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Starlikeness in A Parabolic Region Using Differential Subordination

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Abstract

***In the present paper, we obtain starlikeness of members of class $\mathcal{A}$ in a parabolic region. We use differential subordination to extend certain results in this direction.*.**

Keywords: *Analytic function, Parabolic Starlike function, Uniformly-close-to-convex function, Differential subordination.*
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1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f which are normalized by the conditions $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. Therefore the members of the class $\mathcal{A}$ have the Taylor series expansion of the following form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

A function $f \in \mathcal{A}$ is said to be parabolic starlike in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathbb{E} \tag{1}$$

The class of parabolic starlike functions is denoted by $\mathcal{S}_p$. A function $f \in \mathcal{A}$ is said to be uniformly close-to-convex in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - 1\right|, z \in \mathbb{E} \tag{2}$$

for some $g \in \mathcal{S}_p$. Let UCC denote the class of such functions. Note that the function $g(z) \equiv z \in \mathcal{S}_p$. Therefore, for $g(z) = z$, condition (2) becomes

$$\Re(f'(z)) > |f'(z) - 1|, z \in \mathbb{E} \tag{3}$$

Define the parabolic domain Ω as under:

$$\Omega = \{u + iv : u > \sqrt{(u-1)^2 + v^2}\}. \tag{4}$$

Note that the conditions (1) and (3) are equivalent to the condition that $\frac{zf'(z)}{f(z)}$ and $f'(z)$ take values in the parabolic domain Ω respectively.
Ronning [7] and Ma and Minda [4] showed that the function defined by

$$q(z) = 1 + \frac{2}{\pi^2} log^2 \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \tag{5}$$

maps the open unit disk $\mathbb{E}$ onto the parabolic domain Ω. Therefore, condition (1) is equivalent to

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \prec q(z), z \in \mathbb{E} \tag{6}$$

and condition (3) is same as

$$\Re(f'(z)) \prec q(z), z \in \mathbb{E} \tag{7}$$

where q is given by (5)
Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let p be an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0) \tag{8}$$

A univalent function q is called a dominant of the differential subordination (8) if $p(0) = q(0)$ and $p \prec q$ and for all p satisfying (8). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (8), is said to be the best dominant of the differential subordination (8). The best dominant is unique up to a rotation of $\mathbb{E}$.

For $k \in [0, \infty)$, define the domain

$$\Omega_k = \{u + iv : u^2 > k^2(u-1)^2 + k^2v^2\}. \tag{9}$$

For fixed k, the above domain represents the conic region bounded, successively, by the imaginary axis ($k = 0$), the right branch of hyperbola ($0 < k < 1$), a parabola ($k = 1$), and an ellipse ($k > 1$). Also we note that, for no choice of parameter $k(k > 1)$, Ω_k reduces to a disk. p_k is the univalent function mapping the unit disk $\mathbb{E}$ onto Ω_k, such that $p_k(0) = 1$ and $p_k'(0) > 0$

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & \text{for } k = 0 \\ 1 + \frac{2}{1-k^2} sinh^2(A(k) arctanh\sqrt{z}), & \text{for } k \in (0,1) \\ 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}, & \text{for } k = 1 \\ 1 + \frac{2}{k^2-1} sin^2 \frac{\pi}{2K(t)} F(\frac{\sqrt{z}}{\sqrt{t}}, t), & \text{for } k > 1 \end{cases} \tag{10}$$

where $A(k) = (\frac{2}{\pi} arccos k)$, $F(\omega, t)$ is Legendre elliptic integral of the first kind, $F(\omega, t) = \int_0^\omega \frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}$, $K(t) = F(1, t)$ and $t \in (0, 1)$ is chosen such that $k = cosh(\frac{\pi K'(t)}{2K(t)})$.

Consider the region bounded by a parabola $u = \dfrac{v^2}{2} + \dfrac{1}{2}$ that is the domain $p_1(\mathbb{E})$ with p_1 given by (10). We next consider, the family described by the equality $u = \dfrac{v^2}{2} + \dfrac{2a+1}{2}$ such that $a < \dfrac{1}{2}$ that consists of right handed parabolas with vertex at $\left(\dfrac{2a+1}{2}, 0\right)$ symmetric about the real axis. The family of domains containing point 1 inside and bounded by those parabolas may be characterised as

$$\mathbb{D}_a = \{w : \Re(w - a) > |w - 1 - a|\}$$

or equivalently,

$$\mathbb{D}_a = \{w = u + iv : 2u > v^2 + 2a + 1\}. \tag{11}$$

Kanas [3] solved the problem of finding the largest domain $\mathbb{D}$ for which, under given ϕ and q, the differential subordination $\phi(p(z), zp'(z)) \in \mathbb{D} \Rightarrow p(z) \prec q(z)$, where $\mathbb{D}$ and $q(\mathbb{E})$ are the regions bounded by conic sections, is satisfied. Kanas [3] proved the following results:

Theorem 1.1 *Let p be analytic in $\mathbb{E}$ such that $p(0) = 1$. Also let $a < \frac{1}{2}$. If $p(z) + \dfrac{zp'(z)}{p(z)} \in \mathbb{D}_a$, then*

$$p(z) \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}},$$

where

$$\mathbb{D}_a = \{w : \Re(w - a) > |w - 1 - a|\}$$

or equivalently,

$$\mathbb{D}_a = \{w = u + iv : 2u > v^2 + 2a + 1\} \tag{12}$$

and $a \geq a_0 = -\dfrac{1}{\pi}$.

Theorem 1.2 *Let $f \in \mathcal{A}$ and let $1 + \dfrac{zf''(z)}{f'(z)} \in \mathbb{D}_a$, where*

$$\mathbb{D}_a = \{w : \Re(w - a) > |w - 1 - a|\}$$

or equivalently,

$$\mathbb{D}_a = \{w = u + iv : 2u > v^2 + 2a + 1\} \tag{13}$$

and $a \geq -\dfrac{1}{\pi}$. Then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}.$$

The main objective of the present paper is to extend the above mentioned results of Kanas [3] in the sense that the same operators take values in an extended region to conclude the same results.

2 Main Results

To prove our main result, we use the following lemma of Miller and Mocanu.
Lemma 1[6]. Let q be univalent in the unit disk $\mathbb{E}$ and let θ and ϕ be analytic in domain $\mathbb{D}$ containing $q(\mathbb{E})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q_1(z)$ and suppose that
(i) either h is convex or Q_1 is starlike in $\mathbb{E}$, and
(ii) $\Re \dfrac{zh'(z)}{Q_1(z)} > 0, z \in \mathbb{E}$.
If p is analytic in $\mathbb{E}$ with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[(p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ and q is the best dominant.

Theorem 2.1 *Let p be analytic in $\mathbb{E}$ such that $p(0) = 1$. For $\alpha > 0$, and $p(z)$ be a function such that*

$$p(z) + \alpha\frac{zp'(z)}{p(z)} \prec 1 + \frac{2}{\pi^2}log^2\frac{1+\sqrt{z}}{1-\sqrt{z}} + \alpha\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\frac{log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}{1+\frac{2}{\pi^2}log^2\frac{1+\sqrt{z}}{1-\sqrt{z}}} \tag{14}$$

then

$$p(z) \prec 1 + \frac{2}{\pi^2}log^2\frac{1+\sqrt{z}}{1-\sqrt{z}}$$

Proof. Let us write

$$q(z) = 1 + \frac{2}{\pi^2}log^2\frac{1+\sqrt{z}}{1-\sqrt{z}}.$$

With a little calculation, from (14), we have

$$p(z) + \alpha\frac{zp'(z)}{p(z)} \prec q(z) + \alpha\frac{zq'(z)}{q(z)}.$$

Define the functions θ and ϕ as $\theta(w) = w$, $\phi(w) = \dfrac{\alpha}{w}$. Clearly ϕ is analytic in domain in $\mathbb{D} = \mathbb{C} \setminus \{0\}$. Set $Q_1(z) = \alpha\dfrac{zq'(z)}{q(z)}$ and $h(z) = q(z) + \alpha\dfrac{zq'(z)}{q(z)}$. On differentiation, we obtain:

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{q(z)}{\alpha}.$$

Thus $\Re\left(\dfrac{zQ_1'(z)}{Q_1(z)}\right)$

$$= \Re\left(1 + \frac{\frac{1}{(1-z)^2} + \frac{3z-1}{2\sqrt{z}(1-z)^2}\log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}{\frac{1}{\sqrt{z}(1-z)}\log(\frac{1+\sqrt{z}}{1-\sqrt{z}})} - \frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}{1+\frac{2}{\pi^2}(\log(\frac{1+\sqrt{z}}{1-\sqrt{z}}))^2}\right)$$

$\Re\left(\dfrac{zh'(z)}{Q_1(z)}\right)$

$$= \Re\left(1 + \frac{\frac{1}{(1-z)^2} + \frac{3z-1}{2\sqrt{z}(1-z)^2}\log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}{\frac{1}{\sqrt{z}(1-z)}\log(\frac{1+\sqrt{z}}{1-\sqrt{z}})} - \frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}{1+\frac{2}{\pi^2}(\log(\frac{1+\sqrt{z}}{1-\sqrt{z}}))^2} + \frac{1}{\alpha}\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]\right).$$

We notice that $\Re\left(\dfrac{zQ_1'(z)}{Q_1(z)}\right) > 0$ and $\Re\left(\dfrac{zh'(z)}{Q_1(z)}\right) > 0$ for $\alpha > 0$.
The proof, now, follows from Lemma 1.

Selecting $\alpha = 1$ in the Theorem 2.1, we get following result;

Theorem 2.2 *Let p be analytic in $\mathbb{E}$ such that $p(0) = 1$. Let $p(z)$ be a function such that*

$$p(z) + \frac{zp'(z)}{p(z)} \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}} + \frac{4}{\pi^2} \frac{\sqrt{z}}{1-z} \frac{log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}{1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}}$$

then

$$p(z) \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}$$

Remark 2.3 *Comparing Theorem 2.2 and Theorem 1.1 in Figure 1. By Theorem 2.2 we observe that the operator $p(z) + \frac{zp'(z)}{p(z)}$ takes values in the whole shaded portion whereas by Theorem 1.1, the operator $p(z) + \frac{zp'(z)}{p(z)}$ takes values only in the dark shaded region. Therefore, the region of variability of the above said operator has been extended in Theorem 2.2, to conclude the same result.*

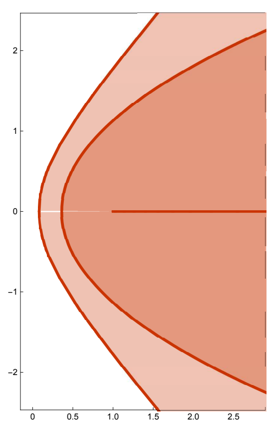

Figure 1

Taking $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 2.2 we obtain :

Corollary 2.4 *For $f \in \mathcal{A}$ and suppose that*

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}} + \frac{4}{\pi^2} \frac{\sqrt{z}}{1-z} \frac{log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}{1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}$$

Remark 2.5 *We notice that the above corollary extends the result of Theorem 1.2 in the same manner as that of Remark 2.3.*

3 Open Problem

The results obtained in this paper hold for $\alpha > 0$. One may try to find the same results that hold for $\alpha < 0$.

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