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Subclasses of univalent functions with positive coefficients defined by Salagean operator

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Abstract

Making use of Salagean operator, we introduce a new subclasses of univalent functions with positive coefficients. We obtain coefficient bounds, distortion inequalities, extreme points and convolution property are studied. Further, we discuss integral mean property and some neighborhoods results.

Keywords: *Univalent functions, Salagean operator, starlike functions, integral mean, neighborhood.*

1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disc $U = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$ and S denote the subclass of A that are univalent in U.

Salagean [13] introduced the following operator which is popularly know as the Salagean derivative operator as follows:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

and in general,

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \qquad (n \in N_0 = N \cup \{0\}\,,\ N = \{1, 2, 3...\})\,.$$

We easily find from (1) that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \qquad (f \in S;\ n \in N_0) \tag{2}$$

In 1999, Kanas and Wisniowaska [8], (see also [7]) studied the class of $\alpha-$uniformly convex analytic functions, denoted by $\alpha - UCV\ \ (0 \le \alpha < \infty)$ so that $f \in \alpha - UCV$, if and only if

$$Re\left\{1 + (z - \zeta)\frac{f''(z)}{f'(z)}\right\} \ge 0, \tag{3}$$

For $\varphi \in R$ and $\zeta = -\alpha z e^{i\varphi}$, the condition (3) can be written as

$$Re\left\{1 + \left(1 + \alpha e^{i\varphi}\right)\frac{zf''(z)}{f'(z)}\right\} \ge 0, \tag{4}$$

and $\alpha - UCV(\rho)$ denote the subclass of S, satisfying

$$Re\left\{1 + \left(1 + \alpha e^{i\varphi}\right)\frac{zf''(z)}{f'(z)}\right\} \ge \rho, \qquad (0 \le \rho < 1)\,. \tag{5}$$

Further, the class $\alpha - S^*(\rho)$ denote the subclass of S, satisfying

$$Re\left\{\left(1 + \alpha e^{i\varphi}\right)\frac{zf'(z)}{f(z)} - \alpha e^{i\varphi}\right\} \ge \rho, \qquad (0 \le \rho < 1) \tag{6}$$

Also, let V be the subclass of S consisting of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} |a_k|\, z^k. \tag{7}$$

Now, we will define a new subclass of S as follows:

Definition 1.1 *A function $f \in S$ is said to be in the class $\alpha - S_n^*(\gamma, \beta)$ if the following condition*

$$Re\left\{\left(1+\alpha e^{i\varphi}\right)\frac{(1-\beta)\,D^{n+1}f(z)+\beta D^{n+2}f(z)}{(1-\beta)\,D^{n}f(z)+\beta D^{n+1}f(z)}-\alpha e^{i\varphi}\right\}<\gamma, \tag{8}$$

is satisfied, where $1 < \gamma \leq \frac{4+\alpha}{3}, 0 \leq \beta \leq 1, \varphi \in R$ and $0 \leq \alpha < \infty$.

Also,

$$\alpha - PS_n^*(\gamma, \beta) = \alpha - S_n^*(\gamma, \beta) \cap V \tag{9}$$

Remark

(i) Putting $\beta = 0$, we obtain $\alpha - S_n^*(\gamma, 0) = \alpha - S_{p,n}^*(\gamma)$ and $\alpha - PS_n^*(\gamma, 0) = \alpha - PS_{p,n}^*(\gamma)$ which were studied by Dixit and Dixit [3];

(ii) Putting $\beta = 0$ and $n = 0$, we obtain $\alpha - S_0^*(\gamma, 0) = \alpha - S_p^*(\gamma)$ which was studied by Porwal and Dixit [11];

(iii) Putting $\beta = 1$ and $n = 0$, we obtain $\alpha - S_0^*(\gamma, 1) = \alpha - UCV^*(\gamma)$ which was studied by Porwal and Dixit [11];

(iv) Putting $\beta = 0$, $n = 0$ and $\alpha = 0$, we obtain $0 - S_0^*(\gamma, 0) = L(\gamma)$ and $0 - PS^*(\gamma, \beta) = U(\gamma)$ which were studied by Uralegaddi et al. [14];

(v) Putting $\beta = 1, n = 0$ and $\alpha = 0$, we obtain $0 - S_0^*(\gamma, 1) = M(\gamma)$ and $0 - PS_0^*(\gamma, 1) = V(\gamma)$ which were studied by Uralegaddi et al. [14];

(vi) Putting $\alpha = 0$, we obtain $0 - PS_n^*(\gamma, \beta) = A(n, \gamma, \beta)$ which was studied by Dixit et al. ([4] with $g(z) = \frac{z}{1-z}$);

(vii) Putting $\alpha = 0$ and $\beta = 0$,we obtain $0 - PS_n^*(\gamma, 0) = A^*(n, \gamma)$which was studied by Dixit and Chandra [2].

Several authors such as [7, 8, 9] studied the classes of $\alpha-$uniformly convex and starlike functions. In the present paper, using Salagean derivative operator, an attempt has been made to have unified study of mentioned classes of functions with positive coefficients. Further, we discuss integral mean property and some neighborhoods results.

2 Coefficient inequalities

Unless otherwise mentioned, we assume in the reminder of this paper that $0 \leq \alpha < \infty$, $1 < \gamma \leq \frac{4+\alpha}{3}$, $0 \leq \beta \leq 1$, $n \in N_0 = N \cup \{0\}$. The following theorems lay the foundation of our systematic study of the class $\alpha - PS_n^*(\gamma, \beta)$ defined in the preceding section.

Theorem 2.1 *Let* $f(z) \in S$ *be given by* (1) *be in* S *if*

$$\sum_{k=2}^{\infty} k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma] |a_k| \leq (\gamma - 1), \tag{10}$$

then $f \in \alpha - S_n^*(\gamma, \beta)$.

Proof. Suppose that (10) is true for $z \in U$. It suffices to show that

$$\left| \frac{(1+\alpha e^{i\varphi}) \left[\frac{(1-\beta)D^{n+1}f(z)+\beta D^{n+2}f(z)}{(1-\beta)D^n f(z)+\beta D^{n+1}f(z)}\right] - \alpha e^{i\varphi} - 1}{(1+\alpha e^{i\varphi}) \left[\frac{(1-\beta)D^{n+1}f(z)+\beta D^{n+2}f(z)}{(1-\beta)D^n f(z)+\beta D^{n+1}f(z)}\right] - \alpha e^{i\varphi} - (2\gamma - 1)} \right| < 1. \tag{11}$$

Then, L. H. S. of (11)

$$= \left| \frac{(1+\alpha e^{i\varphi}) [[(1-\beta) D^{n+1} f(z) + \beta D^{n+2} f(z)] - [(1-\beta) D^n f(z) + \beta D^{n+1} f(z)]]}{\begin{array}{c}(1+\alpha e^{i\varphi}) [(1-\beta) D^{n+1} f(z) + \beta D^{n+2} f(z)] - \\ [(1+\alpha e^{i\varphi}) + (2\gamma - 2)] [(1-\beta) D^n f(z) + \beta D^{n+1} f(z)]\end{array}} \right|$$

$$= \left| \frac{(1+\alpha e^{i\varphi}) \sum\limits_{k=2}^{\infty} k^n [1+\beta(k-1)] [k-1] a_k z^k}{\begin{array}{c}(1+\alpha e^{i\varphi}) \left[\sum\limits_{k=2}^{\infty} k^n [1+\beta(k-1)] [k-1] a_k z^k\right] \\ -2(\gamma-1) - 2(\gamma-1) \left[\sum\limits_{k=2}^{\infty} k^n [1+\beta(k-1)] a_k z^k\right]\end{array}} \right|.$$

The last assertion is bounded above by (1) if

$$\sum_{k=2}^{\infty} k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma] |a_k| \leq (\gamma - 1),$$

which completes the proof. ■

Theorem 2.2 *Let $f(z)$ be given by* (7), *then* $f(z) \in \alpha - PS_n^*(\gamma,\beta)$ *if and only if*

$$\sum_{k=2}^{\infty} k^n\left[1+\beta(k-1)\right]\left[k(1+\alpha)-\alpha-\gamma\right]|a_k| \le (\gamma-1). \tag{12}$$

The result is sharp.

Proof. The if part follows from Theorem 2.1. To prove the only if part, let $f \in \alpha - PS_n^*(\gamma,\beta)$, then by (9), we have

$$Re\left\{\left(1+\alpha e^{i\varphi}\right)\frac{(1-\beta)D^{n+1}f(z)+\beta D^{n+2}f(z)}{(1-\beta)D^n f(z)+\beta D^{n+1}f(z)}-\alpha e^{i\varphi}\right\}<\gamma.$$

which is equivalent to

$$Re\left\{\left(1+\alpha e^{i\varphi}\right)\frac{(1-\beta)\left(z+\sum\limits_{k=2}^{\infty}k^{n+1}a_kz^k\right)+\beta\left(z+\sum\limits_{k=2}^{\infty}k^{n+2}a_kz^k\right)}{(1-\beta)\left(z+\sum\limits_{k=2}^{\infty}k^{n}a_kz^k\right)+\beta\left(z+\sum\limits_{k=2}^{\infty}k^{n+1}a_kz^k\right)}-\alpha e^{i\varphi}\right\}<\gamma.$$

The above condition must hold for all values of z, $|z| = r < 1$, upon choosing the values of z on positive real axis, where $0 \le z = r < 1$ and

$$Re\left(-\alpha e^{i\varphi}\right) \ge -\left|\alpha e^{i\varphi}\right| = -\alpha,$$

the above inequality reduces to

$$\begin{aligned}
&(1+\alpha)\left[(1-\beta)\left(r+\sum_{k=2}^{\infty}k^{n+1}|a_k|r^k\right)+\beta\left(r+\sum_{k=2}^{\infty}k^{n+2}|a_k|r^k\right)\right]\\
&-\alpha\left[(1-\beta)\left(r+\sum_{k=2}^{\infty}k^{n}|a_k|r^k\right)+\beta\left(r+\sum_{k=2}^{\infty}k^{n+1}|a_k|r^k\right)\right]\\
\le\ &\gamma\left[(1-\beta)\left(r+\sum_{k=2}^{\infty}k^{n}|a_k|r^k\right)+\beta\left(r+\sum_{k=2}^{\infty}k^{n+1}|a_k|r^k\right)\right]
\end{aligned}$$

Letting $r \to 1^-$, we have

$$\sum_{k=2}^{\infty} k^n\left[1+\beta(k-1)\right]\left[k(1+\alpha)-\alpha-\gamma\right]|a_k| \le (\gamma-1).$$

The proof is completed.

We note that the assertion (12) of Theorem 2.2 is sharp, the extremal function being

$$f(z) = z+\frac{(\gamma-1)}{k^n\left[1+\beta(k-1)\right]\left[k(1+\alpha)-\alpha-\gamma\right]}z^k, \qquad (k \ge 2).$$

■

Remark

(i) Putting $\beta = 0$ in Theorem 2.2 we obtain the result obtained by Dixit and Dixit [3, *Theorem* 2.2];

(ii) Putting $\beta = 0$ and $n = 1$, we obtain the result obtained by Porwal and Dixit [11, *Theorem* 2.3];

(iii) Putting $\beta = 0$ and $n = 0$, we obtain the result obtained by Porwal and Dixit [11, *Theorem* 2.4];

(iv) Putting $\alpha = 0$, we obtain the result obtained by Dixit et al. $\left([11] \text{ with } g(z) = \frac{z}{1-z}\right)$;

(v) Putting $\alpha = \beta = n = 0$, we obtain the result obtained by Uralegaddi et al. [14].

Corollary 2.3 *Let the function* $f(z)$ *be defined by* (7) *belong to the class* $\alpha - PS_n^*(\gamma, \beta)$. *Then*

$$|a_k| \leq \frac{(\gamma - 1)}{k^n [1 + \beta(k-1)][k(1+\alpha) - \alpha - \gamma]}. \tag{13}$$

3 Distortion Bounds

In this section, we shall prove distortion theorems for the functions belonging to the class $\alpha - PS_n^*(\gamma, \beta)$.

Theorem 3.1 *Let the function* $f(z) \in \alpha - PS_n^*(\gamma, \beta)$ *then, for* $|z| = r < 1$, *we have*

$$|f(z)| \leq r + \frac{(\gamma - 1)}{2^n (1+\beta)[2 + \alpha - \gamma]} r^2 \tag{14}$$

and

$$|f(z)| \geq r - \frac{(\gamma - 1)}{2^n (1+\beta)[2 + \alpha - \gamma]} r^2, \tag{15}$$

with equality for

$$f(z) = z + \frac{(\gamma - 1)}{2^n (1+\beta)[2 + \alpha - \gamma]} z^2. \tag{16}$$

Proof. Since $f(z) \in \alpha - PS_n^*(\gamma, \beta)$, then Theorem 2.2 gives

$$2^n (1+\beta)[2 + \alpha - \gamma] \sum_{k=2}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} k^n [1 + \beta(k-1)][k(1+\alpha) - \alpha - \gamma] |a_k| \leq (\gamma - 1).$$

Thus, we have

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{(\gamma-1)}{2^n (1+\beta) [2+\alpha-\gamma]} z^2. \tag{17}$$

From (2) and (17) we obtain

$$\begin{aligned} |f(z)| &\leq |z| + |z|^2 \sum_{k=2}^{\infty} |a_k| \\ &\leq r + \frac{(\gamma-1)}{2^n (1+\beta) [2+\alpha-\gamma]} r^2. \end{aligned}$$

The proof of assertion (15) is similar, so we omit it. ■

Theorem 3.2 *Let the function $f(z) \in \alpha - PS_n^* (\gamma, \beta)$ then, for $|z| = r < 1$, we have*

$$|f'(z)| \leq 1 + \frac{(\gamma-1)}{2^{n-1} (1+\beta) [2+\alpha-\gamma]} r \tag{18}$$

and

$$|f'(z)| \geq 1 - \frac{(\gamma-1)}{2^{n-1} (1+\beta) [2+\alpha-\gamma]} r. \tag{19}$$

The equalities in (18) and (19) *are attained for the function $f(z)$ given by (*16*).*

Proof. We have

$$|f'(z)| \leq 1 + \sum_{k=2}^{\infty} k |a_k| \left|z^{k-1}\right| \leq 1 + r \sum_{k=2}^{\infty} k |a_k|.$$

Since $f(z) \in \alpha - PS_n^* (\gamma, \beta)$, we have

$$2^{n-1} (1+\beta) (2+\alpha-\gamma) \sum_{k=2}^{\infty} k |a_k| \leq \sum_{k=2}^{\infty} k^n [1+\beta(k-1)] [k(1+\alpha) - \alpha - \gamma] |a_k| \leq (\gamma-1).$$

Thus, we have

$$\sum_{k=2}^{\infty} k |a_k| \leq \frac{(\gamma-1)}{2^{n-1} (1+\beta) [2+\alpha-\gamma]},$$

hence

$$|f'(z)| \leq 1 + \frac{(\gamma-1)}{2^{n-1} (1+\beta) [2+\alpha-\gamma]} r.$$

The proof of assertion (19) is similar, so we omit it. ■

4 Extreme Points

Theorem 4.1 *Let* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{(\gamma - 1)}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]} z^k, \quad (k \geq 2), \tag{20}$$

then $f(z) \in \alpha - PS_n^*(\gamma, \beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z),$$

where $\lambda_k \geq 0$ *and* $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = z + \sum_{k=2}^{\infty} \lambda_k \frac{(\gamma - 1)}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]} z^k.$$

Then, from Theorem 2.2, we have

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]}{(\gamma - 1)} \lambda_k \frac{(\gamma - 1)}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]} \\ = & \sum_{k=2}^{\infty} \lambda_k = (1 - \lambda_1) < 1. \end{aligned}$$

Then $f(z) \in \alpha - PS_n^*(\gamma, \beta)$. Conversely, suppose that $f(z) \in \alpha - PS_n^*(\gamma, \beta)$, then, since

$$|a_k| \leq \frac{(\gamma - 1)}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]},$$

we may set

$$\lambda_k = \frac{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma] |a_k|}{(\gamma - 1)}$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k.$$

Thus clearly, from (20), we have

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z).$$

This completes the proof of the theorem. ■

Corollary 4.2 *The extreme points of the class* $\alpha - PS_n^*(\gamma,\beta)$ *are given by*

$$f_1(z) = z$$

and

$$f_k(z) = z + \frac{(\gamma-1)}{k^n[1+\beta(k-1)][k(1+\alpha)-\alpha-\gamma]} z^k, \quad (k \geq 2).$$

Theorem 4.3 *The class* $\alpha - PS_n^*(\gamma,\beta)$ *is convex set.*

Proof. Suppose that each of the functions $f_i(z), (i = 1,2)$ given by

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{k,i} z^k, \quad (a_{k,i} \geq 0)$$

is in the class $\alpha - PS_n^*(\gamma,\beta)$. It sufficient to show that the function $g(z)$ define by

$$g(z) = \eta f_1(z) + (1-\eta) f_2(z), \qquad (0 \leq \eta < 1)$$

is also in the class $\alpha - PS_n^*(\gamma,\beta)$. Since

$$\begin{aligned} g(z) &= \eta\left(z + \sum_{k=2}^{\infty} a_{k,1} z^k\right) + (1-\eta)\left(z + \sum_{k=2}^{\infty} a_{k,2} z^k\right) \\ &= z + \sum_{k=2}^{\infty}[\eta a_{k,1} + (1-\eta) a_{k,2}] z^k \end{aligned}$$

with the aid of Theorem 2.2, we have

$$\begin{aligned} &\sum_{k=2}^{\infty} k^n[1+\beta(k-1)][k(1+\alpha)-\alpha-\gamma][\eta a_{k,1} + (1-\eta) a_{k,2}] \\ = \ &\eta \sum_{k=2}^{\infty} k^n[1+\beta(k-1)][k(1+\alpha)-\alpha-\gamma] a_{k,1} \\ &+ (1-\eta)\sum_{k=2}^{\infty} k^n[1+\beta(k-1)][k(1+\alpha)-\alpha-\gamma] a_{k,2} \\ \leq \ &\eta(\gamma-1) + (1-\eta)(\gamma-1) = (\gamma-1). \end{aligned}$$

which completes the proof of the theorem. ■

5 Theorems involving Hadamard product

Let $f(z)$ be defined by (7) and let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (b_k \geq 0). \tag{21}$$

The Hadamard product of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{22}$$

The following result present an interesting property of Hadamard product.

Theorem 5.1 *Let $f_1(z), f_2(z), ..., f_p(z)$ be defined as follows*

$$f_s(z) = z + \sum_{k=2}^{\infty} a_{k,s} z^k, \quad (a_{k,s} \geq 0) \tag{23}$$

be in the class $\alpha - PS_n^*(\gamma_s, \beta)$, $(s = 1, 2, ..., p)$ and $(0 \leq \alpha < 1)$, then

$$f_1 * f_2 * ... * f_p \in \alpha - PS_n^*(\gamma, \beta), \tag{24}$$

where $\gamma = \max\{\gamma_s, \ s = 1, 2, ..., p\}$.

Proof. Since $f_s(z) \in \alpha - PS_n^*(\gamma_s, \beta)$, $(s = 1, 2, ..., p)$, by using Theorem 2.2 we have, for $\gamma = \max\{\gamma_s, \ s = 1, 2, ..., p\}$

$$\sum_{k=2}^{\infty} k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma] a_{k,s} \leq (\gamma_s - 1) \tag{25}$$

and

$$\sum_{k=2}^{\infty} |a_{k,s}| \leq \frac{(\gamma_s - 1)}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]}, \tag{26}$$

for each $s = 1, 2, ..., p$. Using (25) for any s and (26) for the rest, we have

$$\begin{aligned}
& \sum_{k=2}^{\infty} k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma_s] \prod_{s=1}^{p} a_{k,s} \\
\leq \ & \prod_{s=1}^{p} (\gamma_s - 1) \left[\frac{1}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]} \right]^{p-1} \\
\leq \ & \prod_{s=1}^{p} (\gamma_s - 1) \leq (\gamma - 1)^p \\
\leq \ & (\gamma - 1) \qquad \text{since } \gamma_s > 1 \text{ for } s = 1, 2, 3, ..., p
\end{aligned}$$

Consequently, we have the assertion (24) with the aid of Theorem 2.2. The proof of Theorem 5.1 is completed. ■

Theorem 5.2 *Let the function $f(z)$ be defined by* (7) *and $g(z)$ defined by* (21) *be in the classes $\alpha - PS_n^*(\gamma_1, \beta)$ and $\alpha - PS_n^*(\gamma_2, \beta)$ respectively. Then the Hadamard product*

$$(f * g)(z) \in \alpha - PS_n^*\left(\left((\gamma - 1)^2 + 1\right), \beta\right), \tag{27}$$

where $\gamma = \max\{\gamma_1, \gamma_2\}$.
Proof. *Since $f(z) \in \alpha - PS_n^*(\gamma_1, \beta)$ and $g(z) \in \alpha - PS_n^*(\gamma_2, \beta)$ in view of Theorem* 2.2 *we have*

$$\begin{aligned}
& \sum_{k=2}^{\infty} k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma] a_k b_k \\
\leq & \sum_{k=2}^{\infty} k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma] |a_k| \frac{(\gamma_2 - 1)}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]} \\
\leq & \frac{(\gamma_2 - 1)}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]} \sum_{k=2}^{\infty} k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma] |a_k| \\
\leq & \frac{(\gamma_2 - 1)(\gamma_1 - 1)}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]} \\
\leq & (\gamma - 1)^2 = \left(\gamma^2 - 2\gamma + 2\right) - 1.
\end{aligned}$$

Since $1 < \gamma \leq \frac{4+\alpha}{3}$ therefor $1 < \gamma^2 - 2\gamma + 2 \leq \frac{4+\alpha}{3}$. Hence by Theorem 2.2 the Hadamard product $(f * g)(z) \in \alpha - PS_n^*(\gamma^2 - 2\gamma + 2, \beta)$. ■

6 Integral Mean inequality

Definition 6.1 *For $f, g \in A$ we say that the function f is subordinate to g, if there exists a Schwarz function w, with $w(0) = 0$ and $|w(z)| < 1$; $z \in U$; such that $f(z) = g(w(z))$ for all $z \in U$. This subordination is usually denoted by $f(z) \prec g(z)$. It is well-known that, if the function g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.*

In 1925 Littlewood prove the following subordination theorem.

Theorem 6.2 [*10*] *If f and g are analytic in U with $f \prec g$, then*

$$\int_0^{2\pi} |f(z)|^\mu \, d\theta \leq \int_0^{2\pi} |g(z)|^\mu \, d\theta. \ \left(\mu > 0, z = re^{i\theta}, 0 < r < 1\right).$$

We will make use of Theorem 6.2 *to prove the following theorem:*

Theorem 6.3 *Let $f(z) \in \alpha - PS_n^*(\gamma, \beta)$ and $f_k(z)$ is defined by* (20) *. If there exist an analytic function $w(z)$ given by*

$$[w(z)]^{k-1} = \frac{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]}{(\gamma - 1)} \sum_{k=2}^{\infty} a_k z^{k-1},$$

then, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^\mu d\theta \leq \int_0^{2\pi} |f_k(z)|^\mu d\theta. \qquad (\ \mu > 0).$$

Proof. We must show that

$$\int_0^{2\pi} \left| 1 + \sum_{k=2}^{\infty} a_k z^{k-1} \right|^\mu d\theta \leq \int_0^{2\pi} \left| 1 + \frac{(\gamma - 1)}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]} z^{k-1} \right|^\mu d\theta.$$

By applying Littlewood's subordition theorem, it would suffice to show that

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 + \frac{(\gamma - 1)}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]} z^{k-1}.$$

By setting

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} = 1 + \frac{(\gamma - 1)}{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]} [w(z)]^{k-1},$$

we find that

$$[w(z)]^{k-1} = \frac{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]}{(\gamma - 1)} \sum_{k=2}^{\infty} a_k z^{k-1},$$

which readily yields $w(0) = 0$.

Furthermore using (10) we obtain

$$\begin{aligned} |w(z)|^{k-1} &\leq \left| \frac{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]}{(\gamma - 1)} \sum_{k=2}^{\infty} a_k z^{k-1} \right| \\ &\leq \frac{k^n [1 + \beta(k-1)] [k(1+\alpha) - \alpha - \gamma]}{(\gamma - 1)} \sum_{k=2}^{\infty} a_k \left| z^{k-1} \right| \\ &\leq \left| z^{k-1} \right| < 1. \end{aligned}$$

This completes the proof of the theorem. ■

7 Neighborhoods for the class $\alpha - PS^*_{n,m}(\gamma,\beta)$

Let $V(m)$ denote the subclass of V consisting of a function of the form

$$f(z) = z + \sum_{k=m+1}^{\infty} |a_k| z^k \tag{28}$$

and $\alpha - PS^*_{n,m}(\gamma,\beta)$ a subclass of $\alpha - PS^*_{n}(\gamma,\beta)$ which is consisting of the functions given by (28).

Following the earlier investigations by Goodman [6], Ruscheweyh [12], Altintas and Owa [1] and El-Ashwah [5], we define (m,δ)-neighborhood of $f(z) \in V(m)$ by

$$N_{m,\delta}(f) = \left\{ g : g \in V(m),\ g(z) = z + \sum_{k=m+1}^{\infty} b_k z^k \text{ and } \sum_{k=m+1}^{\infty} k|a_k - b_k| \leq \delta. \right\} \tag{29}$$

In particular, if

$$e(z) = z,$$

we immediately have

$$N_{m,\delta}(e) = \left\{ g : g \in V(m),\ g(z) = z + \sum_{k=m+1}^{\infty} b_k z^k \text{ and } \sum_{k=m+1}^{\infty} k|b_k| \leq \delta. \right\} \tag{30}$$

Lemma 7.1 *Let the function $f(z) \in V(m)$ be defined by (28). Then $f(z)$ is in the class $\alpha - PS^*_{n,m}(\gamma,\beta)$ if and only if*

$$\sum_{k=m+1}^{\infty} k^n [1 + \beta(k-1)][k(1+\alpha) - \alpha - \gamma] |a_k| \leq (\gamma - 1). \tag{31}$$

Theorem 7.2 *Let*

$$\delta = \frac{(\gamma - 1)}{(m+1)^{n-1}(1 + m\beta)[m(1+\alpha) - (\gamma - 1)]} \tag{32}$$

Then

$$\alpha - PS^*_{n,m}(\gamma,\beta) \subset N_{m,\delta}(e).$$

Proof. Let the function $f(z) \in \alpha - PS^*_{n,m}(\gamma,\beta)$. Then, in view of (31), we have

$$(m+1)^n (1 + m\beta)[m(1+\alpha) - (\gamma - 1)] \sum_{k=m+1}^{\infty} |a_k| \tag{33}$$

$$\leq \sum_{k=m+1}^{\infty} k^n [1 + \beta(k-1)][k(1+\alpha) - \alpha - \gamma] |a_k| \leq (\gamma - 1), \tag{34}$$

which readily yields

$$\sum_{k=m+1}^{\infty}|a_k| \leq \frac{(\gamma-1)}{(m+1)^n(1+m\beta)[m(1+\alpha)-(\gamma-1)]} \tag{35}$$

Making use of (31) again, in conjunction with (35) , we get

$$\begin{aligned}&(m+1)^n(1+m\beta)(1+\alpha)\sum_{k=m+1}^{\infty}k|a_k|\\ \leq\ &\gamma-1+(m+1)^n(1+\beta m)(\alpha+\gamma)\sum_{k=2}^{\infty}|a_k| \leq \frac{(\gamma-1)(1+\alpha)(m+1)}{[m(1+\alpha)-(\gamma-1)]}\end{aligned}$$

$$\sum_{k=m+1}^{\infty}ka_k \leq \frac{(\gamma-1)}{(m+1)^{n-1}(1+m\beta)[m(1+\alpha)-(\gamma-1)]}=\delta \tag{36}$$

The proof is completed. ■

We will determine the neighborhood for the class $\alpha-PS_{n,m}^{*(\rho)}(\gamma,\beta)$ which define as follows.

Definition 7.3 *A function $f(z)\in V(m)$ is said to be in the class $\alpha-PS_{n,m}^{*}(\gamma,\beta)$ if there exist a function $g(z)\in\alpha-PS_{n,m}^{*}(\gamma,\beta)$ such that*

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\rho \qquad (z\in U,\ 0\leq\rho<1). \tag{37}$$

Theorem 7.4 *If $g(z)\in\alpha-PS_{n,m}^{*(\rho)}(\gamma,\beta)$ and*

$$\rho=1-\frac{\delta(m+1)^n(1+m\beta)[m(1+\alpha)-(\gamma-1)]}{(m+1)^{n+1}(1+m\beta)[m(1+\alpha)-(\gamma-1)]-(m+1)(\gamma-1)}, \tag{38}$$

then

$$N_{m,\delta}(g)\subset\alpha-PS_{n,m}^{*}(\gamma,\beta) \tag{39}$$

where

$$\delta\leq(m+1)\left\{1-(\gamma-1)\{(m+1)^n(1+m\beta)[m(1+\alpha)-(\gamma-1)]\}^{-1}\right\}.$$

Proof. Suppose that $f(z)\in N_{m,\delta}(g)$. We find from (29) that

$$\sum_{k=m+1}^{\infty}k|a_k-b_k|\leq\delta, \tag{40}$$

which ready implies that

$$\sum_{k=m+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{m+1} \tag{41}$$

Next, since $g(z) \in \alpha - PS_{n,m}^{*}(\gamma, \beta)$, we have

$$\sum_{k=m+1}^{\infty} b_k \leq \frac{(\gamma - 1)}{(m+1)^n (1 + m\beta) [m(1+\alpha) - (\gamma - 1)]}, \tag{42}$$

so that

$$\begin{aligned} \left| \frac{f(z)}{g(z)} - 1 \right| &\leq \frac{\sum\limits_{k=m+1}^{\infty} |a_k - b_k|}{1 - \sum\limits_{k=m+1}^{\infty} b_k} && (43) \\ &\leq \frac{\delta}{m+1} \frac{(m+1)^n (1+m\beta) [m(1+\alpha) - (\gamma-1)]}{(m+1)^n (1+m\beta) [m(1+\alpha) - (\gamma-1)] - (\gamma-1)} \\ &= 1 - \rho. && (44) \end{aligned}$$

Thus by above definition $g(z) \in \alpha - PS_{n,m}^{*(\rho)}(\gamma, \beta)$ for ρ given by (38). ■

8 Open problem

In the present paper, some geometric properties have been discussed, the differential subordination results still open,e.g. factor sequence.

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