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Coefficient estimates of some classes of rational functions

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Abstract

The purpose of the present paper is to introduce several new subclasses of the function class σ of analytic and bi-univalent functions in the open unit disk U. Furthermore, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these new subclasses

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1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions f in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, $\mathbb{C}$ being, as, usual, the set of complex numbers. Further, by $\wp$ we shall denote the subclass of all functions in $\mathcal{A}$ which are univalent in Δ. If the functions f and g are analytic in Δ, then f is said to be subordinate to g, written $f(z) \prec g(z)$, provided there is an analytic function $w(z)$ defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ so that $f(z) = g(w(z))$. Some of the important and well-investigated subclasses of the univalent function class $\wp$ include (for example) the class $S(\alpha)$ of starlike functions of order α in Δ and the class $C(\alpha)$ of convex functions of order α in Δ. By definition, we have

$$S(\alpha) = \left\{ f : f \in \wp \ \ and \ \ \Re \frac{zf'(z)}{f(z)} > \alpha \quad (z \in \Delta, \ 0 \leq \alpha < 1) \right\} \tag{1}$$

and

$$C(\alpha) = \left\{ f : f \in \wp \ \ and \ \ zf^{'}(z) \in S(\alpha) \ (z \in \Delta, \ 0 \leq \alpha < 1) \right\}. \tag{2}$$

In [12], the authors introduced the class $S(\phi)$ of the so-called Ma and Minda starlike functions and the class $C(\phi)$ of Ma and Minda convex functions, unifying several previously studied classes related to those of starlike and convex functions. The class $S(\phi)$ consists of all the functions $f \in \mathcal{A}$ satisfying subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$, whereas $C(\phi)$ is formed with functions $f \in \mathcal{A}$ for which the subordination $1+ \frac{zf''(z)}{f'(z)} \prec \phi(z)$ holds.

It is well known that for each $f \in \wp$, the koebe one-quarter Theorem [7] ensures the image of Δ under f contains a disk of radius 1/4. Thus every univalent function $f \in \wp$ has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \ (|z| < 1)$$

and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f), \ r_0(f) \leq 1/4).$$

A function $f \in \mathcal{A}$ is said to bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let σ denote the class of bi-univalent functions defined in the unit disk Δ.The class of bi-univalent functions was first introduced and studied by Lewin [11], where it was proved that $|a_2| < 1.51$.Brannan and Clunie [3] improved Lewin's result to $|a_2| < \sqrt{2}$ and later Netanyahu [16] proved that $|a_2| < \frac{3}{4}$. Brannan and Taha [4] and Taha [26] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The pioneering work by Srivastava et al. [23] actually revived the study of bi-univalent functions in recent years. In fact, ever since the publication of their widely-cited paper [23], several results on coefficient bound estimates for the initial and other coefficients were proved for various subclasses of the bi-univalent function σ (see, for example, [1, 2, 5, 6, 8, 9, 10, 13, 15, 19, 21, 24, 27, 28, 22, 25]).

In [14], Mitrinovic essentially investigated certain geometric properties of functions ψ of the form

$$\psi(z) = \frac{z}{g(z)}, \quad g(z) = 1 + \sum_{n=1}^{\infty} a_n z^n. \tag{3}$$

In [20], Reade et al. derived coefficient conditions that guarantee the univalence, starlikeness or convexity of rational functions of the form (3), these results have been improved and generalized in [17]. In this paper, estimates on the initial coefficients for several subclasses of the bi-univalent function class σ of rational form (3) are obtained. Several related classes are also considered.

In order to derive our main results, we require the following lemma.

Lemma 1.1 *(see [18]) If* $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$*is an analytic function in* Δ *with positive real part, then*

$$|c_n| \leq 2 \qquad (n \in \mathbb{N} = \{1, 2, ...\})\,. \tag{4}$$

2 Coefficients estimates

Let ϕ be an analytic function with positive real part in the unit disk Δ, satisfying $\phi(0) = 1,\ \phi'(0) > 0$, and $\phi(\Delta)$ is symmetric with respect to the the real axis, such a function has a Taylor series of the form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...\,(B_1 > 0)\,. \tag{5}$$

A function $\psi(z) \in \mathcal{A}$ with $\mathrm{Re}(\psi'(z)) > 0$ is known to be univalent. This motivates the following class of functions.

Definition 2.1 *A function* $\psi \in \sigma$ *given by (3)is said to be in the class* $\Re_\sigma(\phi)$ *if it satisfies the following conditions:*

$$\left[(1+e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - e^{i\gamma}\right] \prec \phi(z)\,(z \in \Delta, \gamma \in \mathbb{R})$$

and

$$\left[(1+e^{i\gamma})\frac{wg'(w)}{g(w)} - e^{i\gamma}\right] \prec \phi(w) \quad (w \in \Delta, \gamma \in \mathbb{R})\,,$$

where $g(w) := \psi^{-1}(w)$.

Theorem 2.2 *Let* $\psi(z) \in \Re_\sigma(\phi)$ *be of the form* (3)*. Then*

$$|a_1| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|1+e^{i\gamma}|\,|B_1^2 + (1+e^{i\gamma})(B_1 - B_2)|}} \quad \text{and} \quad |a_2| \leq \frac{B_1}{2\,|1+e^{i\gamma}|} \tag{6}$$

Proof. Let $\psi(z) \in \Re_\sigma(\phi)$ and $g = \psi^{-1}$. Then there exist two functions u and v, analytic in Δ, with $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(w)| < 1$, $z, w \in \Delta$, such that

$$\left[(1+e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - e^{i\gamma}\right] = \phi(u(z))$$

and

$$\left[(1+e^{i\gamma})\frac{wg'(w)}{g(w)} - e^{i\gamma}\right] = \phi(v(w)). \tag{7}$$

Next, define the functions p_1 and p_2 by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + ...$$

and

$$p_2(w) = \frac{1+v(w)}{1-v(w)} = 1 + b_1 w + b_2^2 w^2 + ...,$$

or, equivalently,

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + ...\right], \tag{8}$$

and

$$v(w) = \frac{p_2(w)-1}{p_2(w)+1} = \frac{1}{2}\left[b_1 w + \left(b_2 - \frac{b_1^2}{2}\right) w^2 + ...\right]. \tag{9}$$

Then p_1 and p_2 analytic in Δ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \Delta \longrightarrow \Delta$, the functions p_1 and p_2 have a positive real part in Δ, $|b_i| \le 2$ and $|c_i| \le 2$.

Clearly, upon substituting from (8) and (9) into (7), if we make use of (5), we find that

$$\begin{aligned}\left[(1+e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - e^{i\gamma}\right] &= \phi(\frac{p_1(z)-1}{p_1(z)+1}) = 1 + \frac{1}{2}B_1 c_1 z \\ &+ \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right] z^2 + ..., \end{aligned} \tag{10}$$

and

$$\begin{aligned}\left[(1+e^{i\gamma})\frac{wg'(w)}{g(w)} - e^{i\gamma}\right] &= \phi(\frac{p_2(w)-1}{p_2(w)+1}) = 1 + \frac{1}{2}B_1 b_1 w \\ &+ \left[\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2\right] w^2 + ... \end{aligned} \tag{11}$$

Since $\psi \in \sigma$ has the Maclaurin's series given by

$$\psi(z) = z - a_1 z^2 + (a_1^2 - a_2) z^3 + ..., \tag{12}$$

a computation shows that its inverse $g = \psi^{-1}$ has the expansion

$$g(w) = \psi^{-1}(w) = w + a_1 w^2 + (a_1^2 + a_2) w^3 + \cdots . \tag{13}$$

Since

$$\left[(1+e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - e^{i\gamma}\right] = 1 - (1+e^{i\gamma})a_1 z + (1+e^{i\gamma})(a_1^2 - 2a_2)z^2 + \cdots$$

and

$$\left[(1+e^{i\gamma})\frac{wg'(w)}{g(w)} - e^{i\gamma}\right] = 1 + (1+e^{i\gamma})a_1 w + (1+e^{i\gamma})(a_1^2 + 2a_2)w^2 + \cdots .$$

Using (12) and (13) in (10) and (11) respectively, we get

$$-(1+e^{i\gamma})a_1 = \frac{1}{2}B_1 c_1 \tag{14}$$

$$(1+e^{i\gamma})(a_1^2 - 2a_2) = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2, \tag{15}$$

$$(1+e^{i\gamma})a_1 = \frac{1}{2}B_1 b_1 \tag{16}$$

and

$$(1+e^{i\gamma})(a_1^2 + 2a_2) = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2. \tag{17}$$

From (14) and (16), we have

$$c_1 = -b_1. \tag{18}$$

Adding (15) and (17), then using (14) and (18), we get

$$a_1^2 = \frac{B_1^3(c_2 + b_2)}{4(1+e^{i\gamma})\left[B_1^2 + (1+e^{i\gamma})(B_1 - B_2)\right]},$$

and now, by applying Lemma 1.1 for the coefficients b_2 and c_2, the last equation gives the bound of $|a_1|$ from (6). By subtracting (17) from (15), further computations using (18) lead to

$$a_2 = \frac{1}{8(1+e^{i\gamma})}B_1(b_2 - c_2).$$

The bound of $|a_2|$, as asserted in (6), is now a consequence of Lemma 1.1, and this completes our proof.

If we set

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\eta} = 1 + 2\eta z + 2\eta^2 z^2 + ... \, (0 < \eta \le 1, \ z \in \Delta) \, .$$

in Definition 2.1 of the bi-univalent function class $\Re_\sigma(\phi)$, we obtain a new class $\Re_\sigma(\eta)$ given by Definition 2.3 below.

Definition 2.3 *For* $0 < \eta \le 1$, *a function* $\psi \in \sigma$ *given by (3) is said to be in the class* $\Re_\sigma(\eta)$ *if it satisfies the following conditions:*

$$\left[(1+e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - e^{i\gamma}\right] \prec \left(\frac{1+z}{1-z}\right)^{\eta} (z \in \Delta, \gamma \in \mathbb{R})$$

and

$$\left[(1+e^{i\gamma})\frac{wg'(w)}{g(w)} - e^{i\gamma}\right] \prec \left(\frac{1+w}{1-w}\right)^{\eta} (w \in \Delta, \gamma \in \mathbb{R}) \, ,$$

where $g(w) := \psi^{-1}(w)$.

Using the parameter setting of Definition 2.3 in Theorem 2.2, we get the following corollary.

Corollary 2.4 *For* $0 < \eta \le 1$, *let the function* $\psi \in \Re_\sigma(\eta)$ *be of the form (3). Then*

$$|a_1| \le \frac{\eta}{\sqrt{|1+e^{i\gamma}| \, |2\eta + (1+e^{i\gamma})(1-\eta)|}} \quad \text{and } |a_2| \le \frac{\eta}{|1+e^{i\gamma}|} .$$

Let

$$\phi(z) = \frac{1+(1-2\nu)z}{1-z} = 1 + 2(1-\nu)z + 2(1-\nu)z^2 + ... \, (0 < \nu \le 1, \ z \in \Delta) \, .$$

in Definition 2.1 of the bi-univalent function class $\Re_\sigma(\phi)$, we obtain a new class $\mathcal{H}_\sigma(\nu)$ given by Definition 2.5 below.

Definition 2.5 *For* $0 < \nu \le 1$, *a function* $\psi \in \sigma$ *given by (3) is said to be in the class* $\mathcal{H}_\sigma(\nu)$ *if the following conditions holds true:*

$$\left[(1+e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - e^{i\gamma}\right] \prec \frac{1+(1-2\nu)z}{1-z} \, (z \in \Delta, \gamma \in \mathbb{R})$$

and

$$\left[(1+e^{i\gamma})\frac{wg'(w)}{g(w)} - e^{i\gamma}\right] \prec \frac{1+(1-2\nu)w}{1-w} \, (w \in \Delta, \gamma \in \mathbb{R}) \, ,$$

where $g(w) := \psi^{-1}(w)$.

Using the parameter setting of Definition 2.5 in Theorem 2.2 we get the following corollary.

Corollary 2.6 *For* $0 < \nu \leq 1$, *let the function* $\psi \in \Re_\sigma(\nu)$ *be given by (3). Then*

$$|a_1| \leq \sqrt{\frac{2\,(1-\nu)}{|1+e^{i\gamma}|}} \quad \textit{and} \quad |a_2| \leq \frac{(1-\nu)}{|1+e^{i\gamma}|}.$$

Definition 2.7 *A function* $\psi \in \sigma$ *is given by (3) is said to be in the class* $S_\sigma(\lambda, \mu, \phi)$ *if it satisfies the following subordination conditions:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu} + \lambda\psi'(z)\left(\frac{\psi(z)}{z}\right)^{\mu-1} \prec \phi(z)\,(0<\mu<1; 0\leq\lambda\leq 1 \textit{ and } z \in \Delta)$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec \phi(w)\,(0<\mu<1; 0\leq\lambda\leq 1 \textit{ and } w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

For functions in the class $S_\sigma(\lambda, \mu, \phi)$, the following coefficient estimates are obtained.

Theorem 2.8 *Let* $\psi(z) \in S_\sigma(\lambda, \mu, \phi)$ *be of the form (3). Then*

$$|a_1| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2\left[\frac{\mu(\mu+2\lambda+1)}{2}+\lambda\right]+(B_1-B_2)\left[\mu(1-2\lambda)-\lambda\right]^2\right|}}, \tag{19}$$

and

$$|a_2| \leq \frac{B_1}{(\mu+2\lambda)}. \tag{20}$$

Proof. Let $\psi \in S_\sigma(\lambda, \mu, \phi)$, there are two Schwarz functions u and v defined by (8) and (9) respectively, such that

$$\begin{aligned}(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu} + \lambda\psi'(z)\left(\frac{\psi(z)}{z}\right)^{\mu-1} &= \phi(u(z)) \quad \text{and} \qquad (21)\\ (1-\lambda)\left(\frac{\psi(w)}{w}\right)^{\mu} + \lambda\psi'(w)\left(\frac{\psi(w)}{w}\right)^{\mu-1} &= \phi(v(w)).\end{aligned}$$

Since

$$\begin{aligned}&(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu}+\lambda\psi'(z)\left(\frac{\psi(z)}{z}\right)^{\mu-1}\\=\ &1-\left[\mu(1-2\lambda)-\lambda\right]a_1z+\left[\left(\frac{\mu(\mu+2\lambda+1)}{2}+\lambda\right)a_1^2-(2\lambda+\mu)a_2\right]z^2+\cdots\end{aligned}$$

and

$$\begin{aligned}&(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\\=\ &1+\left[\mu(1-2\lambda)-\lambda\right]a_1w+\left[\left(\frac{\mu(\mu+2\lambda+1)}{2}+\lambda\right)a_1^2+(2\lambda+\mu)a_2\right]w^2+\cdots.\end{aligned}$$

Then (12), (13) and (21) yields

$$-\left[\mu(1-2\lambda)-\lambda\right]a_1=\frac{1}{2}B_1c_1 \tag{22}$$

$$\left(\frac{\mu(\mu+2\lambda+1)}{2}+\lambda\right)a_1^2-(2\lambda+\mu)a_2=\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2c_1^2, \tag{23}$$

$$\left[\mu(1-2\lambda)-\lambda\right]a_1=\frac{1}{2}B_1b_1 \tag{24}$$

and

$$\left(\frac{\mu(\mu+2\lambda+1)}{2}+\lambda\right)a_1^2+(2\lambda+\mu)a_2=\frac{1}{2}B_1(b_2-\frac{b_1^2}{2})+\frac{1}{4}B_2b_1^2. \tag{25}$$

From (22) and (24), we get

$$c_1=-b_1, \tag{26}$$

and after some further calculations using (23)-(26) we find

$$a_1^2=\frac{B_1^3(c_2+b_2)}{4\left[B_1^2(\frac{\mu(\mu+2\lambda+1)}{2}+\lambda)+(B_1-B_2)\left[\mu(1-2\lambda)-\lambda\right]^2\right]},$$

and

$$a_2=\frac{B_1(b_2-c_2)}{4(\mu+2\lambda)}.$$

Applying Lemma 1.1, the estimates in (19) and (20) follow.

Definition 2.9 *For* $0<\eta\le 1$, *a function* $\psi\in\sigma$ *given by (3) is said to be in the class* $S_\sigma(\lambda,\mu,\eta)$ *if it satisfies the following subordination conditions:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu}+\lambda\psi'(z)\left(\frac{\psi(z)}{z}\right)^{\mu-1}\prec\left(\frac{1+z}{1-z}\right)^{\eta}(0<\mu<1;0\le\lambda\le 1\ \text{and}\ z\in\Delta),$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\prec\left(\frac{1+w}{1-w}\right)^{\eta}(0<\mu<1;0\le\lambda\le 1\ \text{and}\ w\in\Delta),$$

where $g(w):=\psi^{-1}(w)$.

Using the parameter setting of Definition 2.9 in Theorem 2.8 we get the following corollary.

Corollary 2.10 *For and* $0<\eta\le 1$, *let the function* $\psi\in S_\sigma(\lambda,\mu,\eta)$ *be of the form (3). Then*

$$|a_1|\le\frac{2\eta}{\sqrt{-\eta\left[(\mu(1-2\lambda)-\lambda)^2-2(\frac{\mu(\mu+2\lambda)}{2}+\lambda)\right]+[\mu(1-2\lambda)-\lambda]^2}}$$

and

$$|a_2|\le\frac{2\eta}{(\mu+2\lambda)}.$$

Let

$$\phi(z)=\frac{1+(1-2\nu)z}{1-z}=1+2(1-\nu)z+2(1-\nu)z^2+...(0<\nu\le 1,\ z\in\Delta).$$

Definition 2.11 *For* $0<\nu\le 1$, *a function* $\psi\in\sigma$ *given by (3) is said to be in the class* $S_\sigma(\lambda,\mu,\nu)$ *if it satisfies the following subordination conditions:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu}+\lambda\psi'(z)\left(\frac{\psi(z)}{z}\right)^{\mu-1}\prec\frac{1+(1-2\nu)z}{1-z}(0<\mu<1;0\le\lambda\le 1\ \text{and}\ z\in\Delta)$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\prec\frac{1+(1-2\nu)w}{1-w}(0<\mu<1;0\le\lambda\le 1\ \text{and}\ w\in\Delta),$$

where $g(w)=\psi^{-1}(w)$.

Using the parameter setting of Definition 2.11 in Theorem 2.8 we get the following corollary.

Corollary 2.12 *For* $0 < \nu \leq 1$, *let the function* $\psi \in S_\sigma(\lambda, \mu, \nu)$ *be of the form (3). Then*

$$|a_1| \leq \sqrt{\frac{4\,(1-\nu)}{(\mu(\mu+2\lambda)+2\lambda)}} \quad \textit{and} \ |a_2| \leq \frac{2\,(1-\nu)}{(\mu+2\lambda)}.$$

Definition 2.13 *A function* $\psi \in \sigma$ *given by (3) is said to be in the class* $M_\sigma(\lambda, \mu, \phi)$, *if it satisfies the following subordinations conditions:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu} + \lambda\psi'(z)\left(\frac{z}{\psi(z)}\right)^{\mu-1} \prec \phi(z)\,(0 < \mu < 1; 0 \leq \lambda \leq 1 \textit{ and } z \in \Delta)$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{w}{g(w)}\right)^{\mu-1} \prec \phi(w), (0 < \mu < 1; 0 \leq \lambda \leq 1 \textit{ and } w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

A function in the class $M_\sigma(\lambda, \mu, \phi)$ is called bi-Mocanu convex function of Ma-Minda type. This class unifies the classes $S(\alpha)$ and $C(\alpha)$. For functions in the class $M_\sigma(\lambda, \mu, \phi)$, the following coefficients estimates hold.

Theorem 2.14 *Let* $\psi(z) \in M_\sigma(\lambda, \mu, \phi)$ *be of the form (3). Then*

$$|a_1| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|4B_1^2[(\mu(\mu+1)+4\lambda(3-2\mu)] - [\mu(1-2\lambda)+3\lambda]^2(B_1-B_2)|}}, \tag{27}$$

and

$$|a_2| \leq \frac{B_1}{4[\mu(2\lambda-1)-4\lambda)]}. \tag{28}$$

Proof. If $\psi \in M_\sigma(\lambda, \mu, \phi)$, then there exist are two Schwarz functions u and v defined by (8) and (9) respectively, such that

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu} + \lambda\psi'(z)\left(\frac{z}{\psi(z)}\right)^{\mu-1} = \phi(u(z)), \tag{29}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{w}{g(w)}\right)^{\mu-1} = \phi(v(w)). \tag{30}$$

Since

$$\begin{aligned}
&(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu}+\lambda\psi'(z)\left(\frac{z}{\psi(z)}\right)^{\mu-1}\\
=\;&1-[\mu(1-2\lambda)+3\lambda]a_1 z\\
&+\left[\left(\frac{(\mu(\mu+1)}{2}+2\lambda(3-2\mu)\right)a_1^2+[\mu(2\lambda-1)-4\lambda]a_2\right]z^2+...
\end{aligned}$$

and

$$\begin{aligned}
&(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{w}{g(w)}\right)^{\mu-1}\\
=\;&1+[\mu(1-2\lambda)+3\lambda]a_1 w\\
&+\left[\left(\frac{(\mu(\mu+1)}{2}+2\lambda(3-2\mu)\right)a_1^2-[\mu(2\lambda-1)-4\lambda)]a_2\right]w^2+...,
\end{aligned}$$

from (10), (11), (29) and (30), it follows that

$$-[\mu(1-2\lambda)+3\lambda]a_1=\frac{1}{2}B_1c_1, \tag{31}$$

$$\left(\frac{(\mu(\mu+1)}{2}+2\lambda(3-2\mu)\right)a_1^2+[\mu(2\lambda-1)-4\lambda)]a_2=\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2c_1^2, \tag{32}$$

$$[\mu(1-2\lambda)+3\lambda]a_1=\frac{1}{2}B_1b_1, \tag{33}$$

and

$$\left(\frac{\mu(\mu+1)}{2}+2\lambda(3-2\mu)\right)a_1^2-[\mu(2\lambda-1)-4\lambda)]a_2=\frac{1}{2}B_1(b_2-\frac{b_1^2}{2})+\frac{1}{4}B_2b_1^2, \tag{34}$$

Equations (31) and (33) yields

$$c_1=-b_1, \tag{35}$$

and after some further calculations using (32)-(34) we find

$$a_1^2=\frac{B_1^3(c_2+b_2)}{4B_1^2[(\mu(\mu+1)+4\lambda(3-2\mu)]+(\mu(1-2\lambda)+3\lambda)^2(B_1-B_2)},$$

and

$$a_2=\frac{B_1(b_2-c_2)}{4[\mu(2\lambda-1)-4\lambda)]}.$$

Applying Lemma 1.1, the estimates in (27) and (28) follow.

Let

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^{\eta}=1+2\eta z+2\eta^2z^2+...\,(0<\eta\leq 1,\ z\in\Delta)\,.$$

Definition 2.15 *For $0 < \eta \leq 1$, a function $\psi \in \sigma$ given by (3) is said to be in the class $M_\sigma(\lambda, \mu, \eta)$ if the following subordinations conditions hold:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu}+\lambda\psi'(z)\left(\frac{z}{\psi(z)}\right)^{\mu-1} \prec \left(\frac{1+z}{1-z}\right)^{\eta} \left(0<\mu<1; 0\leq\lambda\leq 1 \text{ and } z\in\Delta\right),$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{w}{g(w)}\right)^{\mu-1} \prec \left(\frac{1+w}{1-w}\right)^{\eta} \left(0<\mu<1; 0\leq\lambda\leq 1 \text{ and } w\in\Delta\right),$$

$g(w) := \psi^{-1}(w)$.

Using the parameter setting of Definition 2.15 in Theorem 2.14 we get the following corollary.

Corollary 2.16 *For $0 < \eta \leq 1$, let the function $\psi \in M_\sigma(\lambda, \mu, \eta)$ be of the form (3). Then*

$$|a_1| \leq \frac{2\eta}{\sqrt{\eta\left[8(\mu(\mu+1)+4\lambda(3-2\mu))-(\mu(1-2\lambda)+3\lambda)^2\right]+\left[\mu(1-2\lambda)+3\lambda\right]^2}}$$

and

$$|a_2| \leq \frac{\eta}{2[\mu(2\lambda-1)-4\lambda)]}.$$

Let

$$\phi(z) = \frac{1+(1-2\nu)z}{1-z} = 1+2(1-\nu)z+2(1-\nu)z^2+...\left(0<\nu\leq 1,\ z\in\Delta\right).$$

Definition 2.17 *For $0 < \nu \leq 1$, a function $\psi \in \sigma$ given by (3) is said to be in the class $M_\sigma(\lambda, \mu, \nu)$ if the following subordinations hold:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu}+\lambda\psi'(z)\left(\frac{z}{\psi(z)}\right)^{\mu-1} \prec \frac{1+(1-2\nu)z}{1-z} \left(0<\mu<1; 0\leq\lambda\leq 1 \text{ and } z\in\Delta\right),$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{w}{g(w)}\right)^{\mu-1} \prec \frac{1+(1-2\nu)w}{1-w} \left(0<\mu<1; 0\leq\lambda\leq 1 \text{ and } w\in\Delta\right),$$

where $g(w) := \psi^{-1}(w)$.
Using the parameter setting of Definition 2.17 in Theorem 2.14 we get the following corollary.

Corollary 2.18 *For $0 < \nu \leq 1$, let the function $\psi \in M_\sigma(\lambda, \mu, \nu)$ be of the form (3). Then*

$$|a_1| \leq \sqrt{\frac{2(1-\nu)}{4\lambda(2\mu-3)+(\mu(\mu+1)}} \quad \text{and } |a_2| \leq \frac{(1-\nu)}{2[\mu(2\lambda-1)-4\lambda)]}.$$

3 Open Problem

The authors suggest to study the class of functions $\psi \in \sigma$ which satisfy the following conditions:

$$\left[(1+\beta e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - \beta e^{i\gamma}\right] \prec \phi(z)\,(z \in \Delta, \beta \geq 0, \gamma \in \mathbb{R})$$

and

$$\left[(1+\beta e^{i\gamma})\frac{wg'(w)}{g(w)} - \beta e^{i\gamma}\right] \prec \phi(w) \quad (w \in \Delta, \beta \geq 0, \gamma \in \mathbb{R}),$$

where $g(w) := \psi^{-1}(w)$.

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