alpha}{p}.$$

Therefore, for $r \to 1^-$, we have the desired inequality (9). This completes the proof of Theorem 2.2.

2.2 Growth and Covering Results for the Class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$

Theorem 2.3 *Let $f_n = h + \overline{g_n} \in \mathcal{H}_p$, where h and g_n be of the form (4), with $\frac{p(2\beta-1)}{p-\alpha}|b_p| < 1$ and $\beta \geq 1$. If $f \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$ then for $|z| = r < 1$, the following inequalities holds true:*

$$(1+|b_p|)r^p - \frac{p-\alpha}{(\beta+p)\left(\frac{p+l+\lambda}{p+l}\right)^n}\left(1-\frac{p(2\beta-1)}{p-\alpha}|b_p|\right)r^{p+1} \leq |f_n(z)|, \tag{11}$$

$$|f_n(z)| \leq (1+|b_p|)r^p + \frac{p-\alpha}{(\beta+p)\left(\frac{p+l+\lambda}{p+l}\right)^n}\Big(1-\frac{p(2\beta-1)}{p-\alpha}|b_p|\Big)r^{p+1}. \tag{12}$$

Proof. Let $f_n \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$, where $f_n(z) = h(z) + \overline{g_n(z)}$. In view of (4), we get

$$f_n(z) = z^p + (-1)^n|b_p|\overline{z}^p - \sum_{k=p+1}^{\infty}|a_k|z^k + (-1)^n\sum_{k=p+1}^{\infty}|b_k|\overline{z}^k.$$

$$
\begin{aligned}
|f_n(z)| &= \left| z^p + (-1)^n |b_p| \bar{z}^p - \sum_{k=p+1}^{\infty} |a_k| z^k + (-1)^n \sum_{k=p+1}^{\infty} |b_k| \bar{z}^k \right| \\
&\leq (1+|b_p|) r^p + r^{p+1} \sum_{k=p+1}^{\infty} (|a_k| + |b_k|) \\
&= (1+|b_p|) r^p + \frac{(p-\alpha) r^{p+1}}{(\beta+p)\left(\frac{p+l+\lambda}{p+l}\right)^n} \sum_{k=p+1}^{\infty} \left(\frac{\beta+p}{p-\alpha}\right)\left(\frac{p+l+\lambda}{p+l}\right)^n (|a_k| + |b_k|) \\
&\leq (1+|b_p|) r^p + \frac{(p-\alpha) r^{p+1}}{(\beta+p)\left(\frac{p+l+\lambda}{p+l}\right)^n} \Bigg\{ \sum_{k=p+1}^{\infty} \frac{((k-p)\beta+p)}{p-\alpha} \left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n |a_k| \\
&+ \sum_{k=p+1}^{\infty} \frac{|(k+p)\beta-p|}{p-\alpha} \left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n |b_k| \Bigg\} \\
&\leq (1+|b_p|) r^p + \frac{p-\alpha}{(\beta+p)\left(\frac{p+l+\lambda}{p+l}\right)^n} \left[1 - \frac{p(2\beta-1)}{p-\alpha} |b_p| \right] r^{p+1},
\end{aligned}
$$

which is the desired inequality (12). The left hand inequality (11) can be proved on similar lines and hence omitted. This completes the proof of Theorem 2.3.

The inequality (11) of Theorem 2.3, upon simplification yields:

$$\frac{A-B}{C} < |f_n(z)|,$$

where $A := (\beta+p)(p+l+\lambda)^n - (p+l)^n(p-\alpha)$, $B := ((\beta+p)(p+l+\lambda)^n - p(p+l)^n(2\beta-1))|b_p|$ and $C := (\beta+p)(p+l+\lambda)^n$. Thus, we have the following covering result:

Theorem 2.4 *Let $f_n(z) = h(z) + \overline{g_n(z)}$ be in $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$ where h and g_n are given by* (4)*. Then*

$$\left\{ w : |w| < \left(\frac{A-B}{C}\right) \right\} \subset f_n(\Delta).$$

2.3 Extreme Points for the Class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$

We recall that, a function $f \in \mathcal{F} \subset S_H^0$ is called an *extreme point* of $\mathcal{F}$ if, for $0 < \mu < 1$, $f = \mu f_1 + (1-\mu) f_2$ implies $f_1 = f_2 = f$ for all f_1 and f_2 in $\mathcal{F}$. The intersection of closed convex subsets of S_H^0 which contain $\mathcal{F}$ is called the *closed convex hull* of $\mathcal{F}$. We denote by $cl\mathcal{F}$, the set of all closed convex hull of $\mathcal{F}$.

Theorem 2.5 *Let $f_n = h + \overline{g}_n \in \mathcal{H}_p$ be defined by (4). Then, for $\beta > 1/2$, $f_n \in cl\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$ if and only if $f_n = \sum_{k=p}^{\infty}(x_k h_k(z) + y_k g_k(z))$, where $h_p(z) = z^p$, $x_k, y_k \geq 0$, and $x_p = 1 - \sum_{k=p+1}^{\infty} x_k - \sum_{k=p}^{\infty} y_k$,*

$$h_k(z) = z^p - \frac{p-\alpha}{((k-p)\beta + p)\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n} z^k, \qquad (k = p+1, p+2, \cdots) \quad \textit{and}$$

$$g_k(z) = z^p - (-1)^n \frac{p-\alpha}{|(k+p)\beta - p|\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n} \overline{z}^k, \quad (k = p, p+1, p+2, \cdots).$$

In particular $\{h_k\}, \{g_k\}$ are the extreme points of the class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$.

Proof. Suppose that

$$f_n(z) = \sum_{k=p}^{\infty} (x_k h_k(z) + y_k g_k(z)).$$

By proper substitutions and application of Theorem 2.2, we get

$$\begin{aligned}
&\sum_{k=p+1}^{\infty} ((k-p)\beta + p)\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n \left(\frac{p-\alpha}{((k-p)\beta+p)\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n} x_k\right) \\
&+ \sum_{k=p}^{\infty} |(k+p)\beta - p|\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n \left(\frac{p-\alpha}{|(k+p)\beta - p|\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n} y_k\right) \\
&= (p-\alpha)\left(\sum_{k=p+1}^{\infty} x_k + \sum_{k=p}^{\infty} y_k\right) = (p-\alpha)(1-x_p) \leq p-\alpha.
\end{aligned}$$

This implies that $f_n \in cl\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$. Conversely, let $f_n \in cl\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$,

$$x_k = \frac{((k-p)\beta + p)\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n}{p-\alpha} |a_k|; \quad (k = p+1, p+2, \cdots),$$

and

$$y_k = \frac{|(k+p)\beta - p|\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n}{p-\alpha} |b_k|; \quad (k = p, p+1, \cdots).$$

From Theorem 2.2, we have $0 \leq x_k \leq 1, \quad (k = p+1, p+2, \cdots)$ and $0 \leq y_k \leq 1, \ (k = p, p+1, \cdots)$. Thus, we have

$$\begin{aligned} f_n(z) = z^p - &\sum_{k=p+1}^{\infty} |a_k| z^k - (-1)^n \sum_{k=p}^{\infty} |b_k| \overline{z}^k \\ &= z^p - \sum_{k=p+1}^{\infty} \frac{p-\alpha}{((k-p)\beta + p)\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n} x_k z^k \\ &\qquad - (-1)^n \sum_{k=p}^{\infty} \frac{p-\alpha}{|(k+p)\beta - p|\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n} y_k \overline{z}^k, \end{aligned}$$

which upon simplification becomes

$$\begin{aligned} f_n(z) &= z^p - \sum_{k=p+1}^{\infty} (z^p - h_k(z)) x_k z^k - \sum_{k=p}^{\infty} (z^p - g_k(z)) y_k \overline{z}^k \\ &= \left(1 - \sum_{k=p+1}^{\infty} x_k - \sum_{k=p}^{\infty} y_k\right) z^p + \sum_{k=p+1}^{\infty} h_k(z) x_k z^k + \sum_{k=p}^{\infty} g_k(z) y_k \overline{z}^k \\ &= \sum_{k=p}^{\infty} [x_k h_k(z) + y_k g_k(z)]. \end{aligned}$$

This completes the proof of Theorem 2.5.

2.4 Inclusion Properties for the Class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$

For the functions $f_m, F_m \in \mathcal{H}_p$ which are of the forms:

$$f_m(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k + (-1)^n \sum_{k=p}^{\infty} |b_k| \overline{z}^k \tag{13}$$

and

$$F_m(z) = z^p - \sum_{k=p+1}^{\infty} |A_k| z^k + (-1)^n \sum_{k=p}^{\infty} |B_k| \overline{z}^k, \tag{14}$$

we define the function $(f_m * F_m)(z)$ by

$$(f_m * F_m)(z) = z^p - \sum_{k=p+1}^{\infty} |a_k A_k| z^k + (-1)^n \sum_{k=p}^{\infty} |b_k B_k| \overline{z}^k.$$

In analogy to standard notation we continue to say $f_m * F_m$ the Hadamard product or convolution of f_m and F_m.

The next result shows that the class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$ is closed under convolution.

Theorem 2.6 *Let* $0 \leq \alpha < \alpha_1 < p,\ f_m \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$ *and* $F_m \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha_1, \beta)$. *Then*

$$f_m * F_m \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha_1, \beta) \subset \widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta).$$

Proof. Let $f_m(z)$ given by (13) be in $\widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$ and $F_m(z)$ represented by (14) be in $\widehat{SH}^{n,l}_{p,\lambda}(\alpha_1, \beta)$. To prove $f_m * F_m \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha_1, \beta)$, we show that sufficient condition (9) of Theorem 2.2 is true for $f_m * F_m$. Now, since $F_m(z) \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha_1, \beta)$, we note that $0 \leq A_k \leq 1$ and $0 \leq B_k \leq 1$. Therefore, for the convolution $f_m * F_m$, since $0 \leq \alpha < \alpha_1 < p$ and $f_m \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$ we get

$$\begin{aligned}
&\frac{\sum_{k=p+1}^{\infty}[(k-p)\beta + p]\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n |a_k A_k|}{p-\alpha} + \frac{\sum_{k=p}^{\infty}|(k+p)\beta - p|\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n |b_k B_K|}{p-\alpha} \\
&\leq \frac{\sum_{k=p+1}^{\infty}[(k-p)\beta + p]\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n |a_k|}{p-\alpha} + \frac{\sum_{k=p}^{\infty}|(k+p)\beta - p|\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n |b_k|}{p-\alpha} \\
&\leq \frac{\sum_{k=p+1}^{\infty}[(k-p)\beta + p]\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n |a_k|}{p-\alpha_1} + \frac{\sum_{k=p}^{\infty}|(k+p)\beta - p|\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n |b_k|}{p-\alpha_1} \\
&\leq 1.
\end{aligned}$$

Therefore $f_m * F_m \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha_1, \beta) \subset \widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$. This completes the proof of Theorem 2.6.

Next, to study the properties of convex combination, we recall the following: Let $f_{n_i}(z) \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta),\quad i = 1, 2, 3, \cdots$. Suppose $f_{n_i}(z)$ is written as,

$$f_{n_i}(z) = z^p - \sum_{k=p+1}^{\infty} |a_{k_i}| z^k + (-1)^n \sum_{k=p}^{\infty} |b_{k_i}| \overline{z}^k, \quad (z \in \Delta).$$

The convex combination of $f_{n_i},\quad i = 1, 2, 3, \cdots$ is given by the expression, for $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1$, as:

$$\begin{aligned}
\sum_{i=1}^{\infty} t_i f_{n_i}(z) &= \sum_{i=1}^{\infty} t_i \left(z^p - \sum_{k=p+1}^{\infty} |a_{k_i}| z^k + (-1)^n \sum_{k=p}^{\infty} |b_{k_i}| \overline{z}^k \right) \qquad (15) \\
&= \sum_{i=1}^{\infty} t_i z^p - \sum_{i=1}^{\infty} \sum_{k=p+1}^{\infty} t_i |a_{k_i}| z^k + (-1)^n \sum_{i=1}^{\infty} \sum_{k=p}^{\infty} t_i |b_{k_i}| \overline{z}^k \\
&= z^P - \sum_{k=p+1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{k_i}| \right) z^k + (-1)^n \sum_{k=p}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{k_i}| \right) \overline{z}^k.
\end{aligned}$$

We have the following inclusion Theorem.

Theorem 2.7 *The class* $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$ *is closed under convex combination.*

Proof. Let $f_{n_i}(z) \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$, $i = 1,2,3,...$ To prove that $\sum_{i=1}^{\infty} t_i f_{n_i}(z)$ also belongs to the class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$ we show that necessary and sufficient conditions of Theorem 2.2 is true for (15).
Indeed, Theorem 2.2 yields:

$$\begin{aligned}
&\sum_{k=p+1}^{\infty}((k-p)\beta+p)\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n\left(\sum_{i=1}^{\infty}t_i|a_{k_i}|\right)\\
&\quad+\sum_{k=p}^{\infty}|(k+p)\beta-p|\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^n\left(\sum_{i=1}^{\infty}t_i|b_{k_i}|\right)\\
&\quad=\sum_{i=1}^{\infty}t_i\Big\{\sum_{k=p+1}^{\infty}((k-p)\beta+p)\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^n|a_{k_i}|\\
&\quad+\sum_{k=p}^{\infty}(|(k+p)\beta-p|)\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^n|b_{k_i}|\Big\}\\
&\quad\le\sum_{i=1}^{\infty}t_i(p-\alpha)=(p-\alpha).
\end{aligned}$$

This completes the proof of Theorem 2.7.

The following class preserving property immediately follows from the Theorem 2.2 and hence we state the result without proof.

Theorem 2.8 *If* $f_n(z) = h(z) + \overline{g_n(z)}$ *defined by* (4) *is in* $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$, *then* $I_c(f_n)$, $c > -1$ *also belongs to the class* $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$, *where* $I_c(f_n)$ *is the well known generalized Bernardi-Libera-Livingston integral operator (cf.[9] and [23]) is defined as:*

$$I_c(f_n) = \frac{c+p}{z^c}\int_0^z t^{c-1}f_n(t)dt \quad c > -1.$$

2.5 Neighborhood Properties for the Class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$

The notion of neighborhood properties of harmonic multivalent function is an analogous to the work of Rucheweyh [26]. Let $s \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$ is of the form:

$$s(z) = z^p - \sum_{k=p+1}^{\infty}|A_k|z^k + (-1)^n\sum_{k=p}^{\infty}|B_k|\overline{z}^k;\quad |B_p| < 1;\quad (z \in \Delta). \tag{16}$$

For $f_n(z) = h(z) + \overline{g_n}(z)$ defined by (4) belonging to $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$, the neighborhood of f in $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$ is defined by

$$N_\delta(f,s) := \Big\{ s \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta) : \sum_{k=p+1}^{\infty} k(|(|a_k| - |A_k|)| + |(|b_k| - |B_k|)|) + p|(|b_p| - |B_p|)| \le \delta \Big\}.$$

In particular for $e(z) = z^p$, we have

$$N_\delta(e,s) := \{ s \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta) : \sum_{k=p+1}^{\infty} k(|A_k| + |B_k|) + p|B_p| \le \delta \}.$$

We have the following inclusion result:

Theorem 2.9 *Let $s(z)$ given by (16) be in the class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$. Then for $1/2 \le \beta \le 2$,*

$$\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta) \subset N_\delta(e,s),$$

where $\delta = 2(p-\alpha) + (1-2\beta)p|B_p|$.

Proof. Let $s(z) \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$. From Theorem 2.2, we get

$$\sum_{k=p+1}^{\infty} ((k-p)\beta + p)\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^n |A_k| + \sum_{k=p}^{\infty}((k+p)\beta - p)\Big(\frac{p+l+\lambda(k-p)}{p+l}\Big)^n |B_k| \le p - \alpha.$$

Upon simplification, the above inequality reduced to

$$\beta\Big(\frac{p+l+\lambda}{p+l}\Big)^n \sum_{k=p+1}^{\infty} k(|A_k| + |B_k|) + (2\beta-1)p|B_p| - \Big(\frac{p+l+\lambda}{p+l}\Big)^n \sum_{k=p+1}^{\infty} p(\beta-1)|A_k| + p(1-\beta)|B_k| \le p - \alpha.$$

Note that $\beta\Big(\frac{p+l+\lambda}{p+l}\Big)^n \ge 1$ for all $\beta \ge 1$. Therefore, the above inequality becomes

$$\sum_{k=p+1}^{\infty} k(|A_k| + |B_k|) + p|B_p| \le (p-\alpha) + \Big(\frac{p+l+\lambda}{p+l}\Big)^n \sum_{k=p+1}^{\infty} p(\beta-1)|A_k| + p(1-\beta)|B_k|. \quad (17)$$

Further, applications of (9), we also have

$$\Big(\frac{p+l+\lambda}{p+l}\Big)^n\Big\{\sum_{k=p+1}^{\infty} p|A_k|+2p|B_k|\Big\} \leq (p-\alpha)+(1-2\beta)p|B_p|. \tag{18}$$

It is observed that for $\beta \in [1/2, 2]$, the inequalities (17) and (18) yield

$$\sum_{k=p+1}^{\infty} k(|A_k|+|B_k|)+p|B_p| \leq 2(p-\alpha)+(1-2\beta)p|B_p|.$$

Hence, $s(z) \in N_\delta(e, s)$. This completes the proof of Theorem 2.9 .

2.6 Radius of Starlikeness for the Class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$

Finally, to study the radius of starlikeness, recalling, a function $f \in \mathcal{H}_p$ is said to be starlike of order μ, $(0 \leq \mu < 1)$ if and only if $\Re\Big(\frac{J_H f(z)}{f(z)}\Big) > \mu$, $(z \in \Delta)$, where $J_H f(z) := zh^{'}(z) - \overline{zg^{'}(z)}$, or equivalently following inequality holds true:

$$\left|\frac{J_H f(z)-(1+\mu)f(z)}{J_H f(z)+(1-\mu)f(z)}\right| < 1, \qquad (z \in \Delta). \tag{19}$$

Let $\Delta_r := \{z \in \mathbb{C} : |z| = r < 1\}$, the radius of starlikeness of the class $\mathcal{F}$ is given by

$$r^s_\mu(\mathcal{F}) := \inf_{f \in \mathcal{F}} \{\sup \ \ (r \in [0, 1) : f \text{ is starlike of order } \mu \text{ in } \Delta_r)\}.$$

The following result determine the radius of starlikeness of order μ $(0 \leq \mu < 1)$, for the class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$.

Theorem 2.10 *Let $0 \leq \mu < 1$ and n be an even number. Then the radius of starlikeness of order μ for the class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$ is given by*

$$r^s_\mu = \inf_{k>p}\Big\{\min\Big\{\frac{\gamma_1\eta^n}{p-\mu}, \frac{\gamma_2\eta^n}{p-\mu}\Big\}\Big\}^{\frac{1}{k-p}}, \tag{20}$$

where $\gamma_1 := (k-p)\beta+p$, $\quad \gamma_2 := |(k+p)\beta-p|$ $\quad$ *and* $\quad \eta := \frac{p+l+\lambda(k-p)}{p+l}$.

Proof. Let $f_n(z) = h(z)+\overline{g_n(z)} \in \widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$, where $h(z)$ and $g_n(z)$ be of the form (4). Then, for $|z| = r < 1$, we have

$$\begin{aligned}
&\left|\frac{J_H f(z)-(1-\mu)f(z)}{J_H f(z)+(1-\mu)f(z)}\right| \\
&= \left|\frac{(p-1-\mu)-\sum_{k=p+1}^{\infty}(k-1-\mu)|a_k|z^{k-p}-(-1)^{n-1}\sum_{k=p}^{\infty}(k+1+\mu)|b_k|\overline{z}^{k-p}}{(p+1-\mu)-\sum_{k=p+1}^{\infty}(k+1-\mu)|a_k|z^{k-p}-(-1)^{n-1}\sum_{k=p}^{\infty}(k-1+\mu)|b_k|\overline{z}^{k-p}}\right| \\
&\leq \left|\frac{\Gamma_1-\Gamma_2}{\Gamma_3-\Gamma_4}\right|,
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\Gamma_1 &= (p-1-\mu) - (-1)^{n-1}(p+1+\mu)|b_p| \\
\Gamma_2 &= \sum_{k=p+1}^{\infty} ((k-1-\mu)|a_k| + (-1)^{n-1}(k+1+\mu)|b_k|)r^{k-p} \\
\Gamma_3 &= (p+1-\mu) - (-1)^{n-1}(p-1+\mu)|b_p| \\
\Gamma_4 &= \sum_{k=p+1}^{\infty} ((k+1-\mu)|a_k| + (-1)^{n-1}(k-1+\mu)|b_k|)r^{k-p}.
\end{aligned}$$

From the (19) and (21), it is cleared that f is starlike of order μ in Δ_r if and only if

$$\begin{aligned}
&(p-1-\mu) - \sum_{k=p+1}^{\infty} (k-1-\mu)|a_k|r^{k-p} - (-1)^n \sum_{k=p}^{\infty}(k+1+\mu)|b_k|r^{k-p} \\
&\quad \leq (p+1-\mu) - \sum_{k=p+1}^{\infty} (k+1-\mu)|a_k|r^{k-p} - (-1)^n \sum_{k=p}^{\infty}(k-1+\mu)|b_k|r^{k-p}.
\end{aligned}$$

Which on simplification reduces to

$$\sum_{k=p+1}^{\infty} |a_k|r^{k-p} + \sum_{k=p}^{\infty}(-1)^n|b_k|r^{k-p} \leq 1. \tag{22}$$

But, from Theorem 2.2, we have

$$\sum_{k=p+1}^{\infty} \frac{\gamma_1\eta^n}{p-\alpha}|a_k| + \sum_{k=p}^{\infty} \frac{\gamma_2\eta^n}{p-\alpha}|b_k| \leq 1. \tag{23}$$

Therefore, comparing (22) and (23), we must have

$$r^{k-p} \leq \frac{\gamma_1\eta^n}{p-\alpha} \qquad \text{and} \qquad r^{k-p} \leq \frac{\gamma_2\eta^n}{p-\alpha}, \text{ for } k = p+1, p+2, \cdots .$$

This implies that

$$r \leq \left(\min\left\{\frac{\gamma_1\eta^n}{p-\alpha}, \frac{\gamma_2\eta^n}{p-\alpha}\right\}\right)^{\frac{1}{k-p}}, \text{ for } k = p+1, p+2, \cdots .$$

Thus, the radius of starlikeness of order μ for the class $\widehat{SH}^{n,l}_{p,\lambda}(\alpha,\beta)$ is given by (20). This completes the proof of Theorem 2.10.

3 Open Problems

The author(s) suggest to study the following aspects:

- The determination of the disc, in the domain set, where the functions are fully convex of order α $(0 \leq \alpha < 1)$, fully starlike of order α are interesting future aspects of investigation. In this context, the author(s) have given a theoritical view point of the radius of starlikeness of order μ $(0 \leq \mu < 1)$ for the family $\widehat{SH}^{n,l}_{p,\lambda}(\alpha, \beta)$. The exact, numerical evaluation of such radius is challenging.

- The geometric nature of the image (whether it is fully starlike, fully convex or uniformly convex) of the functions belonging to the defined classes need to settle down.

- Following the Bieberbach Conjecture for the family S_H (or S_H^0), it also interesting to investigate the sharp bounds of n^{th} coefficients of both analytic and co analytic part of function belonging to the newly defined classes.

References

[1] O. P. Ahuja, Planar harmonic univalent and related mappings, J. Inequal. Pure & Appl. Math., 6(2005), no.4, Art. 122, 1–44.

[2] O. P. Ahuja and J. M. Jahangiri, Multivalent harmonic starlike functions, Ann. Univ. Mariae Curie-Skłodowska, Sect. A, 55(2001), no.1, 1–13.

[3] O. P. Ahuja and J. M. Jahangiri, On a linear combination of classes of multivalently harmonic functions, Kyungpook Math. J., 42(2002), no.1, 61–70.

[4] M. K. Aouf and A. O. Mostafa , On a subclass of $n-p$-valent prestarlike functions, Comput. Math. Appl., 55(2008), no.4, 851–861.

[5] R. M. EI-Ashwah and M. K. Aouf, Inclusion and neighborhood properties of some analytic p-valent functions, Gen. Math., 18(2010), no.2, 173–184.

[6] R. M. El-Ashwah and M. K. Aouf, New classes of p-valent harmonic functions, Bull. Math. Anal. Appl., 2(2010), no.3, 53–64.

[7] W. G. Atshan, S. R. Kulkarni, and R. K. Raina, A class of multivalent harmonic functions involving a generalized Ruscheweyh type operator, Mat. Vesnik, 60(2008), no.3, 207–213.

[8] F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, Int. J. Math. Math. Sci., 27(2004), no. (25/28), 1429–1436.

[9] S. D. Bernardi, Convex and starlike univalent function, Trans. Amer. Math. Soc., 135(1969), 429–446.

[10] M. A. Brilleslyper, M. J. Dorff, J. M. McDougall, J. S. Rolf, L. E. Schaubroeck, R. L. Stankewitz, and K. Stephenson, Explorations in complex analysis, The Mathematical Association of America, Washington, DC, (2012).

[11] D. Bshouty and W. Hengartner, Univalent harmonic mappings in the plane, In: Handbook of Complex Analysis: Geometric Function Theory, Vol.-I, Edited by R. K*ü*hanu, 2(2005), 479–506 .

[12] A. Catas, On certain classes of p-valent functions defined by multiplier transformations, In: Proceeding book of the international symposium on geometric functions theory and applications, Istanbul, Turkey, (2007), 241–250.

[13] N. E. Cho and H. M. Srivastava, Argument estimates of certain analytic functions defined by a class of multiplier transformations, Math. Comput. Modelling, 37(2003), no. (1/2), 39–49.

[14] J. Clunie and T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn., Ser. A 1. Math., 9(1984), 3–25.

[15] L. I. Cot$\hat{\imath}$rl*ă*, A new class of Harmonic multivalent functions defined by an integral operator, Acta Universitatis Apulensis, 21(2010), 55–63.

[16] M. Darus and K. AI-Shaqsi, On harmonic univalent functions defined by a generalized Ruscheweyh derivatives operator, Lobachevskii J. Math., 22(2006), 19–26.

[17] K. K. Dixit and S. Porwal, On a subclass of harmonic univalent functions, J. Inequal. Pure & Appl. Math., 10(2009), no. 1, Art. 27, 1–18.

[18] M. Dorff, M. Nowak and M. Woloszkiewicz, Convolutions of harmonic convex mappings, Complex Var. Elliptic Equ., 57(2012), no.5, 489–503.

[19] P. Duren, Harmonic mapping in the plane, Cambridge Tracts in Mathematics, Cambridge University press, Cambridge, (2004).

[20] P. Greiner, Geometric properties of harmonic shears, Comput. Methods Funct. Theory, 4(2004), no. 1, 77–96.

[21] W. Hengartner and G. Schober, Univalent harmonic mappings, Trans. Amer. Math. Soc., 299(1987), 1–31.

[22] J. M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, Ann. Uni. Mariae Curie-Skłodowska, Sect. A, 52(1998), 57–66.

[23] R. J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc., 16(1965), 755–758.

[24] S. Ponnusamy and A. Rasila, Planar harmonic mappings, RMS Mathematics Newsletter, 17(2007), no. 2, 40–57.

[25] S. Ponnusamy, A. Rasila, and M. Vuorinen, Special classes of plannar harmonic univalent mappings, RMS Mathematics Newsletter, 19(2009), no. 1, 17-27.

[26] S. Rucheweyh, Neighborhoods of univalent functions, Proc. Amer. Math. Soc., 81(1981), 521–527.

[27] G. S. Sălăgean, Subclasses of univalent functions, Complex Analysis 5th Romanian-Finnish seminar, Part 1, Bucharest, 1981, 362-372; Lecture Notes in Math. 1013, Springer, Berlin, (1983), Proc. Amer. Math. Soc. , 51(1975), 109–116.

[28] G. Schober, Planar harmonic mappings, Comput. Methods Funct. Theory (Valparaso, 1989), 171–176; Lecture Notes in Math., Springer, Berlin, 1435(1990).

[29] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51(1975), 109–116.

[30] S. Yalcin, H. Botanci, and M. Ozturk, New classes of Sălăgean-type multivalent harmonic functions, Math. Cluj., 48(2006), No. 1, 111–118.