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Inclusion Properties for Subclasses of $p-$Valent Functions Associated with Dziok-Srivastava Operator

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Abstract

The purpose of the present paper is to introduce subclasses of $p-$valent functions defined by Dziok-Srivastava operator. Inclusion relationships are established and integral operator for functions in these subclasses is discussed.

Keywords: *Analytic function, $p-$valent, starlike, convex, Dziok-Srivastava operator.*

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1 Introduction

Denote by $\mathbb{A}(p)$ the class of $p-$valent analytic functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}, z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}). \tag{1.1}$$

Aouf [1] defined the class $\mathbf{P}_k(p,\gamma)$ of functions $g(z)$ satisfying $g(0)=p$ and

$$\int_0^{2\pi}\left|\frac{Re\{g(z)\}-\gamma}{p-\gamma}\right|d\theta\le k\pi \quad (z=re^{i\theta};k\ge 2;0\le\gamma<p). \tag{1.2}$$

which generalizes the classes:

(i) $\mathbf{P}_k(1,\gamma)=\mathbf{P}_k(\gamma)$ (see [6] and [5] who proved that it is a convex set);

(ii) $\mathbf{P}_k(1,0)=\mathbf{P}_k$ (see [7]);

(iii) $\mathbf{P}_2(p,\gamma)=\mathbf{P}(p,\gamma)$ is the class in which $Re\{g(z)\}>\gamma(0\le\gamma<p)$;

(iv) $\mathbf{P}_2(1,\gamma)=\mathbf{P}(\gamma)$ is the class in which $Re\{g(z)\}>\gamma(0\le\gamma<1)$;

(v) $\mathbf{P}_2(1,0)=\mathbf{P}$ is the class in which $Re\{g(z)\}>\gamma$.

From (1.2), we have $g\in\mathbf{P}_k(p,\gamma)$ if and only if there exists $g_1,g_2\in\mathbf{P}(p,\gamma)$ such that

$$g(z)=\left(\frac{k}{4}+\frac{1}{2}\right)g_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)g_2(z),\quad (z\in\mathbb{U}). \tag{1.3}$$

For complex parameters $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ $(\beta_j\notin Z_0=\{0,-1,-2,...\};j=1,...,s)$. Dziok and Srivastava [2] defined the operator

$$H_{p,q,s}(\alpha_1)f(z)=z^p+\sum_{n=1}^{\infty}\Gamma_{n+p}(\alpha_1)\,a_{n+p}z^{n+p}, \tag{1.4}$$

which satisfy

$$z\left(H_{p,q,s}(\alpha_1)f(z)\right)'=\alpha_1H_{p,q,s}(\alpha_1+1)f(z)-(\alpha_1-p)H_{p,q,s}(\alpha_1)f(z), \tag{1.5}$$

where

$$\Gamma_{n+p}(\alpha_1)=\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n\,(n)!},$$

and

$$(\theta)_n=\frac{\Gamma(\theta+n)}{\Gamma(\theta)}=\begin{cases}1 & n=0\\ \theta(\theta+1)...(\theta+n-1) & (n\in\mathbb{N})\end{cases}.$$

For different values of the parameters $H_{p,q,s}(\alpha_1)$ generalizes many other operators see ([2]). For example:

$H_{p,2,1}(c+p,1;c+p+1)f(z)=J_{c,p}f(z)$, where $J_{c,p}f(z)$ is the generalized Libera operator (see [10] and [11]) defined by

$$J_{c,p}(f)(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt$$

$$=\left(z^p+\sum_{n=1}^{\infty}\frac{c+p}{c+p+n}z^{p+n}\right)*f(z)\quad (c>-p,p\in\mathbb{N}), \tag{1.6}$$

For $0 \leq \gamma < p$, $k \geq 2$ and $f \in \mathbb{A}(p)$. Seoudy [12]. (see also Noor [7 with $p = 1$]), defined the following classes by:

$$U_k(p,\gamma) = \left\{ f \in \mathbb{A}(p) : \frac{zf'(z)}{f(z)} \in \mathbf{P}_k(p,\gamma) \right\},$$
$$V_k(p,\gamma) = \left\{ f \in \mathbb{A}(p) : \frac{(zf'(z))'}{f'(z)} \in \mathbf{P}_k(p,\gamma) \right\},$$
$$T_k(p,\gamma,\beta) = \left\{ f \in \mathbb{A}(p) : g(z) \in U_2(p,\gamma), \frac{zf'(z)}{g(z)} \in \mathbf{P}_k(p,\beta) \right\},$$

$$T_k^*(p,\gamma,\beta) = \left\{ f \in \mathbb{A}(p) : g(z) \in V_2(p,\gamma), \frac{(zf'(z))'}{g'(z)} \in \mathbf{P}_k(p,\beta) \right\}. \quad (1.7)$$

Next, by using the linear operator $H_{p,q,s}(\alpha_1)$, we introduce the following classes of analytic functions for $0 \leq \gamma < p$ and $k \geq 2$:

$$U_{k,p,q,s}(\alpha_1,\gamma) = \{ f \in \mathbb{A}(p) : H_{p,q,s}(\alpha_1)f(z) \in U_k(p,\gamma) \},$$
$$V_{k,p,q,s}(\alpha_1,\gamma) = \{ f \in \mathbb{A}(p) : H_{p,q,s}(\alpha_1)f(z) \in V_k(p,\gamma) \},$$
$$T_{k,p,q,s}(\alpha_1,\gamma,\beta) = \{ f \in \mathbb{A}(p) : H_{p,q,s}(\alpha_1)f(z) \in T_k(p,\gamma,\beta) \},$$

$$T^*_{k,p,q,s}(\alpha_1,\gamma,\beta) = \{ f \in \mathbb{A}(p) : H_{p,q,s}(\alpha_1)f(z) \in T_k^*(p,\gamma,\beta) \}. \quad (1.8)$$

which satisfy

$$f(z) \in V_{k,p,q,s}(\alpha_1,\gamma) \Leftrightarrow \frac{zf'(z)}{p} \in U_{k,p,q,s}(\alpha_1,\gamma) \quad (1.9)$$

and

$$f(z) \in T^*_{k,p,q,s}(\alpha_1,\gamma,\beta) \Leftrightarrow \frac{zf'(z)}{p} \in T_{k,p,q,s}(\alpha_1,\gamma,\beta). \quad (1.10)$$

The following lemmas will be required in our investigation.

Lemma 1 [3]. For $u = u_1 + i\ u_2$, $v = v_1 + iv_2$, $D \subset \mathbb{C} \times \mathbb{C}$. Let the complex valued function

$$\phi : D \to \mathbb{C}\ , D \subset \mathbb{C} \times \mathbb{C},$$

and satisfies the following conditions:

(i) $\phi(u,v)$ is continuous in D ;
(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} > 0$;
(iii) for all $(iu_2, v_1) \in D$, and $v_1 \leq \frac{-(1+u_2^2)}{2}$, $Re\{\phi(iu_2,v_1)\} \leq 0$. Let

$$h(z) = 1 + h_1 z + h_2 z^2 + ..., \quad (1.11)$$

be regular in $\mathbb{U}$. Such that $\left(h\left(z\right), zh^{'}\left(z\right)\right) \in D$.If $Re\left\{\phi\left(h\left(z\right), zh^{'}\left(z\right)\right)\right\} > 0$, then $Re\left\{h\left(z\right)\right\} > 0$.

Lemma 2 [9]. Let $p\left(z\right)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $Re\{p(z)\} > 0, z \in \mathbb{U}$. Then for $s > 0$ and $\mu \in \mathbb{C}\backslash\{-1\}$,

$$Re\left\{p(z) + \frac{szp^{'}(z)}{p(z)+\mu}\right\} > 0\left(|z| < r_0\right), \tag{1.12}$$

where r_0 is given by

$$r_0 = \frac{|\mu+1|}{\sqrt{A + \left(A^2 - |\mu^2 - 1|\right)^{\frac{1}{2}}}}, \quad A = 2(s+1)^2 + |\mu|^2 - 1, \tag{1.13}$$

and this radius is the best possible.

Lemma 3 [8]. Let ϕ be convex and let g be starlike in $\mathbb{U}$. Then, for F analytic in $\mathbb{U}$ with $F(0) = 1, \left(\left(\phi * Fg\right)/\left(\phi * g\right)\right)$ is contained in the convex hull of $F(\mathbb{U})$.

In this paper, we obtain several inclusion properties of the classes $U_{k,p,q,s}(\alpha,\gamma)$ and $V_{k,p,q,s}(\alpha,\gamma)$ associated with the operator $H_{p,q,s}(\alpha_1)$.

2 Main Results

Unless otherwise mentioned, we assume throughout this paper that $k \geq 2$, $0 \leq \gamma < p$, $p \in \mathbb{N}, q$ and $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, q \leq s+1$, $\alpha_1 \in \mathbb{C}$ and $\beta_j \notin Z_0$.

Theorem 1. The following inclusion relation holds:

$$U_{k,p,q,s}(\alpha_1+1,\gamma) \subset U_{k,p,q,s}(\alpha_1,\gamma). \tag{2.1}$$

Proof. Let $f(z) \in U_{k,p,q,s}(\alpha_1+1,\gamma)$ and

$$\frac{z\left(H_{p,q,s}(\alpha_1)f(z)\right)^{'}}{H_{p,q,s}(\alpha_1)f(z)} = \left(p-\gamma\right)h(z) + \gamma$$

$$= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{\left(p-\gamma\right)h_1(z)+\gamma\right\} - \left(\frac{k}{4}-\frac{1}{2}\right)\left\{\left(p-\gamma\right)h_2(z)+\gamma\right\}, \tag{2.2}$$

where h_i is analytic in $\mathbb{U}$ with $h_i(0) = 1, i = 1, 2$. Using the identity (1.5) in (2.2) and differentiating, we obtain

$$\frac{z\left(H_{p,q,s}(\alpha_1+1)f(z)\right)^{'}}{H_{p,q,s}(\alpha_1+1)f(z)} = \left\{\gamma + \left(p-\gamma\right)h(z) + \frac{(p-\gamma)zh^{'}(z)}{(p-\gamma)h(z)+\gamma+\alpha_1-p}\right\} \in \mathbf{P}_k(p,\gamma). \tag{2.3}$$

This implies that

$$\Re\left\{(p-\gamma)\,h_i(z)+\frac{(p-\gamma)\,zh_i'(z)}{(p-\gamma)\,h_i(z)+\gamma+\alpha_1-p}\right\}>0. \tag{2.4}$$

We form the functional $\phi\,(u,v)$ by choosing $u=h_i(z)$ and $v=zh_i'(z)$:

$$\phi\,(u,v)=(p-\gamma)\,u+\frac{(p-\gamma)\,v}{(p-\gamma)\,u+\gamma+\alpha_1-p} \tag{2.5}$$

Clearly, conditions (i) and (ii) of Lemma 1 are satisfied. Now, we verify condition (iii) as follows:

$$\begin{aligned} Re\left\{\phi\,(iu_2,v_1)\right\} &= Re\left\{iu_2\,(p-\gamma)+\frac{(p-\gamma)\,v_1}{(p-\gamma)\,iu_2+\gamma+\alpha_1-p}\right\} \\ &\leq -\frac{(\gamma+\alpha_1-p)\,(1+u_2^2)\,(p-\gamma)}{2\left[(\gamma+\alpha_1-p)^2+(p-\gamma)^2\,u_2^2\right]}<0. \end{aligned} \tag{2.6}$$

Therefore applying Lemma 1, $h_i\in\mathbf{P}(i=1,2)$ and consequently $h\in\mathbf{P}_k$. that is $H_{p,q,s}(\alpha_1)f(z)\in U_k(p,\gamma)$.

Theorem 2. The following inclusion relation holds:

$$V_{k,p,q,s}(\alpha_1+1,\gamma)\subset V_{k,p,q,s}(\alpha_1,\gamma). \tag{2.7}$$

Proof. Applying (1.9) and Theorem 1, we observe that

$$\begin{aligned} f(z) &\in V_{k,p,q,s}(\alpha_1+1,\gamma) \\ &\Leftrightarrow \frac{zf'(z)}{p}\in U_{k,p,q,s}(\alpha_1+1,\gamma)\Longrightarrow\frac{zf'(z)}{p}\in U_{k,p,q,s}(\alpha_1,\gamma) \\ &\Leftrightarrow f(z)\in V_{k,p,q,s}(\alpha_1,\gamma), \end{aligned}$$

the proof of Theorem 2 is completed.

Theorem 3. The following inclusion relation holds:

$$T_{k,p,q,s}\,(\alpha_1+1,\gamma,\beta)\subset T_{k,p,q,s}\,(\alpha_1,\gamma,\beta)\,. \tag{2.8}$$

Proof. Let $f(z)\in T_{k,p,q,s}\,(\alpha_1+1,\gamma,\beta)$. Then, in view of the definition of the class $T_{k,p,q,s}\,(\alpha_1+1,\gamma,\beta)$, there exists a function $g(z)\in U_{2,p,q,s}(\alpha_1+1,\gamma)$ such that

$$\frac{z\left(H_{p,q,s}(\alpha_1+1)f(z)\right)'}{H_{p,q,s}(\alpha_1+1)g(z)}\in\mathbf{P}_k(p,\beta)\qquad(z\in\mathbb{U})\,.$$

Now let

$$\frac{z\left(H_{p,q,s}(\alpha_1)f(z)\right)'}{H_{p,q,s}(\alpha_1)g(z)}=H(z)=(p-\beta)\,h(z)+\beta, \tag{2.9}$$

where $h(z)$ is analytic in $\mathbb{U}$ with $h(0)=1$. Using (1.5) in (2.9), we have

$$\alpha_1 H_{p,q,s}(\alpha_1+1)f(z)-(\alpha_1-p)\,H_{p,q,s}(\alpha_1)f(z)=\left[(p-\beta)\,h(z)+\beta\right]H_{p,q,s}(\alpha_1)g(z). \tag{2.10}$$

Differentiating (2.10) leads to

$$\alpha_1 z\left(H_{p,q,s}(\alpha_1+1)f(z)\right)'-(\alpha_1-p)\,z\left(H_{p,q,s}(\alpha_1)f(z)\right)'$$
$$=(p-\beta)\,zh'(z)H_{p,q,s}(\alpha_1)g(z)+\left[(p-\beta)\,h(z)+\beta\right]z\left(H_{p,q,s}(\alpha_1)g(z)\right)'. \tag{2.11}$$

Since $g(z)\in U_{2,p,q,s}(\alpha_1+1,\gamma)$, by Theorem 1, $g(z)\in U_{2,p,q,s}(\alpha_1,\gamma)$, then we have

$$\frac{z\left(H_{p,q,s}(\alpha_1)g(z)\right)'}{H_{p,q,s}(\alpha_1)g(z)}=(p-\gamma)\,p(z)+\gamma,$$

where $p(z)=1+c_1z+c_2z^2+...$ is analytic in $\mathbb{U}$ with $p(0)=1$. Then by using (1.5), we have

$$\alpha_1\frac{H_{p,q,s}(\alpha_1+1)g(z)}{H_{p,q,s}(\alpha_1)g(z)}=(p-\gamma)\,p(z)+\gamma+\alpha_1-p. \tag{2.12}$$

From (2.11) and (2.12), we obtain

$$\frac{z\left(H_{p,q,s}(\alpha_1+1)f(z)\right)'}{H_{p,q,s}(\alpha_1+1)g(z)}=(p-\beta)\,h(z)+\beta+\frac{(p-\beta)zh'(z)}{(p-\gamma)p(z)+\gamma+\alpha_1-p}\in\mathbf{P}_k(p,\beta). \tag{2.13}$$

Now, we will show that $H(z)\in\mathbf{P}_k(p,\beta)$ or $h_i\in\mathbf{P}(i=1,2)$. From (2.2) and (2.13) we have

$$\begin{aligned}\frac{z(H_{p,q,s}(\alpha_1+1)f(z))'}{H_{p,q,s}(\alpha_1+1)g(z)} &= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{(p-\beta)\,h_1(z)+\beta+\frac{(p-\beta)zh_1'(z)}{(p-\gamma)p(z)+\gamma+\alpha_1-p}\right\}\\ &\quad-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{(p-\beta)\,h_2(z)+\beta+\frac{(p-\beta)zh_2'(z)}{(p-\gamma)p(z)+\gamma+\alpha_1-p}\right\},\end{aligned}$$

this implies that

$$Re\left\{(p-\beta)\,h_i(z)+\frac{(p-\beta)\,zh_i'(z)}{(p-\gamma)\,p(z)+\gamma+\alpha_1-p}\right\}>0\quad(z\in\mathbb{U};i=1,2).$$

We form the functional $\phi(u,v)$ by choosing $u=h_i(z)$, $v=zh_i'(z)$,

$$\phi(u,v)=(p-\beta)\,u+\frac{(p-\beta)\,v}{(p-\gamma)\,p(z)+\gamma+\alpha_1-p}.$$

Clearly, conditions (i) and (ii) of Lemma 1 are satisfied in $D\subseteq\mathbb{C}\backslash Q^*\times\mathbb{C}$, where $Q^*=\left\{z\in\mathbb{C}\text{ and }Re(p(z))=p_1>\frac{p-\gamma-\alpha_1}{p-\gamma}\right\}$ and $p(z)=p_1+ip_2$.

Now, we verify condition (iii) as follows:

$$\begin{aligned} Re\left\{\phi(iu_2,v_1)\right\} &= Re\left\{(p-\beta)\,iu_2+\frac{(p-\beta)\,v_1}{(p-\gamma)\,(p_1+ip_2)+\gamma+\alpha_1-p}\right\} \\ &\leq -\frac{(p-\beta)\left[(p-\gamma)\,p_1+\gamma+\alpha_1-p\right](1+u_2^2)}{2\left\{\left[(p-\gamma)\,p_1+\gamma+\alpha_1-p\right]^2+\left[p_2\,(p-\gamma)\right]^2\right\}}<0. \end{aligned}$$

By applying Lemma 1, $h_i(z)\in\mathbf{P}(i=1,2)$. This completes the proof of Theorem 3.

Theorem 4**.** The following inclusion relation holds:

$$T^{*}_{k,p,q,s}\left(\alpha_1+1,\gamma,\beta\right)\subset T^{*}_{k,p,q,s}\left(\alpha_1,\gamma,\beta\right). \tag{2.14}$$

Proof. By applying (1.10) and Theorem 3, it follows that

$$\begin{aligned} f(z) &\in T^{*}_{k,p,q,s}\left(\alpha_1+1,\gamma,\beta\right) \\ &\Leftrightarrow \frac{zf'(z)}{p}\in T_{k,p,q,s}\left(\alpha_1+1,\gamma,\beta\right)\Longrightarrow\frac{zf'(z)}{p}\in T_{k,p,q,s}\left(\alpha_1,\gamma,\beta\right) \\ &\Leftrightarrow f(z)\in T^{*}_{k,p,q,s}\left(\alpha_1,\gamma,\beta\right) \end{aligned}$$

the proof of Theorem 4 is completed.

Theorem 5. If $f(z)\in U_{k,p,q,s}(\alpha_1,\gamma)$, Then $J_{c,p}(f)(z)$ is given by (1.6) also belongs to the class $U_{k,p,q,s}(\alpha_1,\gamma)$.

Proof. Let $f(z)\in U_{k,p,q,s}(\alpha_1,\gamma)$ and set

$$\frac{z\left(H_{p,q,s}(\alpha_1)J_{c,p}(f)(z)\right)'}{H_{p,q,s}(\alpha_1)J_{c,p}(f)(z)}=(p-\gamma)\,h(z)+\gamma$$

$$=\left(\frac{k}{4}+\frac{1}{2}\right)\left\{(p-\gamma)\,h_1(z)+\gamma\right\}-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{(p-\gamma)\,h_2(z)+\gamma\right\}, \tag{2.15}$$

where h_i is analytic in $\mathbb{U}$ with $h_i(0)=1$. From (1.6), we have

$$z\left(H_{p,q,s}(\alpha_1)J_{c,p}(f)(z)\right)'=(p+c)\,H_{p,q,s}(\alpha_1)f(z)-cH_{p,q,s}(\alpha_1)J_{c,p}(f)(z) \tag{2.16}$$

Then, by using(2.15) and (2.16), we obtain

$$(c+p)\,\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)J_{c,p}(f)(z)}-c=(p-\gamma)\,h(z)+\gamma. \tag{2.17}$$

and

$$\left(\frac{z\left(H_{p,q,s}(\alpha_1)f(z)\right)'}{H_{p,q,s}(\alpha_1)f(z)}\right)=(p-\gamma)h(z)+\gamma+\frac{(p-\gamma)zh'(z)}{(p-\gamma)\,h(z)+\gamma+c}\in\mathbf{P}_k(p,\gamma). \tag{2.18}$$

This implies that

$$Re\left\{(p-\gamma)h_i(z)+\frac{(p-\gamma)zh_i'(z)}{(p-\gamma)\,h_i(z)+\gamma+c}\right\}>0 \quad (z\in\mathbb{U}; i=1,2). \tag{2.19}$$

We form the functional $\phi\,(u,v)$ by choosing $u=h_i(z)$ and $v=zh_i'(z)$:

$$\phi\,(u,v)=(p-\gamma)u+\frac{(p-\gamma)v}{(p-\gamma)\,u+\gamma+c}. \tag{2.20}$$

Clearly, conditions (i) and (ii) of Lemma 1 are satisfied. Now, we verify condition (iii) as follows:

$$\begin{aligned} Re\left\{\phi\,(iu_2,v_1)\right\} &= Re\left\{(p-\gamma)iu_2+\frac{(p-\gamma)v_1}{(p-\gamma)\,iu_2+\gamma+c}\right\} \\ &\leq -\frac{(\gamma+c)\,(1+u_2^2)\,(p-\gamma)}{2\left[(\gamma+c)^2+(p-\gamma)^2\,u_2^2\right]}<0. \end{aligned} \tag{2.21}$$

Therefore applying Lemma 1, $h_i\in\mathbf{P}(i=1,2)$ and consequently $h\in\mathbf{P}_k$. that is $J_{c,p}(f)(z)\in U_{k,p,q,s}(\alpha_1,\gamma)$.

Theorem 6. If $f(z)\in V_{k,p,q,s}(\alpha_1,\gamma)$, then $J_{c,p}(f)(z)\in V_{k,p,q,s}(\alpha_1,\gamma)$.

Proof. By applying Theorem 3, it follows that

$$\begin{aligned} f(z) &\in V_{k,p,q,s}(\alpha_1,\gamma) \\ &\Leftrightarrow \frac{zf'(z)}{p}\in U_{k,p,q,s}(\alpha_1,\gamma)\Longrightarrow J_{c,p}(\frac{zf'(z)}{p})\in U_{k,p,q,s}(\alpha_1,\gamma) \\ &\Leftrightarrow \frac{z\left(J_{c,p}(f)(z)\right)'}{p}\in U_{k,p,q,s}(\alpha_1,\gamma) \\ &\Leftrightarrow J_{c,p}(f)(z)\in V_{k,p,q,s}(\alpha_1,\gamma), \end{aligned}$$

the proof of Theorem 6 is completed.

Theorem 7. Let $c>-p$ and $0\leq\gamma<p$. If $f(z)\in T_{k,p,q,s}\left(\alpha_1,\gamma,\beta\right)$, then

$$J_{c,p}f(z)\in T_{k,p,q,s}\left(\alpha_1,\gamma,\beta\right).$$

Proof. Let $f(z)\in T_{k,p,q,s}\left(\alpha_1,\gamma,\beta\right)$. Then, by (1.8), there exists a function $g(z)\in U_{2,p,q,s}(\alpha_1,\gamma)$ $(0\leq\gamma<p)$ such that

$$\frac{z\left(H_{p,q,s}(\alpha_1)f(z)\right)'}{H_{p,q,s}(\alpha_1)g(z)}\in\mathbf{P}_k(p,\beta) \qquad (0\leq\beta,\gamma<p)\,.$$

By putting

$$\frac{z\left(H_{p,q,s}(\alpha_1)J_{c,p}f(z)\right)'}{H_{p,q,s}(\alpha_1)J_{c,p}g(z)}=\beta+(p-\beta)\,h(z), \tag{2.22}$$

where $h(z)$of the form (1.11). From (2.16) and (3.22) that

$$(c+p)\,z\left(H_{p,q,s}(\alpha_1)f(z)\right)^{'}=z\left(H_{p,q,s}(\alpha_1)J_{c,p}g(z)\right)^{'}\left[\beta+(p-\beta)\,h(z)\right]$$
$$+\,(p-\beta)\,zh^{'}(z)H_{p,q,s}(\alpha_1)J_{c,p}g(z)+cz\left(H_{p,q,s}(\alpha_1)J_{c,p}f(z)\right)^{'}. \qquad (2.23)$$

Now, by applying the identity (2.16) for the function $g(z)$ in (3.23), we get

$$\frac{z\left(H_{p,q,s}(\alpha_1)f(z)\right)^{'}}{H_{p,q,s}(\alpha_1)g(z)}=\left[\beta+(p-\beta)\,h(z)\right]+\frac{H_{p,q,s}(\alpha_1)J_{c,p}g(z)}{H_{p,q,s}(\alpha_1)g(z)}.\frac{(p-\beta)\,zh^{'}(z)}{(c+p)}. \qquad (2.24)$$

Since $g(z)\in U_{2,p,q,s}(\alpha_1,\gamma)$, we conclude from Theorem 5 that $J_{c,p}g(z)\in U_{2,p,q,s}(\alpha_1,\gamma)$.

Let

$$\frac{z\left(H_{p,q,s}(\alpha_1)J_{c,p}g(z)\right)^{'}}{H_{p,q,s}(\alpha_1)J_{c,p}g(z)}=\gamma+(p-\gamma)\,p(z),\, Re\left\{p\left(z\right)\right\}>0\ .$$

Thus (3.24) can be written as:

$$\frac{z\left(H_{p,q,s}(\alpha_1)f(z)\right)^{'}}{H_{p,q,s}(\alpha_1)g(z)}-\beta=(p-\beta)\,h(z)+\frac{(p-\beta)\,zh^{'}(z)}{(c+\gamma)+(p-\gamma)\,p(z)}. \qquad (2.25)$$

We form the functional $\phi(u,v)$ by choosing $u=h(z)$, $v=zh^{'}(z)$, in (3.25) as follows

$$\phi(u,v)=(p-\beta)\,u+\frac{(p-\beta)\,v}{(c+\gamma)+(p-\gamma)\,p(z)}.$$

Clearly, conditions (i) and (ii) of Lemma 1 are satisfied $D\subseteq\mathbb{C}\times\mathbb{C}$. Now, we verify condition (iii) as follows:

$$\begin{aligned} Re\left\{\phi(iu_2,v_1)\right\} &= Re\left\{(p-\beta)\,iu_2+\frac{(p-\beta)\,v_1}{(c+\gamma)+(p-\gamma)\left[p_1\left(x,y\right)+ip_2\left(x,y\right)\right]}\right\} \\ &= \frac{(p-\beta)\left[(c+\gamma)+(p-\gamma)\,p_1(x,y)\right]v_1}{\left[(c+\gamma)+(p-\gamma)\,p_1(x,y)\right]^2+\left[(p-\gamma)\,p_2(x,y)\right]^2} \\ &\le -\frac{(p-\beta)\left[(c+\gamma)+(p-\gamma)\,p_1(x,y)\right](1+u_2^2)}{2\left\{\left[(c+\gamma)+(p-\gamma)\,p_1(x,y)\right]^2+\left[(p-\gamma)\,p_2(x,y)\right]^2\right\}} \\ &< 0. \end{aligned}$$

Hence $Re\left\{h\left(z\right)\right\}>0$ $(z\in\mathbb{U})$ and $J_{c,p}f(z)\in T_{k,p,q,s}\left(\alpha_1,\gamma,\beta\right)$. This completes the proof of Theorem 7.

Similarly, we can prove the following result.

Theorem 8. Let $c>-p$ and $0\le\gamma<p$. If $f(z)\in T_{k,p,q,s}^{*}\left(\alpha_1,\gamma,\beta\right)$, then

$$J_{c,p}f(z)\in T_{k,p,q,s}^{*}\left(\alpha_1,\gamma,\beta\right).$$

Theorem 9. If $f(z) \in V_{k,p,q,s}(\alpha_1+1,\gamma)$, then $f(z) \in V_{k,p,q,s}(\alpha_1,\gamma)$ for

$$|z| < r_0 = r_0 = \frac{|\mu+1|}{\sqrt{A+(A^2-|\mu^2-1|)^{\frac{1}{2}}}}, \tag{2.26}$$

where $A = 2(s+1)^2 + |\mu|^2 - 1$, with $\mu = \frac{\gamma+\alpha_1-p}{p-\gamma} \neq -1$ and $s = \frac{1}{p-\gamma}$. This radius is the best possible.

Proof. We begin by setting

$$\frac{z\left(H_{p,q,s}(\alpha_1)f(z)\right)'}{H_{p,q,s}(\alpha_1)f(z)} = (p-\gamma)\,h(z)+\gamma$$

$$= \left(\frac{k}{4}+\frac{1}{2}\right)\{(p-\gamma)\,h_1(z)+\gamma\} - \left(\frac{k}{4}-\frac{1}{2}\right)\{(p-\gamma)\,h_2(z)+\gamma\}, \tag{2.27}$$

where h_i is analytic in $\mathbb{U}$ with $h_i(0)=1, i=1,2$. Using the identity (1.5) in (2.27).

$$\begin{aligned}
&\frac{1}{p-\gamma}\left\{\frac{z\left(H_{p,q,s}(\alpha_1+1)f(z)\right)'}{H_{p,q,s}(\alpha_1+1)f(z)}-\gamma\right\} \\
&= h(z)+\frac{(1/(p-\gamma))\,zh'(z)}{h(z)+((\gamma+\alpha_1-p)/(p-\gamma))} \\
&= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z)+\frac{(1/(p-\gamma))\,zh_1'(z)}{h_1(z)+((\gamma+\alpha_1-p)/(p-\gamma))}\right\} \\
&-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_2(z)+\frac{(1/(p-\gamma))\,zh_2'(z)}{h_2(z)+((\gamma+\alpha_1-p)/(p-\gamma))}\right\},
\end{aligned} \tag{2.28}$$

where $Re\{h_i(z)\} > 0$ for $i=1,2$. Applying Lemma 2 with $s = \frac{1}{p-\gamma}$ and $\mu = \frac{\gamma+\alpha_1-p}{p-\gamma} \neq -1$, we get

$$Re\left\{h_i(z)+\frac{(1/(p-\gamma))\,zh_i'(z)}{h_i(z)+((\gamma+\alpha_1-p)/(p-\gamma))}\right\} > 0 \text{ for } |z| < r_0, \tag{2.29}$$

where r_0 is given by (2.26). This completes the proof of Theorem 9.

Theorem 10. Let Ψ be a convex function and $f(z) \in U_{2,p,q,s}(\alpha_1,\gamma)$. Then $G \in U_{2,p,q,s}(\alpha_1,\gamma)$, where $G = \Psi * f$.

Proof. Let $G = \Psi * f$. Then

$$H_{p,q,s}(\alpha_1)G(z) = H_{p,q,s}(\alpha_1)(\Psi * f) = \Psi(z)H_{p,q,s}(\alpha_1)f(z). \tag{2.30}$$

Also, $f(z) \in U_{2,p,q,s}(\alpha_1, \gamma)$. Therefore $H_{p,q,s}(\alpha_1)f(z) \in U_2(p,\gamma)$. By logarithmic differentiation of $(2,30)$ and after some simplification, we obtain

$$\frac{z\left(H_{p,q,s}(\alpha_1)G(z)\right)'}{pH_{p,q,s}(\alpha_1)G(z)} = \frac{\Psi(z) * F(z)H_{p,q,s}(\alpha_1)f(z)}{\Psi(z) * H_{p,q,s}(\alpha_1)f(z)}, \tag{2.31}$$

where $F = z\left(H_{p,q,s}(\alpha_1)f(z)\right)' / pH_{p,q,s}(\alpha_1)G(z)$ is analytic in $\mathbb{U}$ and $F(0) = 1$. From Lemma 3, we can see that $\left(z\left(H_{p,q,s}(\alpha_1)G(z)\right)' / pH_{p,q,s}(\alpha_1)G(z)\right)$ is contained in the convex hull of $F(\mathbb{U})$. Since $\left(z\left(H_{p,q,s}(\alpha_1)G(z)\right)' / pH_{p,q,s}(\alpha_1)G(z)\right)$ is analytic in $\mathbb{U}$ and

$$F(\mathbb{U}) = \Omega = \left\{\omega : \frac{z\left(H_{p,q,s}(\alpha_1)\omega(z)\right)'}{pH_{p,q,s}(\alpha_1)\omega(z)} \in \mathbf{P}(\gamma)\right\}, \tag{2.32}$$

then $\left(z\left(H_{p,q,s}(\alpha_1)G(z)\right)' / pH_{p,q,s}(\alpha_1)G(z)\right)$ lies in Ω, this implies that $G = \Psi * f \in U_{2,p,q,s}(\alpha_1, \gamma)$.

Remarks.

(i) Putting different values of the parameters of $H_{p,q,s}(\alpha_1)f(z)$ in our results, we obtain corresponding results for many other operators.

(ii) Putting $p = 1$ in the above results, we obtain corresponding results for the operator $H_{q,s}(\alpha_1)$.

Open Problem

The authors suggest to study these classes defined by the operator

$$L^m_{\lambda,p,l} f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{l + p + \lambda(k-p)}{l+p}\right)^m a_k z^k$$

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