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Certain Results on Meromorphic Starlike Functions

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Abstract

In the present paper, we obtain certain results on meromorphic starlike functions using the technique of differential subordination.

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1 Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p} \ (p \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$

which are analytic and p-valent in the punctured unit disc $\mathbb{E}_0 = \mathbb{E} \setminus \{0\}$, where $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \Sigma_p$ is said to be meromorphic p-valent starlike of order α if $f(z) \neq 0$ for $z \in \mathbb{E}_0$ and

$$-\Re\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}).$$

The class of all such meromorphic p-valent starlike functions is denoted by $\mathcal{MS}_p^*(\alpha)$.

A function $f \in \Sigma_p$ is called meromorphic p-valent convex of order α if $f'(z) \neq 0$ and

$$-\Re\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}).$$

The class of all meromorphic p-valent convex functions defined above is denoted by $\mathcal{MK}_p(\alpha)$.
The class $\Sigma_p^\alpha(\gamma)$ consists of functions $f \in \Sigma_p$ with $f(z)f'(z) \neq 0$ satisfying

$$-\Re\frac{1}{p}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \gamma, \quad (z \in \mathbb{E}),$$

where γ is real number and $\gamma < 1$.
The function $f \in \Sigma_p^\alpha(\gamma)$ is called a meromorphic p-valent $\alpha-$convex functions of order γ.
Let $\Sigma = \Sigma_1,\ \mathcal{MS}^*(\alpha) = \mathcal{MS}^*{}_1(\alpha),\ \mathcal{MK}(\alpha) = \mathcal{MK}_1(\alpha)$ and $\Sigma^\alpha(\gamma) = \Sigma_1^\alpha(\gamma)$.
For two functions f and g analytic in $\mathbb{E}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{E}$, and write $f(z) \prec g(z),\ (z \in \mathbb{E})$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ (z \in \mathbb{E})$, such that $f(z) = g(w(z)),\ (z \in \mathbb{E})$.
In particular, if the function g is univalent in $\mathbb{E}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.
In the literature of meromorphic functions, there exist certain results involving the above classes. We state below some of them.
Recently Cho and Owa [2] proved the following results.

Theorem 1.1. *If* $f(z) \in \Sigma$ *satisfies* $f(z)f'(z) \neq$ *in* $\mathbb{E}_0$ *and*

$$\Re\left[\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right] < 2(2-\alpha) - \beta,\ (z \in \mathbb{E}),$$

then

$$-\Re\left[\frac{z^{2-\alpha}f'(z)}{f^\alpha(z)}\right] > \frac{1}{1+2(2-\alpha)-2\beta},\ (z \in \mathbb{E}),$$

where $\alpha \leq 2$ *and* $[2(2-\alpha)-1]/2 \leq \beta < 2-\alpha$.

Nunokawa and Ahuja [1] proved the following result.

Theorem 1.2. *Let* $\alpha < 0$ *and* $\gamma \geq 0$. *If*

$$f \in \Sigma_\gamma^*\left(\frac{2\alpha - 2\alpha^2 + \gamma\alpha}{2(1-\alpha)}\right),$$

then $f \in \mathcal{MS}^*(\alpha)$

Ravichandaran et al. [5] proved the following results.

Theorem 1.3. *Let $q(z)$ be univalent and $q(z) \neq 0$ in $\mathbb{E}$ and*
(i) $\dfrac{zq'(z)}{q(z)}$ *is starlike univalent in* $\mathbb{E}$, *and*
(ii) $\Re\left[1+\dfrac{zq''(z)}{q'(z)}-\dfrac{zq'(z)}{q(z)}-\dfrac{q(z)}{\gamma}\right]>0$ *for* $z\in\mathbb{E},\ \gamma\neq 0$.
If $f(z)\in\Sigma$ *and*

$$-\left[(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\prec q(z)-\gamma\frac{zq'(z)}{q(z)},$$

then

$$-\frac{zf'(z)}{f(z)}\prec q(z)$$

and $q(z)$ is the best dominant.

Theorem 1.4. *Let $\alpha<0,\ \gamma\neq 0$. If $f(z)\in\Sigma$ and*

$$-\left[(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\prec\frac{1+2[1-\gamma+(\alpha-1)\gamma]z+(1-2\alpha)^2z^2}{1-2\alpha z-(1-2\alpha)z^2},$$

then $-\Re\dfrac{zf'(z)}{f(z)}>\alpha$.

Roshian and Ravichandaran [3] proved the following results.

Theorem 1.5. *Let $q(z)$ be univalent in $\mathbb{E}$ and $\dfrac{zq'(z)}{q(z)}$ be starlike in $\mathbb{E}$. If $f\in\Sigma_p$ satisfies*

$$\alpha\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\prec 1+(1-\alpha)p-\frac{zq'(z)}{q(z)},$$

then

$$-\frac{z^{1+(1-\alpha)p}f'(z)}{pf^{\alpha}(z)}\prec q(z),$$

and $q(z)$ is the best dominant.

Theorem 1.6. *Let $-1\leq B<A\leq 1$. If $f\in\Sigma$ satisfies*

$$\alpha\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\prec 2-\alpha-\frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$-\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\prec\frac{(1+Az)}{(1+Bz)}.$$

2 Preliminaries

We shall use the following lemma of Miller and Mocanu [4] to prove our result.

Lemma 2.1. *Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*
(i) h is convex, or
(ii) Q_1 is starlike.
In addition, assume that
(iii) $\Re\left(\frac{zh'(z)}{Q_1(z)}\right) > 0$ for all z in $\mathbb{E}$.
If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)], \ z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

3 Main Theorem

Theorem 3.1. *Let α be a non-zero complex number. Let q, $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ satisfying therein the condition*

$$\Re\left[1 + \frac{zq''(z)}{q'(z)} - \frac{2zq'(z)}{q(z)}\right] > max\left\{0, -\Re\frac{\gamma}{\alpha}\right\} \tag{1}$$

If P, $P(z) \neq 0$ in $\mathbb{E}$, satisfies the differential subordination

$$1 + \frac{\gamma}{P(z)} - \alpha\frac{zP'(z)}{(P(z))^2} \prec 1 + \frac{\gamma}{q(z)} - \alpha\frac{zq'(z)}{(q(z))^2} \tag{2}$$

then $P(z) \prec q(z)$ and q is the best dominant.

Proof. Let us define the functions θ and ϕ as follows:

$$\theta(w) = 1 + \frac{\gamma}{w} \quad \textit{and} \quad \phi(w) = -\frac{\alpha}{w^2}$$

Clearly, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Now, define the function h as follows:

$$h(z) = 1 + \frac{\gamma}{q(z)} - \alpha\frac{zq'(z)}{(q(z))^2} \tag{3}$$

Differentiate (3) and simplifying a little, we get

$$\frac{zh'(z)}{Q_1(z)} = \frac{\gamma}{\alpha} + \frac{zQ_1'(z)}{Q_1(z)}$$

where $Q_1(z) = -\alpha\dfrac{zq'(z)}{(q(z))^2}$. In view of condition (1), Q_1 is starlike in $\mathbb{E}$ and

$$\Re\left(\frac{zh'(z)}{Q_1(z)}\right) > 0,\ z \in \mathbb{E}.$$

Thus conditions (ii) and (iii) of lemma 2.1 are satisfied. Hence $P(z) \prec q(z)$. This completes the proof of our theorem. □

Remark 3.1. *If $\Re\,\dfrac{\gamma}{\alpha} \geq 0$, then the condition (1) of Theorem 3.1 can be stated as*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - 2\frac{zq'(z)}{q(z)}\right) > 0.$$

Letting $\gamma = 0$ in Theorem 3.1, we obtain the following result.

Theorem 3.2. *Let q, $q(z) \neq 0$ in $\mathbb{E}$, be a univalent function satisfying the condition*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - 2\frac{zq'(z)}{q(z)}\right) > 0,\ z \in \mathbb{E}.$$

If an analytic function P, $P(z) \neq 0$ in $\mathbb{E}$, satisfies the differential subordination

$$1 - \alpha\frac{zP'(z)}{(P(z))^2} \prec 1 - \alpha\frac{zq'(z)}{(q(z))^2},\ z \in \mathbb{E},$$

then $P(z) \prec q(z)$ in $\mathbb{E}$ and q is the best dominant.

By selecting $P(z) = -\dfrac{zf'(z)}{f(z)}$ in Theorem 3.1, we obtain the following result.

Theorem 3.3. *Let $\alpha(\neq 0)$ and γ be complex numbers. Let q, $q(z) \neq 0$, $z \in \mathbb{E}$, be a univalent function satisfying the condition (1) of Theorem 3.1. If a meromorphic function $f \in \Sigma$, $-\dfrac{zf'(z)}{f(z)} \neq 0$, satisfies the differential subordination*

$$\frac{(1-\alpha)zf'(z)/f(z) + \alpha(1 + zf''(z)/f'(z)) - \gamma}{zf'(z)/f(z)} \prec 1 + \frac{\gamma}{q(z)} - \alpha\frac{zq'(z)}{(q(z))^2},\ z \in \mathbb{E},$$

then $-\dfrac{zf'(z)}{f(z)} \prec q(z)$ in $\mathbb{E}$ and q is the best dominant.

3.1 Applications to Univalent Functions

By selecting $q(z) = \dfrac{1 + Az}{1 + Bz}$ and $\gamma = 0$ in Theorem 3.3, we get the following criterion for starlikeness.

Corollary 3.4. *Let A and B be real numbers with $-1 \leq B < A \leq 1$. If a meromorphic function $f \in \Sigma, -\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential subordination*

$$\frac{(1-\alpha)zf'(z)/f(z) + \alpha(1 + zf''(z)/f'(z))}{zf'(z)/f(z)} \prec 1 - \alpha\frac{(A-B)z}{(1+Az)^2},$$

where α is non-zero complex number, then $-\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+Az}{1+Bz},\ z \in \mathbb{E}$.

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ and $\alpha = 1$ in Theorem 3.3, we obtain

Corollary 3.5. *Let γ be a complex number with $\Re\,\gamma \geq 0$ and A and B be real numbers satisfying $-1 \leq B < A \leq 1$. If $f \in \Sigma$ satisfies the differential subordination*

$$\frac{1-\gamma + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \gamma\left(\frac{1+Bz}{1+Az}\right) - \frac{(A-B)z}{(1+Az)^2}$$

in $\mathbb{E}$, then $-\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+Az}{1+Bz}$ in $\mathbb{E}$.

Taking $A = 1-2\alpha, B = -1, 0 \leq \alpha < 1$ in Corollary 3.5, we get the following result.

Corollary 3.6. *Let γ, $\Re\,\gamma \geq 0$ be a complex number. If $f \in \Sigma$ satisfies the differential subordination*

$$\frac{1-\gamma + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \gamma\frac{1-z}{1+(1-2\alpha)z} - \frac{2(1-\alpha)z}{[1+(1-2\alpha)z]^2}$$

in $\mathbb{E}$, then $f \in \mathcal{MS}^*(\alpha)$.

Writing $A = 0$ in Corollary 3.5, we obtain the following result:

Corollary 3.7. *Let $f \in \Sigma$ satisfy*

$$\left|\frac{1-\gamma + zf''(z)/f'(z)}{zf'(z)/f(z)} - (1+\gamma)\right| < (1+\gamma)|B|,\ z \in \mathbb{E},\ \gamma \geq 0,\ \ -1 \leq B < 0,$$

then $-\dfrac{zf'(z)}{f(z)} \prec \dfrac{1}{1+Bz}$ in $\mathbb{E}$.

The selection of $B = 0$ in Corollary 3.5, gives us the following:

Corollary 3.8. *Let $f \in \Sigma$ satisfy*

$$\frac{1-\gamma+zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1+\frac{\gamma}{1+Az}-\frac{Az}{(1+Az)^2},\ z \in \mathbb{E},\ \gamma \geq 0,\ 0 < A \leq 1,$$

then $\left|1+\dfrac{zf'(z)}{f(z)}\right| < A,\ z \in \mathbb{E}.$

Remark 3.2. *If we take* $q(z) = \dfrac{2\beta}{1+z}, 0 < \beta < 1,$ *it is easy to verify that it satisfies the condition (1). In addition, setting* $\alpha = 1$ *in Theorem 3.3, we obtain the following result.*

Corollary 3.9. *For* $0 < \beta < 1$ *and* $\gamma > -1$*, if* $f \in \Sigma$ *satisfies*

$$\left|\frac{1-\gamma+zf''(z)/f'(z)}{zf'(z)/f(z)} - \left(1+\frac{\gamma}{2\beta}\right)\right| < \frac{1+\gamma}{2\beta},\ z \in \mathbb{E},$$

then f is meromorphic starlike of order β*.*

Writing $\gamma = 1$ in Corollary 3.5, we obtain the following result:

Corollary 3.10. *If* $f \in \Sigma$ *satisfies the differential subordination*

$$\frac{f''(z)f(z)}{f'^2(z)} \prec 1+\frac{1+Bz}{1+Az}-\frac{(A-B)z}{(1+Az)^2},\ z \in \mathbb{E},\ \ -1 \leq B < A \leq 1,$$

then $-\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+Az}{1+Bz}$ *in* $\mathbb{E}$.

In particular, for $\gamma = 1$ in Corollary 3.7 , we obtain the following result:

Corollary 3.11. *Let* $f \in \Sigma$ *satisfy*

$$\left|\frac{f''(z)f(z)}{f'^2(z)} - 2\right| < 2|B|,\ z \in \mathbb{E},\ \ -1 \leq B < 0,$$

then $-\dfrac{zf'(z)}{f(z)} \prec \dfrac{1}{1+Bz},\ z \in \mathbb{E}.$

Taking $B = -\dfrac{1-\beta}{\beta},\ \dfrac{1}{2} \leq \beta < 1$ in above corollary, we get the following result.

Corollary 3.12. *Let* $f \in \Sigma$ *with* $f(z)f'(z) \neq 0$ *for* $0 < |z| < 1$ *and let* β *be a constant such that* $\dfrac{1}{2} \leq \beta < 1$*. If*

$$\left|\frac{f''(z)f(z)}{f'^2(z)} - 2\right| < 2\left(\frac{1-\beta}{\beta}\right),\ z \in \mathbb{E},$$

then $f \in \mathcal{MS}^*(\beta)$ *and* $-\dfrac{zf'(z)}{f(z)} \prec \dfrac{\beta}{\beta-(1-\beta)z},\ z \in \mathbb{E}.$

Replacing $B=-1$ in Corollary 3.11, we get the following result:

Corollary 3.13. *If $f\in\Sigma$ satisfies,*

$$\frac{f''(z)f(z)}{f'^2(z)}\prec 2(1+z),\ \ z\in\mathbb{E},$$

then $f\in\mathcal{MS}^*\left(\frac{1}{2}\right)$.

3.2 Applications to Multivalent Functions

In Theorem 3.2, if we take $P(z)=-\dfrac{1}{p(k+1)}\left(1+\dfrac{zf''(z)}{f'(z)}+k\dfrac{zf'(z)}{f(z)}\right)$ where $f\in\Sigma_p$, we obtain the following theorem:

Theorem 3.14. *If a function $f\in\Sigma_p,\ P(z)=-\dfrac{1}{p(k+1)}\left(1+\dfrac{zf''(z)}{f'(z)}+k\dfrac{zf'(z)}{f(z)}\right)\neq 0$ in $\mathbb{E}$ with $k+1>0$, satisfies the differential subordination*

$$p(k+1)\frac{z\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)'}{\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)^2}\prec-\frac{zq'(z)}{(q(z))^2},\ \ z\in\mathbb{E},$$

then

$$-\frac{1}{p(k+1)}\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)\prec q(z),$$

and q is the best dominant.

Setting $q(z)=\dfrac{1+z}{1-z}$ in Theorem 3.14, we get:

Corollary 3.15. *If a function $f\in\Sigma_p,-\dfrac{1}{p(k+1)}\left(1+\dfrac{zf''(z)}{f'(z)}+k\dfrac{zf'(z)}{f(z)}\right)\neq 0,\ z\in\mathbb{E},\ k+1>0$, satisfies the differential subordination*

$$p(k+1)\frac{z\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)'}{\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)^2}\prec-\frac{2z}{(1+z)^2},\ \ z\in\mathbb{E},$$

or equivalently

$$1+\frac{z\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)'}{\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)^2}\prec 1-\frac{1}{p(k+1)}\frac{2z}{(1+z)^2}=F_1(z),$$

then

$$-\frac{1}{p(k+1)}\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)\prec\frac{1+z}{1-z},\ \ z\in\mathbb{E}. \tag{4}$$

Remark 3.3. *We observe that the condition (4) is equivalent to*

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)>0,\ z\in\mathbb{E},\ k+1\geq 0,$$

which, in turn,implies that $f\in\mathcal{MS}_p^*(k)$*, a well known class of meromorphic multivalent functions showed that the functions in this class are p-valent meromorphic convex for* $-1<k\leq 0$ *and p-valent meromorphic starlike for* $k>0$.

Remark 3.4. *It can easily be seen that the function* F_1 *(given by Corollary 3.15) is a conformal mapping of the unit disc* $\mathbb{E}$ *with* $F_1(0)=1$ *and*

$$F_1(\mathbb{E})=\mathbb{C}\setminus\{w\in\mathbb{C}:1-\frac{1}{2p(k+1)}\leq\Re(w)<\infty,\ \Im(w)=0\}.$$

On writting $P(z)=-\frac{1}{p}\frac{zf'(z)}{f(z)},\ f\in\Sigma_p$ and $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 3.2, we get the following result.

Theorem 3.16. *Let A and B be real numbers satisfying* $-1\leq B<A\leq 1$*. If a function* $f\in\Sigma_p,\ -\frac{1}{p}\frac{zf'(z)}{f(z)}\neq 0,\ z\in\mathbb{E}$*, satisfies the differential subordination*

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\prec 1+\frac{(B-A)z}{p(1+Az)^2},\ z\in\mathbb{E},$$

then $-\frac{1}{p}\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz}$.

Selecting $A=0$ and $B=-1$ in Theorem 3.16, we get the following result.

Corollary 3.17. *If a function* $f\in\Sigma_p,\ -\frac{1}{p}\frac{zf'(z)}{f(z)}\neq 0,\ z\in\mathbb{E}$*, satisfies the inequality*

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}-1\right|<\frac{1}{p},$$

then

$$-\Re\left(\frac{zf'(z)}{f(z)}\right)>\frac{p}{2}.$$

By taking $A=1$ and $B=0$ in Theorem 3.16, we obtain the following result.

Corollary 3.18. *If a function* $f\in\Sigma_p,\ -\frac{1}{p}\frac{zf'(z)}{f(z)}\neq 0,\ z\in\mathbb{E}$*, satisfies the differential subordination*

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\prec 1-\frac{1}{p}\frac{z}{(1+z)^2}=F_2(z),$$

then

$$\left|\frac{1}{p}\frac{zf'(z)}{f(z)}+1\right|<1.$$

It can be observed that function F_2 is a conformal mapping of the unit disc $\mathbb{E}$ with $F_2(0)=1$ and

$$F_2(\mathbb{E})=\mathbb{C}\setminus\{w\in\mathbb{C}:1-\frac{1}{4p}\le\Re(w)<\infty,\ \Im(w)=0\}.$$

When we take $A=1$ and $B=-1$ in Theorem 3.16, we get the following result.

Corollary 3.19. *If a function $f\in\Sigma_p$, $-\frac{1}{p}\frac{zf'(z)}{f(z)}\neq 0$, $z\in\mathbb{E}$, satisfies the differential subordination*

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\prec 1-\frac{1}{p}\frac{2z}{(1+z)^2}=F_3(z),$$

then $f\in\mathcal{MS}_p^*$ i.e. f is a p-valent meromorphic starlike function.

Remark 3.5. *It is easy to verify that function F_3 is a conformal mapping of the unit disc $\mathbb{E}$ with $F_3(0)=1$ and*

$$F_3(\mathbb{E})=\mathbb{C}\setminus\{w\in\mathbb{C}:1-\frac{1}{2p}\le\Re(w)<\infty,\ \Im(w)=0\}.$$

If we take $q(z)=p\left(\frac{1+z}{1-z}\right)$ and $\alpha=1$ in Theorem 3.2, we obtain:

Corollary 3.20. *Let P be an analytic function in $\mathbb{E}$ with $P(0)=p$. Suppose P satisfies the differential subordination*

$$1-\frac{zP'(z)}{(P(z))^2}\prec 1-\frac{2z}{p(1+z)^2}=F_4(z),$$

then $-\Re(P(z))>0$, $z\in\mathbb{E}$.

Remark 3.6. *We observe that F_4 is a conformal mapping of the unit disc $\mathbb{E}$ with $F_4(0)=1$ and*

$$F_4(\mathbb{E})=\mathbb{C}\setminus\{w\in\mathbb{C}:-\frac{1}{2p}\le\Re(w)<\infty,\ \Im(w)=0\}.$$

4 Open Problem

In the present paper, we have obtained the results on meromorphic starlike functions in terms of differential subordination. One may explore the corresponding results in terms of superordination.

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