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Some Results Related to uniqueness of Linear differential Polynomials

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Abstract

The purpose of the paper is to study the uniqueness problems of linear differential polynomials of meromorphic functions sharing a small function with finite weight and we obtain some results which improve and generalize the related results due to J. L. Zhang and L. Z. Yang [12]. Here, examples have also been given in support of the existence of the result and the sharpness of the conditions invoved in it.

Keywords: *Meromorphic function, linear differential polynomial, uniqueness, sharing small function.*

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1 Introduction

In this article, we use some basic results and symbols of Nevanlinna theory like characteristic function $T(r, f)$, proximity function $m(r, f)$, counting function $N(r, f)$, reduced counting function $\overline{N}(r, f)$, and the first and second theorems (see [1],[6]). We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$, possibly outside a set of finite linear measure.

The order of f is defined by

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Let $k \in \mathbb{N}$ and $a \in \mathbb{C}\cup\{\infty\}$. We use $N_{(k}(r,a;f)$ to denote the counting function of a-points of f with multiplicity $\geq k$, $N_{k)}(r,a;f)$ to denote the counting function of a-points of f with multiplicity $\leq k$. Similarly $\overline{N}_{(k}(r,a;f)$ and $\overline{N}_{k)}(r,a;f)$ are their reduced functions respectively.

For $a \in \mathbb{C}\cup\{\infty\}$ and $p \in \mathbb{N}$ we denote by $N_p(r,a;f)$ the sum

$$\overline{N}(r,a;f) + \overline{N}_{(2}(r,a;f) + ... + \overline{N}_{(p}(r,a;f).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

For $a \in \mathbb{C}\cup\{\infty\}$ and $p \in \mathbb{N}$ we put

$$\delta_p(a;f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,a;f)}{T(r,f)}.$$

Clearly,

$$0 \leq \delta(a;f) \leq \delta_p(a;f) \leq \delta_{p-1}(a;f) \leq ... \leq \delta_2(a;f) \leq \delta_1(a;f) = \Theta(a;f).$$

In 2003, Kit-Wing Yu [10] considered the case that a is a small function, and obtained the following results.

Theorem A. Let f be a non-constant entire function, let k be a positive integer, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0,f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.

Theorem B. Let f be a non-constant, non-entire meromorphic function, let k be a positive integer, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If f and a do not have any common pole, and if $f - a$ and $f^{(k)} - a$ share the value 0 CM and $4\delta(0,f) + 2(8+k)\Theta(\infty,f) > 19 + 2k$, then $f \equiv f^{(k)}$.

In 2004, Liu and Gu [5] obtained the following results.

Theorem C. Let $k \geq 1$ and let f be a non-constant meromorphic function, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM, $f^{(k)}$ and a do not have any common poles of the same multiplicities and $2\delta(0,f) + 4\Theta(\infty,f) > 5$, then $f \equiv f^{(k)}$.

Theorem D. Let $k \geq 1$ and let f be a non-constant entire function, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0,f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.

Lahiri [3] improved Theorem C with weighted shared values and obtained the following theorem.

Theorem E. Let f be a non-constant meromorphic function, k be a positive integer, and let $a \equiv a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If

(i) $a(z)$ has no zero (pole) which is also a zero (pole) of f or $f^{(k)}$ with the same multiplicity,

(ii) $f - a$ and $f^{(k)} - a$ share $(0, 2)$,

(iii) $2\delta_{2+k}(0, f) + (4 + k)\Theta(\infty, f) > 5 + k$,

then $f \equiv f^{(k)}$.

In 2005, Zhang [11] obtained the following result which is an improvement and complement of Theorem D.

Theorem F. Let f be a non-constant meromorphic function, $k\,(\geq 1)$ and $l\,(\geq 0)$ be integers. Also, let $a \equiv a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. Then $f \equiv f^{(k)}$ if one of the following conditions is satisfied,

(i) $l \geq 2$ and

$$(3 + k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4;$$

(ii) $l = 1$ and

$$(4 + k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6;$$

(iii) $l = 0$ (i.e., $f - a$ and $f^{(k)} - a$ share the value 0 IM) and

$$(6 + 2k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10.$$

It is natural to ask what happens if $f^{(k)}$ is replaced by a differential polynomial

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + ... + a_0 f$$

in Theorem E or F, where a_j $(j = 0, 1, ..., k - 1)$ are small meromorphic functions of f. Corresponding to this question,in 2007 Zhang and Yang [12] obtained the following results.

Theorem G. Let f be a non-constant meromorphic function, $k\,(\geq 1)$ and $l\,(\geq 0)$ be integers. Also, let $a = a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $L(f) - a$ share $(0, l)$. Then $f \equiv L(f)$ if one of the following assumptions holds,

(i) $l \geq 2$ and

$$\delta_{2+k}(0,f) + \delta_2(0,f) + 3\Theta(\infty,f) + \delta(a,f) > 4;$$

(ii) $l = 1$ and

$$\delta_{2+k}(0,f) + \delta_2(0,f) + \frac{1}{2}\delta_{1+k}(0,f) + \frac{k+7}{2}\Theta(\infty,f) + \delta(a,f) > \frac{k}{2} + 5;$$

(iii) $l = 0$ (i.e., $f - a$ and $L(f) - a$ share the value 0 IM) and

$$\delta_{2+k}(0,f)+2\delta_{1+k}(0,f)+\delta_2(0,f)+\Theta(0,f)+(6+2k)\Theta(\infty,f)+\delta(a,f) > 2k+10.$$

We now explain the notation of weighted sharing as introduced in [2].

Definition 1.1 ([2]). Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a, g)$, we say that f, g share the value a with the weight k.

We write f, g share (a, k) to mean that f, g share the value a with the weight k. Clearly, if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively. Let h be a non-constant meromorphic function. We denote by

$$P_1(h) = h^{(k)} + a_1 h^{(k-1)} + a_2 h^{(k-2)} + ... + a_{k-1} h^{'} + a_k h \tag{1}$$

and

$$P_2(h) = h^{(k)} + b_1 h^{(k-1)} + b_2 h^{(k-2)} + ... + b_{k-1} h^{'} \tag{2}$$

the differential polynomials of h, where $a_1, a_2, ..., a_k\,(\neq 0), b_1, b_2, ..., b_{k-1}$ are finite complex numbers with $(b_1, b_2, ..., b_{k-1}) \neq (0, 0, ..., 0)$ and k is a positive integer.

The main purpose of this paper is to improve and generalize Theorem G. Further in this paper we provide some examples to show that the conditions in our results are the best possible.

2 Main Results

To prove our main results, we need the following lemma.

Lemma 1 ([7]). Let f be a non-constant meromorphic function. Then

$$N\left(r, \frac{1}{f^{(k)}}\right) \le T(r, f^{(k)}) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f), \tag{3}$$

$$N\left(r, \frac{1}{f^{(k)}}\right) \le N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \tag{4}$$

Suppose that F and G are two non-constant meromorphic functions such that F and G share the value 1 IM. Let z_0 be a 1-point of F of order p, a 1-point of G of order q. We denote by $N_L\left(r, \frac{1}{F-1}\right)$ the counting function of those 1-points of F where $p > q$, by $N_E^{1)}\left(r, \frac{1}{F-1}\right)$ the counting function of those 1-points of F where $p = q = 1$, by $N_E^{(2}\left(r, \frac{1}{F-1}\right)$ the counting function of those 1-points of F where $p = q \ge 2$; each point in these counting functions is counted only once. In the same way, we can define $N_L\left(r, \frac{1}{G-1}\right)$ $N_E^{1)}\left(r, \frac{1}{G-1}\right)$ $N_E^{(2}\left(r, \frac{1}{G-1}\right)$ (see [9]). In particular, if F and G share 1 CM, then

$$N_L\left(r, \frac{1}{F-1}\right) = N_L\left(r, \frac{1}{G-1}\right) = 0. \tag{5}$$

With these notations, if F and G share 1 IM, it is easy to see that

$$\begin{aligned} \overline{N}\left(r, \frac{1}{F-1}\right) &= N_E^{1)}\left(r, \frac{1}{G-1}\right) + N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) \\ &\quad + N_E^{(2}\left(r, \frac{1}{G-1}\right) \\ &= \overline{N}\left(r, \frac{1}{G-1}\right) \end{aligned} \tag{6}$$

Lemma 2 ([8]). Let

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{7}$$

where F and G are two nonconstant meromorphic functions. If F and G share 1 IM and $H \not\equiv 0$, then

$$N_E^{1)}\left(r, \frac{1}{F-1}\right) \le N(r, H) + S(r, F) + S(r, G). \tag{8}$$

Lemma 3 ([12]). Let f be a non-constant meromorphic function, $P(f)$ be defined by (1.1) and p, k be positive integers. If $P(f) \not\equiv 0$, we have

$$N_p(r, 0; P(f)) \le T(r, P(f)) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f),$$

$$N_p(r,0;P(f)) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

The following theorems are the main results of the paper.

Theorem 1. Let f and g be two non-constant meromorphic functions, $k\,(\geq 1)$ and $l\,(\geq 0)$ be integers. Also, let $a = a(z)\,(\not\equiv 0,\infty)$ be a small function with respect to f and g. Suppose that $P_1(f)-a$ and $P_1(g)-a$ share $(0,l)$. If one of the following assumptions holds,

(i) $l \geq 2$ and
$$(3+k)\Theta(\infty,f) + 2\delta_{2+k}(0,f) > 4+k, \tag{9}$$

(ii) $l = 1$ and
$$\left(\frac{7+3k}{2}\right)\Theta(\infty,f) + \frac{5}{2}\delta_{2+k}(0,f) > \frac{3k}{2} + 5, \tag{10}$$

(iii) $l = 0$ (i.e., $P_1(f)-a$ and $P_1(g)-a$ share the value 0 IM) and
$$(6+4k)\Theta(\infty,f) + 5\delta_{2+k}(0,f) > 4k+10. \tag{11}$$

Then $P_1(f) \equiv P_1(g)$ unless $P_1(f)P_1(g) \equiv a^2$.

Proof. Let
$$F = \frac{P_1(f)}{a}, \qquad G = \frac{P_1(g)}{a}. \tag{12}$$

From the conditions of Theorem 1, we know that F and G share $(1,l)$ except the zeros and poles of $a(z)$. From (12), we have

$$T(r,F) = O(T(r,f)) + S(r,f), \quad T(r,G) = O(T(r,f)) + S(r,f), \tag{13}$$
$$\overline{N}(r,F) = \overline{N}(r,G) + S(r,f). \tag{14}$$

It is obvious that f is a transcendental meromorphic function. Let H be defined by (7). We discuss the following two cases.

Case 1. $H \not\equiv 0$, by Lemma 2 we know that (8) holds. From (7) and (14), we have

$$\begin{aligned} N(r,H) \leq\ & \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + N_L\left(r,\frac{1}{F-1}\right) \\ & + N_L\left(r,\frac{1}{G-1}\right) + N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right), \end{aligned} \tag{15}$$

where $N_0\left(r,\frac{1}{F'}\right)$ denotes the counting function corresponding to the zeros of F' which are not the zeros of F and $F-1$. $N_0\left(r,\frac{1}{G'}\right)$ denotes the counting function corresponding to the zeros of G' which are not the zeros of G and $G-1$. From the second fundamental theorem in Nevanlinna's Theory, we have

$$\begin{aligned}T(r,F)+T(r,G)\leq{}&\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G}\right)\\&+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)\\&+S(r,f).\end{aligned}\tag{16}$$

Noting that F and G share 1 IM except the zeros and poles of $a(z)$, we get from (6),

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)={}&2N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{F-1}\right)\\&+2N_L\left(r,\frac{1}{G-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)\\&+S(r,f).\end{aligned}$$

Combining with (8) and (15), we obtain

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\leq{}&N_{(2}\left(r,\frac{1}{F}\right)+N_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)\\&+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)\\&+N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)\\&+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f).\end{aligned}\tag{17}$$

We discuss the following three subcases.

Subcase 1.1. $l\geq 2$. It is easy to see that

$$\begin{aligned}3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)\\\leq N\left(r,\frac{1}{G-1}\right)+S(r,f).\end{aligned}\tag{18}$$

From (17) and (18), we have

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\leq N_{(2}\left(r,\frac{1}{F}\right)+N_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)$$
$$+N\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)$$
$$+S(r,f). \tag{19}$$

Substituting (19) into (16) and by using (14), we have

$$T(r,F)\leq 3\overline{N}(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+S(r,f).$$

Now applying Lemma 3 we have

$$\begin{aligned}T(r,f)&\leq T(r,F)+N_{k+2}\left(r,\frac{1}{F}\right)-N_2\left(r,\frac{1}{F}\right)+S(r,f)+S(r,g)\\
&\leq 3\overline{N}(r,f)+N_2\left(r,\frac{1}{G}\right)+N_{k+2}\left(r,\frac{1}{f}\right)+S(r,f)+S(r,g)\\
&\leq (3+k)\overline{N}(r,f)+N_{k+2}\left(r,\frac{1}{g}\right)+N_{k+2}\left(r,\frac{1}{f}\right)+S(r,f)+S(r,g)\\
&\leq (3+k)\overline{N}(r,f)+2N_{k+2}\left(r,\frac{1}{f}\right)+S(r,f)+S(r,g)\\
&\leq \left((3+k)-(3+k)\Theta(\infty,f)+2-2\delta_{k+2}(0,f)+\epsilon\right)T(r,f)+S(r,f)\\
&+S(r,g)\\
&\leq \left((3+k)-(3+k)\Theta(\infty,f)+2-2\delta_{k+2}(0,f)+\epsilon\right)T(r)+S(r),\end{aligned}$$

i.e.,

$$T(r,f)\leq \left((3+k)-(3+k)\Theta(\infty,f)+2-2\delta_{k+2}(0,f)+\epsilon\right)T(r)+S(r). \tag{20}$$

Similarly we have

$$T(r,g)\leq \left((3+k)-(3+k)\Theta(\infty,f)+2-2\delta_{k+2}(0,f)+\epsilon\right)T(r)+S(r). \tag{21}$$

Combining (20) and (21) we get

$$\left(-4-k+(3+k)\Theta(\infty,f)+2\delta_{k+2}(0,f)-\epsilon\right)T(r)\leq S(r). \tag{22}$$

Since $\epsilon > 0$ is arbitrary, we see that (22) leads to a contradiction.

Subcase 1.2. $l = 1$. Noting that

$$2N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right) + 2N_E^{(2}\left(r, \frac{1}{G-1}\right) + N_E^{1)}\left(r, \frac{1}{F-1}\right) \leq N\left(r, \frac{1}{G-1}\right) + S(r,f),$$

$$\begin{aligned} N_L\left(r, \frac{1}{F-1}\right) &\leq \frac{1}{2}N\left(r, \frac{F}{F'}\right) \leq \frac{1}{2}N\left(r, \frac{F'}{F}\right) \\ &\leq \frac{1}{2}\left(\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r,F)\right) + S(r,f) \\ &\leq \frac{1}{2}\left(N_1\left(r, \frac{1}{F}\right) + \overline{N}(r,f)\right) + S(r,f) \\ &\leq \frac{1}{2}\left(N_{k+1}\left(r, \frac{1}{f}\right) + (k+1)\overline{N}(r,f)\right) + S(r,f), \end{aligned}$$

and by the same reasoning as in Subcase 1.1, we get

$$\begin{aligned} T(r,F) \leq{}& N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 3\overline{N}(r,f) + \frac{1}{2}\left(N_{k+1}\left(r, \frac{1}{f}\right) + (k+1)\overline{N}(r,f)\right) \\ &+ S(r,f) + S(r,g). \end{aligned}$$

$$\begin{aligned} T(r,f) &\leq T(r,F) + N_{k+2}\left(r, \frac{1}{F}\right) - N_2\left(r, \frac{1}{F}\right) + S(r,f) + S(r,g) \\ &\leq N_2\left(r, \frac{1}{G}\right) + 3\overline{N}(r,f) + N_{k+2}\left(r, \frac{1}{f}\right) + \frac{1}{2}\left(N_{k+1}\left(r, \frac{1}{f}\right) + (k+1)\overline{N}(r,f)\right) \\ &\quad + S(r,f) + S(r,g) \\ &\leq (3+k)\overline{N}(r,f) + N_{k+2}\left(r, \frac{1}{g}\right) + N_{k+2}\left(r, \frac{1}{f}\right) + \frac{1}{2}N_{k+1}\left(r, \frac{1}{f}\right) \\ &\quad + \left(\frac{k+1}{2}\right)\overline{N}(r,f) + S(r,f) + S(r,g) \\ &\leq \left(\frac{7+3k}{2}\right)\overline{N}(r,f) + \frac{5}{2}N_{k+2}\left(r, \frac{1}{f}\right) + S(r,f) + S(r,g) \\ &\leq \left(\left(\frac{7+3k}{2}\right) - \left(\frac{7+3k}{2}\right)\Theta(\infty,f) + \frac{5}{2} - \frac{5}{2}\delta_{k+2}(0,f) + \epsilon\right)T(r) + S(r), \end{aligned}$$

i.e.,

$$T(r,f) \leq \left(\left(\frac{7+3k}{2}\right) - \left(\frac{7+3k}{2}\right)\Theta(\infty,f) + \frac{5}{2} - \frac{5}{2}\delta_{k+2}(0,f) + \epsilon\right)T(r) + S(r). \tag{23}$$

Similarly we have

$$T(r,g) \leq \left(\left(\frac{7+3k}{2} \right) - \left(\frac{7+3k}{2} \right) \Theta(\infty, f) + \frac{5}{2} - \frac{5}{2}\delta_{k+2}(0,f) + \epsilon \right) T(r) + S(r). \tag{24}$$

Combining (23) and (24) we get

$$\left(-\frac{3k}{2} - 5 + \left(\frac{7+3k}{2} \right) \Theta(\infty, f) + \frac{5}{2}\delta_{k+2}(0,f) - \epsilon \right) T(r) \leq S(r). \tag{25}$$

Since $\epsilon > 0$ is arbitrary, we see that (25) leads to a contradiction.

Subcase 1.3. $l = 0$. Noting that

$$N_L\left(r, \frac{1}{F-1}\right) + 2N_L\left(r, \frac{1}{G-1}\right) + 2N_E^{(2}\left(r, \frac{1}{G-1}\right) + N_E^{1)}\left(r, \frac{1}{F-1}\right) \leq N\left(r, \frac{1}{G-1}\right) + S(r,f),$$

$$2N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) \leq 2N\left(r, \frac{1}{F'}\right) + N\left(r, \frac{1}{G'}\right),$$

and by the same reasoning as in the Subcase 1.2, we get

$$\begin{aligned} T(r,F) \leq{} & N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 3\overline{N}(r,f) + 2N\left(r, \frac{1}{F'}\right) + N\left(r, \frac{1}{G'}\right) \\ & + S(r,f) + S(r,g). \end{aligned}$$

Now applying Lemma 3 we have

$$\begin{aligned} T(r,f) &\leq T(r,F) + N_{k+2}\left(r, \frac{1}{F}\right) - N_2\left(r, \frac{1}{F}\right) + S(r,f) + S(r,g) \\ &\leq N_2\left(r, \frac{1}{G}\right) + 3\overline{N}(r,f) + 2N\left(r, \frac{1}{F'}\right) + N\left(r, \frac{1}{G'}\right) + N_{k+2}\left(r, \frac{1}{f}\right) \\ &+ S(r,f) + S(r,g) \\ &\leq (3+k)\overline{N}(r,f) + N_{k+2}\left(r, \frac{1}{g}\right) + N_{k+2}\left(r, \frac{1}{f}\right) \\ &+ 2\left[N_1\left(r, \frac{1}{P_1(f)}\right) + \overline{N}(r,f)\right] + N_1\left(r, \frac{1}{P_1(g)}\right) + \overline{N}(r,g) \\ &+ S(r,f) + S(r,g) \\ &\leq (6+k)\overline{N}(r,f) + 2N_{k+2}\left(r, \frac{1}{f}\right) + 2N_{k+1}\left(r, \frac{1}{f}\right) + 2k\overline{N}(r,f) \end{aligned}$$

$$+ N_{k+1}\left(r,\frac{1}{g}\right) + k\overline{N}(r,g) + S(r,f) + S(r,g)$$
$$\leq (6+4k)\overline{N}(r,f) + 5N_{k+2}\left(r,\frac{1}{f}\right) + S(r,f) + S(r,g)$$
$$\leq \left((6+4k) - (6+4k)\Theta(\infty,f) + 5 - 5\delta_{k+2}(0,f) + \epsilon\right) T(r) + S(r),$$

i.e.,

$$T(r,f) \leq \left((6+4k) - (6+4k)\Theta(\infty,f) + 5 - 5\delta_{k+2}(0,f) + \epsilon\right) T(r) + S(r). \tag{26}$$

Similarly we have

$$T(r,g) \leq \left((6+4k) - (6+4k)\Theta(\infty,f) + 5 - 5\delta_{k+2}(0,f) + \epsilon\right) T(r) + S(r). \tag{27}$$

Combining (26) and (27) we get

$$\left(-4k - 10 + (6+4k)\Theta(\infty,f) + 5\delta_{k+2}(0,f) - \epsilon\right) T(r) \leq S(r). \tag{28}$$

Since $\epsilon > 0$ is arbitrary, we see that (28) leads to a contradiction.

Case 2. $H \equiv 0$. By integration, we get from (7) that

$$\frac{1}{G-1} = \frac{A}{F-1} + B, \tag{29}$$

where $A\,(\neq 0)$ and B are constants, from (29) we have

$$\overline{N}(r,F) = \overline{N}(r,G) = \overline{N}(r,f) = S(r,f),\ \Theta(\infty,f) = 1, \tag{30}$$

and

$$G = \frac{(B+1)F + (A-B-1)}{BF + (A-B)}, \quad F = \frac{(B-A)G + (A-B-1)}{BG - (B+1)}. \tag{31}$$

We discuss the following three subcases.

Subcase 2.1. Suppose that $B \neq 0, -1$. From (31) we have $\overline{N}\left(r,\frac{1}{(G-\frac{B+1}{B})}\right) = \overline{N}(r,F)$. From this and the second fundamental theorem, we have

$$T(r,G) \leq \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{(G-\frac{B+1}{B})}\right) + S(r,f)$$
$$\leq \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + S(r,f)$$
$$\leq T(r,G) - T(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + S(r,f).$$

$$T(r,f) \leq N_{k+1}\left(r,\frac{1}{f}\right) + S(r,f).$$

Which contradicts the assumptions of Theorem 1.

Subcase 2.2. Suppose that $B = 0$. From (31) we have

$$G = \frac{F + (A-1)}{A}, \quad F = AG - (A-1). \tag{32}$$

If $A \neq 1$, from (32) we can obtain $\overline{N}\left(r, \frac{1}{\left(G - \frac{A-1}{A}\right)}\right) = \overline{N}\left(r, \frac{1}{F}\right)$. Similarly we can again deduce a contradiction as in Subcase 2.1.

If $A = 1$, then $F \equiv G$, that is

$$P_1(f) \equiv P_1(g). \tag{33}$$

Subcase 2.3. Suppose that $B = -1$, from (31) we have

$$G = \frac{A}{-F + (A+1)}, \quad F = \frac{(A+1)G - A}{G}. \tag{34}$$

If $A \neq -1$, we obtain from (34) that $N\left(r, \frac{1}{\left(G - \frac{A}{A+1}\right)}\right) = N\left(r, \frac{1}{F}\right)$. Similarly, we can deduce a contradiction as in Subcase 2.1.

Hence $A = -1$. From (34), we get $F.G \equiv 1$, that is

$$P_1(f)P_1(g) \equiv a^2.$$

This completes the proof.

Example 1. Let

$f(z) = e^z\left(1 - \frac{1}{2}e^z\right)$ and $g(z) = e^{-z}\left(\frac{1}{2} - e^{-z}\right)$.

Then

$$P_1(f) = -\frac{3}{8}\left(f^{(iv)} + \frac{2}{3}f^{'''} - 5f^{''} - 2f^{'} + 8f\right) = e^z(1 - e^z)$$

and

$$P_1(g) = -\frac{3}{8}\left(g^{(iv)} + \frac{2}{3}g^{'''} - 5g^{''} - 2g^{'} + 8g\right) = e^{-z}(1 - e^{-z}).$$

Since $\overline{N}(r,f) = S(r,f)$, $\Theta(\infty, f) = 1$ and

$$N\left(r,\frac{1}{f}\right)=N\left(r,\frac{1}{e^z(1-\frac{1}{2}e^z)}\right)=N_{k+2}\left(r,\frac{1}{f}\right)\sim\frac{1}{2}T(r,f),$$

$\delta_{k+2}(0,f)=\frac{1}{2}$. Also we know that $P_1(f)$ and $P_1(g)$ share (a,l) $(l\geq 0)$, but none of the inequalities (9), (10) and (11) are satisfied and neither $P_1(f)\equiv P_1(g)$ nor $P_1(f)P_1(g)\equiv 1$. Therefore the conditions of Theorem 1 are essential.

Theorem 2. Let f and g be two non-constant meromorphic functions, $k\,(\geq 1)$ and $l\,(\geq 0)$ be integers. Also, let $a=a(z)\,(\not\equiv 0,\infty)$ be a small function with respect to f and g. Suppose that $P_2(f)-a$ and $P_2(g)-a$ share $(0,l)$. If one of the following assumptions holds,

(i) $l\geq 2$ and

$$(3+k)\Theta(\infty,f)+2\delta_{2+k}(0,f)>4+k, \tag{35}$$

(ii) $l=1$ and

$$\left(\frac{7+3k}{2}\right)\Theta(\infty,f)+\frac{5}{2}\delta_{2+k}(0,f)>\frac{3k}{2}+5, \tag{36}$$

(iii) $l=0$ (i.e., $P_2(f)-a$ and $P_2(g)-a$ share the value 0 IM) and

$$(6+4k)\Theta(\infty,f)+5\delta_{2+k}(0,f)>4k+10. \tag{37}$$

Then $P_2(f)\equiv P_2(g)$ unless $P_2(f)P_2(g)\equiv a^2$.

Proof. The proof of Theorem follows from the proof of Theorem 1. So we omit the detailed proofs.

Example 2. Let

$f(z)=e^z\left(1-\frac{1}{2}e^z\right)$ and $g(z)=e^{-z}\left(\frac{1}{2}-e^{-z}\right)$.

Then

$$P_2(f)=-\frac{3}{8}\left(f^{(iv)}-\frac{2}{3}f^{'''}-5f^{''}+2f^{'}\right)=e^z(1-e^z)$$

and

$$P_2(g)=-\frac{3}{8}\left(g^{(iv)}-\frac{2}{3}g^{'''}-5g^{''}+2g^{'}\right)=e^{-z}(1-e^{-z}).$$

Since $\overline{N}(r,f)=S(r,f)$, $\Theta(\infty,f)=1$ and

$$N\left(r,\frac{1}{f}\right)=N\left(r,\frac{1}{e^z(1-\frac{1}{2}e^z)}\right)=N_{k+2}\left(r,\frac{1}{f}\right)\sim\frac{1}{2}T(r,f),$$

$\delta_{k+2}(0, f) = \frac{1}{2}$. Also we know that $P_2(f)$ and $P_2(g)$ share (a, l) $(l \geq 0)$, but none of the inequalities (35), (36) and (37) are satisfied and neither $P_2(f) \equiv P_2(g)$ nor $P_2(f)P_2(g) \equiv 1$. Therefore, the conditions of Theorem 2 are essential.

3 Open Problems

1. Is it possible to replace the non-constant meromorphic functions by non-constant entire functions?.

2. Whether it is possible to replace the sharing value small function $a(z)$ by a polynomial $p(z)$?.

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