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Coefficient estimates of some classes of rational functions

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Abstract

The purpose of the present paper is to introduce several new subclasses of the function class σ of analytic and bi-univalent functions in the open unit disk U. Furthermore, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these new subclasses

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1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions f in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, $\mathbb{C}$ being, as, usual, the set of complex numbers. Further, by $\wp$ we shall denote the subclass of all functions in $\mathcal{A}$ which are univalent in Δ. If the functions f and g are analytic in Δ, then f is said to be subordinate to g, written $f(z) \prec g(z)$, provided there is an analytic function $w(z)$ defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ so that $f(z) = g(w(z))$. Some of the important and well-investigated subclasses of the univalent function class $\wp$ include (for example) the class $S(\alpha)$ of starlike functions of order α in Δ and the class $C(\alpha)$ of convex functions of order α in Δ. By definition, we have

$$S(\alpha)=\left\{f: f\in\wp \;\; and \;\; \Re\frac{zf'(z)}{f(z)}>\alpha \quad (z\in\Delta,\ 0\le\alpha<1)\right\} \tag{1}$$

and

$$C(\alpha)=\left\{f: f\in\wp \;\; and \;\; zf^{'}(z)\in S(\alpha)\ (z\in\Delta,\ 0\le\alpha<1)\right\}. \tag{2}$$

In [12], the authors introduced the class $S(\phi)$ of the so-called Ma and Minda starlike functions and the class $C(\phi)$ of Ma and Minda convex functions, unifying several previously studied classes related to those of starlike and convex functions. The class $S(\phi)$ consists of all the functions $f\in\mathcal{A}$ satisfying subordination $\dfrac{zf'(z)}{f(z)}\prec\phi(z)$, whereas $C(\phi)$ is formed with functions $f\in\mathcal{A}$ for which the subordination $1+\dfrac{zf''(z)}{f'(z)}\prec\phi(z)$ holds.

It is well known that for each $f\in\wp$, the koebe one-quarter Theorem [7] ensures the image of Δ under f contains a disk of radius 1/4. Thus every univalent function $f\in\wp$ has an inverse f^{-1} which satisfies

$$f^{-1}(f(z))=z\ (|z|<1)$$

and

$$f(f^{-1}(w))=w, \quad (|w|<r_0(f),\ r_0(f)\le 1/4).$$

A function $f\in\mathcal{A}$ is said to bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let σ denote the class of bi-univalent functions defined in the unit disk Δ.The class of bi-univalent functions was first introduced and studied by Lewin [11], where it was proved that $|a_2|<1.51$.Brannan and Clunie [3] improved Lewin's result to $|a_2|<\sqrt{2}$ and later Netanyahu [16] proved that $|a_2|<\frac{3}{4}$. Brannan and Taha [4] and Taha [26] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The pioneering work by Srivastava et al. [23] actually revived the study of bi-univalent functions in recent years. In fact, ever since the publication of their widely-cited paper [23], several results on coefficient bound estimates for the initial and other coefficients were proved for various subclasses of the bi-univalent function σ (see, for example, [1, 2, 5, 6, 8, 9, 10, 13, 15, 19, 21, 24, 27, 28, 22, 25]).

In [14], Mitrinovic essentially investigated certain geometric properties of functions ψ of the form

$$\psi(z)=\frac{z}{g(z)}, \quad g(z)=1+\sum_{n=1}^{\infty}a_n z^n. \tag{3}$$

In [20], Reade et al. derived coefficient conditions that guarantee the univalence, starlikeness or convexity of rational functions of the form (3), these results have been improved and generalized in [17]. In this paper, estimates on the initial coefficients for several subclasses of the bi-univalent function class σ of rational form (3) are obtained. Several related classes are also considered.

In order to derive our main results, we require the following lemma.

Lemma 1.1 *(see [18]) If* $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ *is an analytic function in* Δ *with positive real part, then*

$$|c_n| \leq 2 \qquad (n \in \mathbb{N} = \{1, 2, ...\}). \tag{4}$$

2 Coefficients estimates

Let ϕ be an analytic function with positive real part in the unit disk Δ, satisfying $\phi(0) = 1$, $\phi'(0) > 0$, and $\phi(\Delta)$ is symmetric with respect to the the real axis, such a function has a Taylor series of the form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \, (B_1 > 0). \tag{5}$$

A function $\psi(z) \in \mathcal{A}$ with $\mathrm{Re}(\psi'(z)) > 0$ is known to be univalent. This motivates the following class of functions.

Definition 2.1 *A function* $\psi \in \sigma$ *given by (3)is said to be in the class* $\Re_\sigma(\phi)$ *if it satisfies the following conditions:*

$$\left[(1 + e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - e^{i\gamma}\right] \prec \phi(z) \, (z \in \Delta, \gamma \in \mathbb{R})$$

and

$$\left[(1 + e^{i\gamma})\frac{wg'(w)}{g(w)} - e^{i\gamma}\right] \prec \phi(w) \quad (w \in \Delta, \gamma \in \mathbb{R}),$$

where $g(w) := \psi^{-1}(w)$.

Theorem 2.2 *Let* $\psi(z) \in \Re_\sigma(\phi)$ *be of the form* (3)*. Then*

$$|a_1| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|1 + e^{i\gamma}|\,|B_1^2 + (1 + e^{i\gamma})(B_1 - B_2)|}} \quad \text{and} \quad |a_2| \leq \frac{B_1}{2\,|1 + e^{i\gamma}|} \tag{6}$$

Proof. Let $\psi(z) \in \Re_\sigma(\phi)$ and $g = \psi^{-1}$. Then there exist two functions u and v, analytic in Δ, with $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(w)| < 1$, $z, w \in \Delta$, such that

$$\left[(1+e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - e^{i\gamma}\right] = \phi(u(z))$$

and

$$\left[(1+e^{i\gamma})\frac{wg'(w)}{g(w)} - e^{i\gamma}\right] = \phi(v(w)). \tag{7}$$

Next, define the functions p_1 and p_2 by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + ...$$

and

$$p_2(w) = \frac{1+v(w)}{1-v(w)} = 1 + b_1 w + b_2^2 w^2 + ...,$$

or, equivalently,

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + ...\right], \tag{8}$$

and

$$v(w) = \frac{p_2(w)-1}{p_2(w)+1} = \frac{1}{2}\left[b_1 w + \left(b_2 - \frac{b_1^2}{2}\right) w^2 + ...\right]. \tag{9}$$

Then p_1 and p_2 analytic in Δ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \Delta \longrightarrow \Delta$, the functions p_1 and p_2 have a positive real part in Δ, $|b_i| \leq 2$ and $|c_i| \leq 2$.

Clearly, upon substituting from (8) and (9) into (7), if we make use of (5), we find that

$$\begin{aligned}\left[(1+e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - e^{i\gamma}\right] &= \phi(\frac{p_1(z)-1}{p_1(z)+1}) = 1 + \frac{1}{2}B_1 c_1 z \\ &+ \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right] z^2 + ..., \quad (10)\end{aligned}$$

and

$$\begin{aligned}\left[(1+e^{i\gamma})\frac{wg'(w)}{g(w)} - e^{i\gamma}\right] &= \phi(\frac{p_2(w)-1}{p_2(w)+1}) = 1 + \frac{1}{2}B_1 b_1 w \\ &+ \left[\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2\right] w^2 + ...(11)\end{aligned}$$

Since $\psi \in \sigma$ has the Maclaurin's series given by

$$\psi(z) = z - a_1 z^2 + (a_1^2 - a_2)z^3 + ..., \tag{12}$$

a computation shows that its inverse $g = \psi^{-1}$ has the expansion

$$g(w) = \psi^{-1}(w) = w + a_1 w^2 + (a_1^2 + a_2)w^3 + \cdots . \tag{13}$$

Since

$$\left[(1+e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - e^{i\gamma}\right] = 1 - (1+e^{i\gamma})a_1 z + (1+e^{i\gamma})(a_1^2 - 2a_2)z^2 + \cdots$$

and

$$\left[(1+e^{i\gamma})\frac{wg'(w)}{g(w)} - e^{i\gamma}\right] = 1 + (1+e^{i\gamma})a_1 w + (1+e^{i\gamma})(a_1^2 + 2a_2)w^2 + \cdots .$$

Using (12) and (13) in (10) and (11) respectively, we get

$$-(1+e^{i\gamma})a_1 = \frac{1}{2}B_1 c_1 \tag{14}$$

$$(1+e^{i\gamma})(a_1^2 - 2a_2) = \frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2 c_1^2, \tag{15}$$

$$(1+e^{i\gamma})a_1 = \frac{1}{2}B_1 b_1 \tag{16}$$

and

$$(1+e^{i\gamma})(a_1^2 + 2a_2) = \frac{1}{2}B_1(b_2 - \frac{b_1^2}{2}) + \frac{1}{4}B_2 b_1^2. \tag{17}$$

From (14) and (16), we have

$$c_1 = -b_1. \tag{18}$$

Adding (15) and (17), then using (14) and (18), we get

$$a_1^2 = \frac{B_1^3(c_2 + b_2)}{4(1+e^{i\gamma})\left[B_1^2 + (1+e^{i\gamma})(B_1 - B_2)\right]},$$

and now, by applying Lemma 1.1 for the coefficients b_2 and c_2, the last equation gives the bound of $|a_1|$ from (6). By subtracting (17) from (15), further computations using (18) lead to

$$a_2 = \frac{1}{8(1+e^{i\gamma})}B_1(b_2 - c_2).$$

The bound of $|a_2|$, as asserted in (6), is now a consequence of Lemma 1.1, and this completes our proof.

If we set

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^{\eta}=1+2\eta z+2\eta^2z^2+...\,(0<\eta\leq 1,\ z\in\Delta)\,.$$

in Definition 2.1 of the bi-univalent function class $\Re_\sigma(\phi)$, we obtain a new class $\Re_\sigma(\eta)$ given by Definition 2.3 below.

Definition 2.3 *For* $0<\eta\leq 1$, *a function* $\psi\in\sigma$ *given by (3) is said to be in the class* $\Re_\sigma(\eta)$ *if it satisfies the following conditions:*

$$\left[(1+e^{i\gamma})\frac{z\psi'(z)}{\psi(z)}-e^{i\gamma}\right]\prec\left(\frac{1+z}{1-z}\right)^{\eta}(z\in\Delta,\gamma\in\mathbb{R})$$

and

$$\left[(1+e^{i\gamma})\frac{wg'(w)}{g(w)}-e^{i\gamma}\right]\prec\left(\frac{1+w}{1-w}\right)^{\eta}(w\in\Delta,\gamma\in\mathbb{R})\,,$$

where $g(w):=\psi^{-1}(w)$.

Using the parameter setting of Definition 2.3 in Theorem 2.2, we get the following corollary.

Corollary 2.4 *For* $0<\eta\leq 1$, *let the function* $\psi\in\Re_\sigma(\eta)$ *be of the form (3). Then*

$$|a_1|\leq\frac{\eta}{\sqrt{|1+e^{i\gamma}|\,|2\eta+(1+e^{i\gamma})(1-\eta)|}}\quad\text{and }|a_2|\leq\frac{\eta}{|1+e^{i\gamma}|}.$$

Let

$$\phi(z)=\frac{1+(1-2\nu)z}{1-z}=1+2(1-\nu)z+2(1-\nu)z^2+...\,(0<\nu\leq 1,\ z\in\Delta)\,.$$

in Definition 2.1 of the bi-univalent function class $\Re_\sigma(\phi)$, we obtain a new class $\mathcal{H}_\sigma(\nu)$ given by Definition 2.5 below.

Definition 2.5 *For* $0<\nu\leq 1$, *a function* $\psi\in\sigma$ *given by (3) is said to be in the class* $\mathcal{H}_\sigma(\nu)$ *if the following conditions holds true:*

$$\left[(1+e^{i\gamma})\frac{z\psi'(z)}{\psi(z)}-e^{i\gamma}\right]\prec\frac{1+(1-2\nu)z}{1-z}\,(z\in\Delta,\gamma\in\mathbb{R})$$

and

$$\left[(1+e^{i\gamma})\frac{wg'(w)}{g(w)}-e^{i\gamma}\right]\prec\frac{1+(1-2\nu)w}{1-w}\,(w\in\Delta,\gamma\in\mathbb{R})\,,$$

where $g(w):=\psi^{-1}(w)$.

Using the parameter setting of Definition 2.5 in Theorem 2.2 we get the following corollary.

Corollary 2.6 *For* $0 < \nu \leq 1$, *let the function* $\psi \in \Re_\sigma(\nu)$ *be given by (3). Then*

$$|a_1| \leq \sqrt{\frac{2(1-\nu)}{|1+e^{i\gamma}|}} \quad \textit{and} \quad |a_2| \leq \frac{(1-\nu)}{|1+e^{i\gamma}|}.$$

Definition 2.7 *A function* $\psi \in \sigma$ *is given by (3) is said to be in the class* $S_\sigma(\lambda, \mu, \phi)$ *if it satisfies the following subordination conditions:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu} + \lambda\psi'(z)\left(\frac{\psi(z)}{z}\right)^{\mu-1} \prec \phi(z)\,(0 < \mu < 1; 0 \leq \lambda \leq 1 \textit{ and } z \in \Delta)$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec \phi(w)\,(0 < \mu < 1; 0 \leq \lambda \leq 1 \textit{ and } w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

For functions in the class $S_\sigma(\lambda, \mu, \phi)$, the following coefficient estimates are obtained.

Theorem 2.8 *Let* $\psi(z) \in S_\sigma(\lambda, \mu, \phi)$ *be of the form (3). Then*

$$|a_1| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2\left[\frac{\mu(\mu+2\lambda+1)}{2} + \lambda\right] + (B_1 - B_2)\left[\mu(1-2\lambda) - \lambda\right]^2\right|}}, \tag{19}$$

and

$$|a_2| \leq \frac{B_1}{(\mu + 2\lambda)}. \tag{20}$$

Proof. Let $\psi \in S_\sigma(\lambda, \mu, \phi)$, there are two Schwarz functions u and v defined by (8) and (9) respectively, such that

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu} + \lambda\psi'(z)\left(\frac{\psi(z)}{z}\right)^{\mu-1} = \phi(u(z)) \quad \text{and} \tag{21}$$

$$(1-\lambda)\left(\frac{\psi(w)}{w}\right)^{\mu} + \lambda\psi'(w)\left(\frac{\psi(w)}{w}\right)^{\mu-1} = \phi(v(w)).$$

Since

$$
\begin{aligned}
&(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu}+\lambda\psi'(z)\left(\frac{\psi(z)}{z}\right)^{\mu-1}\\
=\;& 1-[\mu(1-2\lambda)-\lambda]\,a_1 z+\left[\left(\frac{\mu(\mu+2\lambda+1)}{2}+\lambda\right)a_1^2-(2\lambda+\mu)a_2\right]z^2+\cdots
\end{aligned}
$$

and

$$
\begin{aligned}
&(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\\
=\;& 1+[\mu(1-2\lambda)-\lambda]\,a_1 w+\left[\left(\frac{\mu(\mu+2\lambda+1)}{2}+\lambda\right)a_1^2+(2\lambda+\mu)a_2\right]w^2+\cdots.
\end{aligned}
$$

Then (12), (13) and (21) yields

$$-[\mu(1-2\lambda)-\lambda]\,a_1=\frac{1}{2}B_1c_1 \tag{22}$$

$$\left(\frac{\mu(\mu+2\lambda+1)}{2}+\lambda\right)a_1^2-(2\lambda+\mu)a_2=\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2c_1^2, \tag{23}$$

$$[\mu(1-2\lambda)-\lambda]\,a_1=\frac{1}{2}B_1b_1 \tag{24}$$

and

$$\left(\frac{\mu(\mu+2\lambda+1)}{2}+\lambda\right)a_1^2+(2\lambda+\mu)a_2=\frac{1}{2}B_1(b_2-\frac{b_1^2}{2})+\frac{1}{4}B_2b_1^2. \tag{25}$$

From (22) and (24), we get

$$c_1=-b_1, \tag{26}$$

and after some further calculations using (23)-(26) we find

$$a_1^2=\frac{B_1^3(c_2+b_2)}{4\left[B_1^2(\frac{\mu(\mu+2\lambda+1)}{2}+\lambda)+(B_1-B_2)\,[\mu(1-2\lambda)-\lambda]^2\right]},$$

and

$$a_2=\frac{B_1(b_2-c_2)}{4(\mu+2\lambda)}.$$

Applying Lemma 1.1, the estimates in (19) and (20) follow.

Definition 2.9 *For* $0 < \eta \leq 1$, *a function* $\psi \in \sigma$ *given by (3) is said to be in the class* $S_\sigma(\lambda, \mu, \eta)$ *if it satisfies the following subordination conditions:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu}+\lambda\psi'(z)\left(\frac{\psi(z)}{z}\right)^{\mu-1} \prec \left(\frac{1+z}{1-z}\right)^{\eta} (0 < \mu < 1; 0 \leq \lambda \leq 1 \text{ and } z \in \Delta),$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec \left(\frac{1+w}{1-w}\right)^{\eta} (0 < \mu < 1; 0 \leq \lambda \leq 1 \text{ and } w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

Using the parameter setting of Definition 2.9 in Theorem 2.8 we get the following corollary.

Corollary 2.10 *For and* $0 < \eta \leq 1$, *let the function* $\psi \in S_\sigma(\lambda, \mu, \eta)$ *be of the form (3). Then*

$$|a_1| \leq \frac{2\eta}{\sqrt{-\eta\left[(\mu(1-2\lambda)-\lambda)^2 - 2(\frac{\mu(\mu+2\lambda)}{2}+\lambda)\right] + [\mu(1-2\lambda)-\lambda]^2}}$$

and

$$|a_2| \leq \frac{2\eta}{(\mu+2\lambda)}.$$

Let

$$\phi(z) = \frac{1+(1-2\nu)z}{1-z} = 1 + 2(1-\nu)z + 2(1-\nu)z^2 + \ldots (0 < \nu \leq 1, \ z \in \Delta).$$

Definition 2.11 *For* $0 < \nu \leq 1$, *a function* $\psi \in \sigma$ *given by (3) is said to be in the class* $S_\sigma(\lambda, \mu, \nu)$ *if it satisfies the following subordination conditions:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu}+\lambda\psi'(z)\left(\frac{\psi(z)}{z}\right)^{\mu-1} \prec \frac{1+(1-2\nu)z}{1-z} (0 < \mu < 1; 0 \leq \lambda \leq 1 \text{ and } z \in \Delta)$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec \frac{1+(1-2\nu)w}{1-w} (0 < \mu < 1; 0 \leq \lambda \leq 1 \text{ and } w \in \Delta),$$

where $g(w) = \psi^{-1}(w)$.

Using the parameter setting of Definition 2.11 in Theorem 2.8 we get the following corollary.

Corollary 2.12 *For* $0 < \nu \leq 1$, *let the function* $\psi \in S_\sigma(\lambda, \mu, \nu)$ *be of the form (3). Then*

$$|a_1| \leq \sqrt{\frac{4\,(1-\nu)}{(\mu(\mu+2\lambda)+2\lambda)}} \quad \textit{and } |a_2| \leq \frac{2\,(1-\nu)}{(\mu+2\lambda)}.$$

Definition 2.13 *A function* $\psi \in \sigma$ *given by (3) is said to be in the class* $M_\sigma(\lambda, \mu, \phi)$, *if it satisfies the following subordinations conditions:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu} + \lambda\psi'(z)\left(\frac{z}{\psi(z)}\right)^{\mu-1} \prec \phi(z)\,(0 < \mu < 1; 0 \leq \lambda \leq 1 \textit{ and } z \in \Delta)$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{w}{g(w)}\right)^{\mu-1} \prec \phi(w), (0 < \mu < 1; 0 \leq \lambda \leq 1 \textit{ and } w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

A function in the class $M_\sigma(\lambda, \mu, \phi)$ is called bi-Mocanu convex function of Ma-Minda type. This class unifies the classes $S(\alpha)$ and $C(\alpha)$. For functions in the class $M_\sigma(\lambda, \mu, \phi)$, the following coefficients estimates hold.

Theorem 2.14 *Let* $\psi(z) \in M_\sigma(\lambda, \mu, \phi)$ *be of the form (3). Then*

$$|a_1| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|4B_1^2[(\mu(\mu+1)+4\lambda(3-2\mu)] - [\mu(1-2\lambda)+3\lambda]^2(B_1-B_2)|}}, \tag{27}$$

and

$$|a_2| \leq \frac{B_1}{4[\mu(2\lambda-1)-4\lambda)]}. \tag{28}$$

Proof. If $\psi \in M_\sigma(\lambda, \mu, \phi)$, then there exist are two Schwarz functions u and v defined by (8) and (9) respectively, such that

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu} + \lambda\psi'(z)\left(\frac{z}{\psi(z)}\right)^{\mu-1} = \phi(u(z)), \tag{29}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{w}{g(w)}\right)^{\mu-1} = \phi(v(w)). \tag{30}$$

Since

$$\begin{aligned}
&(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu}+\lambda\psi'(z)\left(\frac{z}{\psi(z)}\right)^{\mu-1} \\
= \; & 1-[\mu(1-2\lambda)+3\lambda]a_1 z \\
&+\left[\left(\frac{(\mu(\mu+1)}{2}+2\lambda(3-2\mu)\right)a_1^2+[\mu(2\lambda-1)-4\lambda]a_2\right]z^2+...
\end{aligned}$$

and

$$\begin{aligned}
&(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{w}{g(w)}\right)^{\mu-1} \\
= \; & 1+[\mu(1-2\lambda)+3\lambda]a_1 w \\
&+\left[\left(\frac{(\mu(\mu+1)}{2}+2\lambda(3-2\mu)\right)a_1^2-[\mu(2\lambda-1)-4\lambda)]a_2\right]w^2+...,
\end{aligned}$$

from (10), (11), (29) and (30), it follows that

$$-[\mu(1-2\lambda)+3\lambda]a_1=\frac{1}{2}B_1c_1, \tag{31}$$

$$\left(\frac{(\mu(\mu+1)}{2}+2\lambda(3-2\mu)\right)a_1^2+[\mu(2\lambda-1)-4\lambda)]a_2=\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2c_1^2, \tag{32}$$

$$[\mu(1-2\lambda)+3\lambda]a_1=\frac{1}{2}B_1b_1, \tag{33}$$

and

$$\left(\frac{\mu(\mu+1)}{2}+2\lambda(3-2\mu)\right)a_1^2-[\mu(2\lambda-1)-4\lambda)]a_2=\frac{1}{2}B_1(b_2-\frac{b_1^2}{2})+\frac{1}{4}B_2b_1^2, \tag{34}$$

Equations (31) and (33) yields

$$c_1=-b_1, \tag{35}$$

and after some further calculations using (32)-(34) we find

$$a_1^2=\frac{B_1^3(c_2+b_2)}{4B_1^2[(\mu(\mu+1)+4\lambda(3-2\mu)]+(\mu(1-2\lambda)+3\lambda)^2(B_1-B_2)},$$

and

$$a_2=\frac{B_1(b_2-c_2)}{4[\mu(2\lambda-1)-4\lambda)]}.$$

Applying Lemma 1.1, the estimates in (27) and (28) follow.

Let

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^{\eta}=1+2\eta z+2\eta^2z^2+...\,(0<\eta\leq 1,\; z\in\Delta).$$

Definition 2.15 *For $0 < \eta \leq 1$, a function $\psi \in \sigma$ given by (3) is said to be in the class $M_\sigma(\lambda, \mu, \eta)$ if the following subordinations conditions hold:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu} + \lambda\psi'(z)\left(\frac{z}{\psi(z)}\right)^{\mu-1} \prec \left(\frac{1+z}{1-z}\right)^{\eta} \left(0 < \mu < 1; 0 \leq \lambda \leq 1 \text{ and } z \in \Delta\right),$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{w}{g(w)}\right)^{\mu-1} \prec \left(\frac{1+w}{1-w}\right)^{\eta} \left(0 < \mu < 1; 0 \leq \lambda \leq 1 \text{ and } w \in \Delta\right),$$

$g(w) := \psi^{-1}(w)$.

Using the parameter setting of Definition 2.15 in Theorem 2.14 we get the following corollary.

Corollary 2.16 *For $0 < \eta \leq 1$, let the function $\psi \in M_\sigma(\lambda, \mu, \eta)$ be of the form (3). Then*

$$|a_1| \leq \frac{2\eta}{\sqrt{\eta\left[8(\mu(\mu+1) + 4\lambda(3-2\mu)) - (\mu(1-2\lambda)+3\lambda)^2\right] + [\mu(1-2\lambda)+3\lambda]^2}}$$

and

$$|a_2| \leq \frac{\eta}{2[\mu(2\lambda-1) - 4\lambda)]}.$$

Let

$$\phi(z) = \frac{1+(1-2\nu)z}{1-z} = 1 + 2(1-\nu)z + 2(1-\nu)z^2 + ... \left(0 < \nu \leq 1,\ z \in \Delta\right).$$

Definition 2.17 *For $0 < \nu \leq 1$, a function $\psi \in \sigma$ given by (3) is said to be in the class $M_\sigma(\lambda, \mu, \nu)$ if the following subordinations hold:*

$$(1-\lambda)\left(\frac{\psi(z)}{z}\right)^{\mu} + \lambda\psi'(z)\left(\frac{z}{\psi(z)}\right)^{\mu-1} \prec \frac{1+(1-2\nu)z}{1-z} \left(0 < \mu < 1; 0 \leq \lambda \leq 1 \text{ and } z \in \Delta\right),$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{w}{g(w)}\right)^{\mu-1} \prec \frac{1+(1-2\nu)w}{1-w} \left(0 < \mu < 1; 0 \leq \lambda \leq 1 \text{ and } w \in \Delta\right),$$

where $g(w) := \psi^{-1}(w)$.
Using the parameter setting of Definition 2.17 in Theorem 2.14 we get the following corollary.

Corollary 2.18 *For $0 < \nu \leq 1$, let the function $\psi \in M_\sigma(\lambda, \mu, \nu)$ be of the form (3). Then*

$$|a_1| \leq \sqrt{\frac{2\,(1-\nu)}{4\lambda(2\mu-3) + (\mu(\mu+1)}} \quad \text{and} \quad |a_2| \leq \frac{(1-\nu)}{2[\mu(2\lambda-1) - 4\lambda)]}.$$

3 Open Problem

The authors suggest to study the class of functions $\psi \in \sigma$ which satisfy the following conditions:

$$\left[(1+\beta e^{i\gamma})\frac{z\psi'(z)}{\psi(z)} - \beta e^{i\gamma}\right] \prec \phi(z)\,(z \in \Delta, \beta \geq 0, \gamma \in \mathbb{R})$$

and

$$\left[(1+\beta e^{i\gamma})\frac{wg'(w)}{g(w)} - \beta e^{i\gamma}\right] \prec \phi(w) \quad (w \in \Delta, \beta \geq 0, \gamma \in \mathbb{R})\,,$$

where $g(w) := \psi^{-1}(w)$.

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References

[1] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl. Math. Lett., 25(2012), 344-351.

[2] M. K. Aouf, R. M. El-Ashwah and A. M. Abd-Eltawab, New subclasses of bi-univalent functions involving Dziok-Srivastava operator, ISRN Math. Anal., 2013(2013), Article ID 387178, 5 pages.

[3] D. A. Brannan and J. G. Clunie (Eds.), Aspects of Contemporary Complex Analysis (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1-20, 1979), Academic Press, New York and London, 1980.

[4] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, Studia Univ. Babes-Bolyai Math., 31(2)(1986), 70-77.

[5] S. Bulut, Coefficient estimates for a class of analytic and bi-univalent functions, Novi Sad J. Math., 43(2)(2013), 59–65 .

[6] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, J. Class. Anal., 2(1)(2013), 49–60 .

[7] P. L. Duren, Univalent Functions, Grundlehren der Math. Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[8] R. M. El-Ashwah, Subclasses of bi-univalent functions defined by convolution, J. Egypt. Math. Soc., 22(2014), 348–351.

[9] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett., 24 (2011), 1569-1573.

[10] A. Y. Lashin, On certain subclasses of analytic and bi-univalent functions, J. Egypt. Math. Soc., 24(2016), 220–225.

[11] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc., 18(1967), 63-68.

[12] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions in proceedings of the Conference on Complex Analysis, Tianjin, 1992, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, (1994), 157-169.

[13] N. Magesh and J. Yamini, Coefficient bounds for certain subclasses of bi-univalent functions, Int. Math. Forum 8 (2013), 1337–1344 .

[14] S. D. Mitrinovic, On the univalence of rational functions, Univ. Beograd. Publ. Elektrotehn, Fak., Ser. Mat. Fiz., No. 634-No. 677 (1979), 221-227.

[15] G. Murugusundaramoorthy, N. Magesh and V. Prameela, Coefficient bounds for certain subclasses of bi-univalent function, Abst. Appl. Anal., 2013(2013), Article ID 573017, 3 pages.

[16] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational. Mech. Anal. 32(1969), 100-112.

[17] M. Obradovic, S. Ponnusamy, V. Singh and P. Vasundhra, Univalency starlikesess and convexity applied to certain classes of rational functions, Analysis (Munich) 22(3)(2002), 225-242.

[18] C. Pommerenke, Univalent Functions, Vandenhoeck and Rupercht, Göttingen, 1975.

[19] S. Porwal and M. Darus, On a new subclass of bi-univalent functions, J. Egypt. Math. Soc., 21 (3)(2013), 190–193.

[20] M. O. Reade, H. Silverman and P. G. Todorov, On the starlikeness and convexity of a class of analytic functions, Rend. Circ. Mat. Palermo 33(1984), 265-272.

[21] H. M. Srivastava and D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, J. Egypt. Math. Soc., 23(2015), 242–246.

[22] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for some general subclasses of analytic and bi-univalent functions, Afrika Mat. 28 (2017), 693-706.

[23] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclass of analytic and bi-univalent functions, Appl. Math. Lett. 23(10) (2010), 1188–1192.

[24] H. M. Srivastava, G. Murugusundaramoorthy and N. Magesh, Certain subclasses of bi-univalent functions associated with the Hohlov operator, Global J. Math. Anal., 1(2)(2013), 67–73.

[25] H. M. Srivastava, Serap Bulut, Murat Caglar and Nihat Yagmur, Coefficient estimates for general subclass of analytic and bi-univalent functions, Filomat 27: 5(2013), 831-842.

[26] T. S. Taha, Topics in univalent function theory, University of London, 1981 Phd thesis.

[27] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett., 25(2012), 990-994 .

[28] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, Appl. Math. Comput., 218 (2012), 11461-11465.