Int. J. Open Problems Complex Analysis, Vol. 12, No. 1, March 2020
ISSN 2074-2827; Copyright ©ICSRS Publication, 2020
www.i-csrs.org

On certain subclasses of multivalent analytic functions with higher order derivatives

H. E. Darwish, A.Y. Lashin, R.M. El-Ashwah and E. M. Madar

Department of Mathematics,Faculty of Science, Mansoura University
Mansoura 35516, Egypt
e-mail: Darwish333@yahoo.com
e-mail: aylashin@mans.edu.eg

Department of Mathematics, Faculty of Science, Damietta University
New Damietta 34517, Egypt
e-mail: r_elashwah@yahoo.com
e-mail: EntesarMadar@Gmail.com

Received 15 December 2019; Accepted 20 February 2020
(Communicated by Iqbal Jebril)

Abstract

The main object of the present paper is to derive some properties of certain general classes $\mathcal{M}_{p,q}(\alpha, A, B)$ ***and*** $\mathcal{N}_{p,q}(\alpha, A, B)$ ***of multivalent analytic functions with higher order derivatives in the open unit disk.***

Key Words: Analytic functions; p-Valently starlike functions; p-Valently convex functions; Differential subordination.

1 Introduction

Let $\mathcal{A}_p$ be the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, 3, ...\}) \tag{1}$$

which are analytic and p-valent in the open unit disk

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}.$$

If $f(z)$ and $g(z)$ are analytic in Δ, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \quad \text{in } \Delta \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta),$$

if there exists a Schwrz function $w(z)$, which (by definition) is analytic in Δ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta)$$

such that

$$f(z) = g(w(z)) \quad (z \in \Delta).$$

Furthermore, If the function $g(z)$ is univalent in Δ, then we have the following equivalence (cf., e.g., [8]; see also [[10], p.4]):

$$f(z) \prec g(z) \quad (z \in \Delta) \;\Leftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

Definition 1 *For fixed parameters* A, $B(-1 \leq B < A \leq 1)$ *and* $\alpha > 0$, *a function* $f(z) \in \mathcal{A}_p$ *is said to be in the class* $\mathcal{M}_{p,q}(\alpha, A, B)$, *if and only if*

$$(1-\alpha)\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} + \alpha\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}} \prec \frac{1+Az}{1+Bz}, \tag{2}$$

$$(p \in \mathbb{N}, q \in \mathbb{N}_0; = \mathbb{N} \cup \{0\}; p > q),$$

where

$$f^{(q)}(z) = \delta(p,q)z^{p-q} + \sum_{k=p+1}^{\infty} \delta(k,q)a_k z^{k-q}, \tag{3}$$

and

$$\delta(i,j) = \frac{i!}{(i-j)!} = \begin{cases} 1 & (j = 0) \\ i(i-1)\ldots(i-j+1) & (j \neq 0). \end{cases} \tag{4}$$

Definition 2 *For fixed parameters* A, $B(-1 \leq B < A \leq 1)$, *a function* $f(z) \in \mathcal{A}_p$ *is said to be in the class* $\mathcal{N}_{p,q}(\alpha, A, B)$, *if and only if*

$$\frac{1}{p-q}\left[(1-\alpha)\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + \alpha\left(1 + \frac{zf^{(q+2)}}{f^{(q+1)}(z)}\right)\right] \prec \frac{1+Az}{1+Bz}, \tag{5}$$

$(p \in \mathbb{N}, q \in \mathbb{N}_0; p > q)$.

Remark 1 *Let $p = 1, q = 0, A = 1$ and $B = -1$, the class $\mathcal{N}_{p,q}(\alpha, A, B)$ reduce to the class $\alpha-$convex function which introduce by Mocanu [11] and studied by ([12], [13]).*

Recently, many authors defined and investigated many subclasses defined by the higher order derivative (see [1] ,[2], [3], [6] and [19]).

The object of the present paper is to derive some properties for the general classes $\mathcal{M}_{p,q}(\alpha, A, B)$ and $\mathcal{N}_{p,q}(\alpha, A, B)$ by using the method of differential subordination.

2 Preliminaries

To prove our main results, we need the following lemmas.

Lemma 1 ([4], [8] and [10]). *Let $\phi(z)$ be analytic in Δ and $h(z)$ be analytic and convex (univalent) in Δ with $h(0) = \phi(0) = 1$. If*

$$\phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z) \quad (\text{Re}(\gamma) \geq 0; \quad \gamma \neq 0; \ z \in \Delta), \tag{6}$$

then

$$\phi(z) \prec \psi(z) = \gamma z^{-\gamma} \int_0^z t^{\gamma-1} h(t) dt \prec h(z) \quad (z \in \Delta),$$

and $\psi(z)$ is the best dominant of (6).

We denote by $P(\gamma)$ the class of functions $\varphi(z)$ given by

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + \cdots , \tag{7}$$

which are analytic in Δ and satisfy the following inequality:

$$\text{Re}(\varphi(z)) > \gamma \quad (0 \leq \gamma < 1, \ z \in \Delta).$$

Lemma 2 ([7]). *Let the function $\varphi(z)$ given by (7) be in the class $P = P(0)$. Then*

$$\text{Re}(\varphi(z)) \geq \frac{1 - |z|}{1 + |z|} \quad (z \in \Delta).$$

Lemma 3 ([16]).*If $\varphi_j \in P(\gamma_j)$ $(0 \leq \gamma_j < 1; \ j = 1, 2)$, then*

$$\varphi_1 * \varphi_2 \in P(\gamma_3) \quad \gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2).$$

The result is the best possible.

Lemma 4 *([9], [15]). If* $-1 \leq B < A \leq 1,\ \beta > 0,$ *and the complex number* γ *satisfy* $\mathrm{Re}(\gamma) \geq -\beta(1-A)/(1-B),$ *then the following differential equation:*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1+Az}{1+Bz} \quad (z \in \Delta)$$

has a univalent solution in Δ *given by*

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta \int\limits_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B} dt} - \frac{\gamma}{\beta}, & B \neq 0, \\ \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta \int\limits_0^z t^{\beta+\gamma-1}\exp(\beta At) dt} - \frac{\gamma}{\beta}, & B = 0. \end{cases} \tag{8}$$

If the function $\phi(z)$ *given by*

$$\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

is analytic in Δ *and satisfies the following subordination:*

$$\phi(z) + \frac{z\phi'(z)}{\beta(\phi(z)) + \gamma} \prec \frac{1+Az}{1+Bz} \quad (z \in \Delta). \tag{9}$$

Then

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \Delta)$$

and $q(z)$ *is the best dominant* (9).

Lemma 5 *(Wilken and Feng [18]). Let* μ *be a positive measure on* $[0,1]$*. Let* $g(z,t)$ *be a complex-valued defined on* $\Delta \times [0,1]$ *such that* $g(.,t)$ *is analytic in* Δ *for each* $t \in [0,1]$ *and that* $g(z,.)$ *is* $\mu-$*integrable on* $[0,1]$ *for all* $z \in \Delta$*. In addition, suppose that* $\mathrm{Re}\,\{g(z,t)\} > 0,\ g(-r,t)$ *is real and*

$$\mathrm{Re}\left(\frac{1}{g(z,t)}\right) \geq \frac{1}{g(-r,t)} \quad \text{for } |z| \leq r < 1 \text{ and } t \in [0,1].$$

If the function $G(z)$ *is defined by*

$$G(z) := \int\limits_0^1 g(z,t)d\mu(t),$$

then

$$\mathrm{Re}\left(\frac{1}{G(z)}\right) \geq \frac{1}{G(-r)}.$$

For real or complex numbers a, b and $c(c \neq 0, -1, -2, ...)$, the Gauss hypergeometric function is defined by

$$_2F_1(a, b; c; z) = 1 + \frac{a.b}{1.c} z + \frac{a(a+1).b(b+1)}{2!c(c+1)} z^2 + \cdots .$$

We note that the series converges absolutely for $z \in \Delta$ and hence represents an analytic function in the unite disk Δ (see, for details, [[17], Chapter 14]).

Each of the identities (asserted by Lemma 6 below) is fairly well known (cf., e.g., [[17], Chapter 14]).

Lemma 6 *For real or complex numbers a, b and $c(c \neq 0, -1, -2, ...)$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \, {}_2F_1(a, b; c; z) \quad (\mathrm{Re}(c) > \mathrm{Re}(b) > 0); \tag{10}$$

$$_2F_1(a, b; c; z) = (1-z)^{-a} \, {}_2F_1(a, c-b; c; \frac{z}{z-1}); \tag{11}$$

$$_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z); \tag{12}$$

and

$$_2F_1(a, b; \frac{a+b+1}{2}; \frac{1}{2}) = \frac{\sqrt{\pi}\ \Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}. \tag{13}$$

3 Main Results

Theorem 7 *If $f \in \mathcal{M}_{p,q}(\alpha, A, B)$, then*

$$\frac{1}{\delta(p,q)} \frac{f^{(q)}(z)}{z^{(p-q)}} \prec Q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \Delta), \tag{14}$$

where the function $Q(z)$ is given by

$$Q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1} \, {}_2F_1(1, 1; \frac{p-q}{\alpha} + 1; \frac{Bz}{Bz+1}) & (B \neq 0) \\ 1 + \frac{p-q}{(p-q)+\alpha} Az & (B = 0), \end{cases}$$

is the best dominant of (14). Furthermore,

$$\mathrm{Re}\left(\frac{1}{\delta(p,q)} \frac{f^{(q)}(z)}{z^{(p-q)}}\right) > \rho \quad (z \in \Delta), \tag{15}$$

where

$$\rho(p, q, \alpha, A, B) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1} \, {}_2F_1(1, 1; \frac{p-q}{\alpha} + 1; \frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{p-q}{(p-q)+\alpha} A & (B = 0). \end{cases}$$

The result is the best possible.

Proof. Setting

$$\phi(z)=\frac{1}{\delta(p,q)}\frac{f^{(q)}(z)}{z^{(p-q)}}\quad (z\in\Delta). \tag{16}$$

Then the function $\phi(z)$ is analytic in Δ with $\phi(0)=1$. Differentiating (16), we get

$$\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}}=\phi(z)+\frac{z\phi'(z)}{p-q}$$

then

$$\begin{aligned}&(1-\alpha)\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}+\alpha\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}}\\ =\;&\phi(z)+\left(\frac{\alpha}{p-q}\right)z\phi'(z)\prec\frac{1+Az}{1+Bz}\quad (z\in\Delta)\;.\end{aligned} \tag{17}$$

Now, by using Lemma 1 for $\gamma=\frac{p-q}{\alpha}$, we deduce that

$$\phi(z)\prec Q(z)=\frac{p-q}{\alpha}z^{-\frac{p-q}{\alpha}}\int_0^z t^{\frac{p-q}{\alpha}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

$$=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1(1,1;\frac{p-q}{\alpha}+1;\frac{Bz}{Bz+1}) & (B\neq 0)\\ 1+\frac{p-q}{(p-q)+\alpha}Az & (B=0),\end{cases}$$

by change of variables followed by use (11) (with $a=1$, $b=\frac{p-q}{\alpha}$ and $c=b+1$). This proves the assertion (14) of Theorem 7. Next, to prove the assertion (15) of Theorem 7, it suffices to show that

$$\inf_{|z|<1}\{\mathrm{Re}(Q(z))\}=Q(-1).$$

Indeed, for $|z|\leq r<1$,

$$\mathrm{Re}\left(\frac{1+Az}{1+Bz}\right)\geq\frac{1-Ar}{1-Br}.$$

Setting

$$G(s,z)=\frac{1+Asz}{1+Bsz}$$

and

$$d\mu(s)=\frac{p-q}{\alpha}s^{\frac{p-q}{\alpha}-1}ds\quad (0\leq s\leq 1),$$

which is a positive measure on $[0,1]$, we get

$$Q(z)=\int_0^1 G(s,z)d\mu(s),$$

so that

$$\mathrm{Re}(Q(z)) \geq \int_0^1 \frac{1-Asr}{1-Bsr} d\mu(s) = Q(-r) \quad (\,|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (18). The result in (15) is best possible as function $Q(z)$ is the best dominant of (14). ■

By putting $q = 0$ in Theorem 7, we deduce the following consequence.

Corollary 8 *Let $\alpha > 0$ and let the function $f(z) \in \mathcal{A}_p$ satisfy the following inequality:*

$$(1-\alpha)\frac{f(z)}{z^p} + \alpha\frac{f'(z)}{pz^{p-1}} \prec \frac{1+Az}{1+Bz} \quad (z \in \Delta). \tag{18}$$

Then

$$\frac{f(z)}{z^p} \prec Q(z) \quad (z \in \Delta),$$

where the function $Q(z)$ is given by

$$Q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1} \; {}_2F_1(1,1;\frac{p}{\alpha}+1;\frac{Bz}{Bz+1}) & (B \neq 0) \\ 1 + \frac{p}{p+\alpha}Az & (B = 0), \end{cases}$$

is the best dominant of (14). Furthermore,

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) > \rho \qquad (z \in \Delta),$$

where

$$\rho(p,\alpha,A,B) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1} \; {}_2F_1(1,1;\frac{p}{\alpha}+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{p}{p+\alpha}A & (B = 0). \end{cases}$$

The result is the best possible.

By putting $A = 1-2\eta$ and $B = -1$ in Corollary 8, we deduce the following consequence.

Corollary 9 [*14*] *Let $\alpha > 0$ and let the function $f(z) \in \mathcal{A}_p$ satisfy the following inequality:*

$$(1-\alpha)\frac{f(z)}{z^p} + \alpha\frac{f'(z)}{pz^{p-1}} \prec \frac{1+(1-2\eta)z}{1-z} \quad (z \in \Delta).$$

Then

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) > \eta + (1-\eta)\left[{}_2F_1(1,1;\frac{p}{\alpha}+1;\frac{1}{2}) \; - 1\right] \qquad (z \in \Delta).$$

The result is the best possible.

For a function $f(z) \in \mathcal{A}_p$, the generalized Bernardi-Libera-Livingston integral operator $F_{\mu,p} : \mathcal{A}_p \to \mathcal{A}_p$ is defined by (cf., eg., [5])

$$\begin{aligned} F_{\mu,p}f(z) &= \frac{\mu+p}{z^{\mu}}\int_0^z t^{\mu-1}f(t)dt \\ &= \left(z^p + \sum_{k=p+1}^{\infty}\left(\frac{\mu+p}{\mu+k}\right)z^k\right) \star f(z) \quad (\mu > -p;\ z \in \Delta). \quad (19) \end{aligned}$$

It follows from (3) and (19) that

$$z(F_{\mu,p}f(z))^{q+1} = (\mu+p)f^q(z) - (\mu+q)(F_{\mu,p}f(z))^q. \quad (20)$$

Theorem 10 *If $\mu > -(p-q)$. Let $f(z) \in \mathcal{M}_{p,q}(\alpha, A, B)$, for some α, $\alpha > 0$, then the function $F_{\mu,p}f(z)$ defined by (19) satisfies*

$$(1-\alpha)\frac{(F_{\mu,p}f(z))^{(q)}}{\delta(p,q)z^{p-q}} + \alpha\frac{(F_{\mu,p}f(z))^{q+1}}{\delta(p,q+1)z^{p-q-1}} \prec \frac{1+Az}{1+Bz} \qquad (z \in \Delta)\ ,$$

and

$$\frac{(F_{\mu,p}f(z))^{(q)}}{\delta(p,q)z^{p-q}} \prec Q'(z) \prec \frac{1+Az}{1+Bz} \quad (21)$$

where

$$Q'(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1(1,1;(\mu+p)+1;\frac{Bz}{Bz+1}) & (B \neq 0) \\ 1 + \frac{(\mu+p)}{(\mu+p)+1}Az & (B = 0), \end{cases}$$

is the best dominant of (21).

Furthermore,

$$\mathrm{Re}\left(\frac{(F_{\mu,p}f(z))^{(q)}}{\delta(p,q)z^{p-q}}\right) > \sigma \qquad (z \in \Delta), \quad (22)$$

where

$$\sigma(p,q,\alpha,A,B) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1(1,1;(\mu+p)+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{\mu+p}{(\mu+p)+1}A & (B = 0). \end{cases}$$

The result is the best possible.

Proof. It is clear that the function $F_{\mu,p}f(z)$ in $\mathcal{A}_p$. Differentiating both sides of the equality

$$F_{\mu,p}f(z) = \frac{(\mu+p)}{z^{\mu}}\int_0^z t^{\mu-1}f(t)dt. \quad (23)$$

From (20), we have

$$z(F_{\mu,p}f(z))^{q+1} + (\mu+q)(F_{\mu,p}f(z))^q = (\mu+p)f^q(z). \quad (24)$$

Letting

$$\begin{aligned}\phi(z) &= (1-\alpha)\frac{(F_{\mu,p}f(z))^{(q)}}{\delta(p,q)z^{p-q}} + \frac{\alpha}{p-q}\frac{z\left((F_{\mu,p}f(z))^{(q)}\right)'}{\delta(p,q)z^{p-q}} \\ &= 1 + b_1 z + b_2 z^2 + \cdots ,\end{aligned}$$

then (24) becomes

$$\phi(z) + \frac{z\phi'(z)}{(\mu+p)} = (1-\alpha)\frac{f^q(z)}{\delta(p,q)z^{p-q}} + \frac{\alpha}{p-q}\frac{f^{q+1}(z)}{\delta(p,q)z^{p-q-1}}. \tag{25}$$

It follows from (25) that

$$\phi(z) + \frac{z\phi'(z)}{\mu+p} \prec \frac{1+Az}{1+Bz} \qquad (z \in \Delta),$$

using the method of the proof of Theorem 7, we can obtain the assertion of Theorem 10. ■

Theorem 11 *Let* $-1 \leq B_j \leq 1 (j = 1, 2)$. *If each of the functions* $f_j(z) \in \mathcal{A}_p$ *satisfies the following subordination condition*

$$(1-\alpha)\frac{f_j^{(q)}(z)}{\delta(p,q)z^{p-q}} + \alpha\frac{f_j^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}} \prec \frac{1+A_j z}{1+B_j z} \qquad (j = 1, 2; z \in \Delta)\ . \tag{26}$$

If $f(z) \in \mathcal{A}_p$ *is defined by*

$$\frac{f^{(q)}(z)}{\delta(p,q)} = \frac{f_1^{(q)}(z)}{\delta(p,q)} * \frac{f_2^{(q)}(z)}{\delta(p,q)} \tag{27}$$

then

$$\mathrm{Re}\left\{(1-\alpha)\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} + \alpha\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}}\right\} > \gamma, \tag{28}$$

where

$$\gamma = 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}\,{}_2F_1\left(1, 1; \frac{p-q}{\alpha}+1; \frac{1}{2}\right)\ \right].$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. Suppose that each of the functions $f_j(z) \in \mathcal{A}_p$ $(j = 1, 2)$ satisfies the condition (26). Then by letting

$$\varphi_j(z) = (1-\alpha)\frac{f_j^{(q)}(z)}{\delta(p,q)z^{p-q}} + \frac{\alpha}{p-q}\frac{z(f_j^{(q)}(z))'}{\delta(p,q)z^{p-q}} \quad (j = 1, 2)\ , \tag{29}$$

we have

$$\varphi_j(z) \in P(\gamma_j) \quad \left(\gamma_j = \frac{1-A_j}{1-B_j}; \quad j = 1,2\right).$$

From (29), we have

$$\frac{f_j^{(q)}(z)}{\delta(p,q)} = \frac{p-q}{\alpha} z^{-(p-q)(\frac{1}{\alpha}-1)} \int\limits_0^z t^{\frac{p-q}{\alpha}-1} \varphi_j(t)dt \quad (j = 1,2). \tag{30}$$

Now if $f(z) \in \mathcal{A}_p$ is defined by (27), we fined from (30) that

$$\begin{aligned} \frac{f^{(q)}(z)}{\delta(p,q)} &= \frac{f_1^{(q)}(z)}{\delta(p,q)} * \frac{f_2^{(q)}(z)}{\delta(p,q)} \\ &= \left(\frac{p-q}{\alpha} z^{-(p-q)(\frac{1}{\alpha}-1)} \int\limits_0^z t^{\frac{p-q}{\alpha}-1} \varphi_1(t)dt\right) \\ &\quad * \left(\frac{p-q}{\alpha} z^{-(p-q)(\frac{1}{\alpha}-1)} \int\limits_0^z t^{\frac{p-q}{\alpha}-1} \varphi_2(t)dt\right) \\ &= \frac{p-q}{\alpha} z^{-(p-q)(\frac{1}{\alpha}-1)} \int\limits_0^z t^{\frac{p-q}{\alpha}-1} \varphi_0(t)dt \end{aligned} \tag{31}$$

where

$$\varphi_0(t) = \frac{p-q}{\alpha} z^{-(p-q)(\frac{1}{\alpha}-1)} \int\limits_0^z t^{\frac{p-q}{\alpha}-1} (\varphi_1 * \varphi_2)(t)dt. \tag{32}$$

Since $\varphi_1(z) \in P(\gamma_j)$ and $\varphi_2(z) \in P(\gamma_2)$, it follows from Lemma 3 that

$$(\varphi_1 * \varphi_2)(z) \in P(\gamma_3) \quad (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)). \tag{33}$$

According to Lemma 2, we have

$$\mathrm{Re}(\varphi_1 * \varphi_2)(z) > \gamma_3 + (1-\gamma_3)\frac{1-|z|}{1+|z|} = (2\gamma_3 - 1) + \frac{2(1-\gamma_3)}{1+|z|}. \tag{34}$$

Now by using (34) in (32), we get

$$\begin{aligned} &\mathrm{Re}\left\{(1-\alpha)\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} + \frac{\alpha}{p-q}\frac{z(f^{(q)}(z))'}{\delta(p,q)z^{p-q}}\right\} \\ = \;& \mathrm{Re}\left\{\varphi_0(z)\right\} \\ = \;& \frac{p-q}{\alpha}\int\limits_0^1 u^{\frac{p-q}{\alpha}-1}\,\mathrm{Re}(\varphi_1 * \varphi_2)(uz)du \\ > \;& \frac{p-q}{\alpha}\int\limits_0^1 u^{\frac{p-q}{\alpha}-1}\left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u}\right)du \\ = \;& 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{p-q}{\alpha} z^{-(\frac{p-q}{\alpha})}\int\limits_0^1 u^{\frac{p-q}{\alpha}-1}(1+u)^{-1}du\right] \\ = \;& 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}{}_2F_1\left(1,1;\frac{p-q}{\alpha}+1;\frac{1}{2}\right)\right] \\ = \;& \gamma \quad (z \in \Delta) \end{aligned}$$

which completes the proof of the assertion (28).

When $B_1 = B_2 = -1$, we consider the functions $f_j^{(q)}(z) \in \mathcal{A}_p$ $(j = 1, 2)$, are defined by

$$f_j^{(q)}(z) = \frac{p-q}{\alpha} z^{-(p-q)(\frac{1}{\alpha}-1)} \int_0^z t^{\frac{p-q}{\alpha}-1} \left(\frac{1+A_j t}{1-t} \right) dt \quad (j = 1, 2),$$

for which we have

$$\varphi_j(z) = (1-\alpha)\frac{f_j^{(q)}(z)}{\delta(p,q)z^{p-q}} + \frac{\alpha}{p-q}\frac{z(f_j^{(q)}(z))'}{\delta(p,q)z^{p-q}} = \frac{1+A_j t}{1-t} \ (j = 1, 2) \ ,$$

and

$$(\varphi_1 * \varphi_2)(z) = 1 - \frac{(1+A_1)(1+A_2)}{1-z}.$$

Hence, for $f(z) \in \mathcal{A}_p$ given by (27) , we obtain

$$\left\{ (1-\alpha)\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} + \frac{\alpha}{p-q}\frac{z(f^{(q)}(z))'}{\delta(p,q)z^{p-q}} \right\}$$

$$\begin{aligned}
&= \varphi_0(z) \\
&= \frac{p-q}{\alpha} \int_0^1 u^{\frac{p-q}{\alpha}-1} \left(1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-uz} \right) du \\
&= 1 - (1+A_1)(1+A_2) + (1+A_1)(1+A_2)(1-z)^{-1} {}_2F_1\left(1, 1; \frac{p-q}{\alpha}+1; \frac{z}{z-1}\right) \\
&\to 1 - (1+A_1)(1+A_2) + \frac{1}{2}(1+A_1)(1+A_2) {}_2F_1\left(1, 1; \frac{p-q}{\alpha}+1; \frac{1}{2}\right)
\end{aligned}$$

as $z \to -1$,

which evidently completes the proof of Theorem 11. ■

Theorem 12 *If $f \in \mathcal{N}_{p,q}(\alpha, A, B)$, then*

$$\frac{1}{p-q}\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \prec \frac{\alpha}{p-q}\frac{1}{\widetilde{Q}(z)} = \tilde{O}(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \Delta), \qquad (35)$$

where

$$\widetilde{Q}(z) = \begin{cases} \int_0^1 t^{(\frac{p-q}{\alpha}-1)} \left(\frac{1+Btz}{1+Bz}\right)^{\frac{p-q}{\alpha}(A-B)/B} dt, & B \neq 0 \\ \int_0^1 t^{(\frac{p-q}{\alpha}-1)} \exp(\frac{p-q}{\alpha}(t-1)Az)dt, & B = 0 \end{cases} \qquad (36)$$

and $\tilde{O}(z)$ is best dominant of (35). Furthermore, if

$$A \leq -\frac{\alpha}{p-q}B \quad \text{with} \quad -1 \leq B < 0,$$

then

$$\mathrm{Re}\left(\frac{1}{p-q}\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right) > \widetilde{\rho} \quad (z \in \Delta),$$

where

$$\widetilde{\rho}(p,q,A,B) = \frac{1}{{}_2F_1(1, (\frac{p-q}{\alpha}(B-A)/B); (\frac{p-q}{\alpha}+1); B/(B-1))}.$$

The result is best possible.

Proof. Let $f \in \mathcal{N}_{p,q}(\alpha, A, B)$. Let us put

$$g(z) = \left(\frac{f^{(q)}(z)}{\delta(p,q)}\right)^{\frac{1}{p-q}} \tag{37}$$

and

$$r_1 := \sup\{r : g(z) \neq 0 \ \ (0 < |z| < r < 1)\}.$$

Then $g(z)$ is single-valued and analytic in $|z| < r_1$. Carrying out logarithmic differentiation in (37), it follows that the function $\varphi(z)$ given by

$$\varphi(z) = \frac{zg'(z)}{g(z)} = \frac{1}{p-q}\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \tag{38}$$

is analytic in $|z| < r_1$ and $\varphi(0) = 1$. Differentiating logarithmically (38), we get

$$\begin{aligned} &\frac{1}{p-q}\left[(1-\alpha)\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + \alpha\left(1 + \frac{zf^{(q+2)}}{f^{(q+1)}(z)}\right)\right] \\ = \ &\varphi(z) + \frac{\alpha z\varphi'(z)}{(p-q)\varphi(z)} \prec \frac{1+Az}{1+Bz} \quad (|z| < r_1). \end{aligned} \tag{39}$$

Thus, by using Lemma 4, we find that

$$\varphi(z) \prec \tilde{O}(z) \prec \frac{1+Az}{1+Bz} \quad (|z| < r_1), \tag{40}$$

where $\tilde{O}(z)$ is the best dominant of (36) and is given by (8) with

$$\beta = \frac{p-q}{\alpha} \text{ and } \gamma = 0.$$

Since, for $-1 \leq B < A \leq 1$, it easy to see that

$$\operatorname{Re}\left(\frac{1+Az}{1+Bz}\right) > 0 \quad (|z| < r_1),$$

by (40), we have

$$\operatorname{Re}(\varphi(z)) > 0 \ .$$

Now (38) shows that $g(z)$ is starlike (univalent) in $|z| < r_1$. Thus it is not possible that $g(z)$ vanishes on $|z| = r_1$ if $r_1 < 1$. So we conclude that $r_1 = 1$, and hence $\varphi(z)$ is analytic in Δ. Hence (40) implies that

$$\varphi(z) \prec \tilde{O}(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \Delta).$$

This proves the assertion (35) of Theorem 12.

Next, we show that

$$\inf_{|z|<1} \left\{ \operatorname{Re}(\tilde{O}(z)) \right\} = \tilde{O}(-1). \tag{41}$$

Indeed, if we set

$$a = \frac{p-q}{\alpha}(B-A)/B, \ b = \frac{p-q}{\alpha} \ \text{ and } \ c = \frac{p-q}{\alpha} + 1,$$

then $c > b > 0$. From (36), by using (10) to (11), we see that, for $B \neq 0$,

$$\begin{aligned} \widetilde{Q}(z) &= (1+Bz)^a \int_0^1 t^{b-1}(1+Btz)^{-a} dt \\ &= \frac{1}{b} \, {}_2F_1\left(1, a; c; \frac{Bz}{Bz+1}\right). \end{aligned} \tag{42}$$

To prove (41), we need show that

$$\operatorname{Re}\left\{\frac{1}{\widetilde{Q}(z)}\right\} \geq \frac{1}{\widetilde{Q}(-1)} \quad (z \in \Delta).$$

Since

$$A < -\frac{\alpha}{p-q} B \quad \text{with} \quad -1 \leq B < 0,$$

implies that $c > a > 0$, by using Lemma (5), we find from (42) that

$$\widetilde{Q}(z) = \int_0^1 g(z,t) d\mu(t),$$

where

$$g(z,t) = \frac{1+Bz}{1+(1-t)Bz} \quad (0 \leq t \leq 1)$$

and

$$d\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^{a-1}(1-t)^{c-a-1}dt,$$

which is a positive measure on $[0,1]$.

For $-1 \leq B < 0$, it may be noted that $\operatorname{Re}\{g(z,t)\} > 0$, $g(-r,t)$ is real for $0 \leq |z| \leq r < 1$ and $t \in [0,1]$. Therefore, by using Lemma 5, we have

$$\operatorname{Re}\left\{\frac{1}{\widetilde{Q}(z)}\right\} \geq \frac{1}{\widetilde{Q}(-1)} \ (\ |z| \leq r < 1),$$

which, upon letting $r \to 1-$, yields

$$\operatorname{Re}\left\{\frac{1}{\widetilde{Q}(z)}\right\} \geq \frac{1}{\widetilde{Q}(-1)}.$$

Further, by taking

$$A \to \left(-\frac{\alpha}{p-q}B\right)+$$

for the case

$$A = -\frac{\alpha}{p-q}B \ ,$$

and using (35) we get

$$\widetilde{\rho}(p,q,A,B) = \frac{1}{{}_2F_1(1,(\frac{p-q}{\alpha}(B-A)/B);(\frac{p-q}{\alpha}+1);B/(B-1))}.$$

The result is best possible. ■

By putting $q = 0$ in Theorem 12, we deduce the following consequence.

Corollary 13 *Let $\alpha > 0$ and let the function $f(z) \in \mathcal{A}_p$ satisfy the following inequality:*

$$\frac{1}{p}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} \prec \frac{1+Az}{1+Bz}.$$

Then

$$\frac{1}{p}\frac{zf'(z)}{f(z)} \prec \frac{\alpha}{p}\frac{1}{\bar{Q}(z)} = \tilde{O}(z) \ \prec \frac{1+Az}{1+Bz} \quad (z \in \Delta),$$

where

$$\bar{Q}(z) = \begin{cases} \int\limits_0^1 t^{(\frac{p}{\alpha}-1)} \left(\frac{1+Btz}{1+Bz}\right)^{\frac{p}{\alpha}(A-B)/B} dt, & B \neq 0 \\ \int\limits_0^1 t^{(\frac{p}{\alpha}-1)} \exp(\frac{p}{\alpha}(t-1)Az)dt, & B = 0 \end{cases}$$

and $\tilde{O}(z)$ is best dominant of (35). Furthermore, if

$$A \leq -\frac{\alpha}{p}B \quad \textit{with} \quad -1 \leq B < 0,$$

then

$$\mathrm{Re}\left(\frac{1}{p}\frac{zf'(z)}{f(z)}\right) > \tilde{\rho}$$

where

$$\bar{\rho}(p, A, B) = \frac{1}{{}_2F_1(1, (\frac{p}{\alpha}(B-A)/B); (\frac{p}{\alpha}+1); B/(B-1))}.$$

The result is best possible.

By putting $A = 1-2\eta$ $(0 < \eta < 1)$ and $B = -1$ in Corollary 13, we deduce the following consequence.

Corollary 14 *Let $\alpha > 0$ and let the function $f(z) \in \mathcal{A}_p$ satisfy the following inequality:*

$$\frac{1}{p}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f(z)}\right)\right\} \prec \frac{1+(1-2\eta)z}{1-z} \quad (z \in \Delta).$$

Then

$$\mathrm{Re}\left(\frac{1}{p}\frac{zf'(z)}{f(z)}\right) > \left[{}_2F_1(1, \frac{2p}{\alpha}(1-\eta); \frac{p}{\alpha}+1; \frac{1}{2})\right]^{-1} \quad (z \in \Delta).$$

The result is the best possible.

4 Inclusion Properties Involving the Higher Order Derivatives

Theorem 15 *Let $0 \leq \alpha_1 < \alpha_2$. Then*

$$\mathcal{M}_{p,q}(\alpha_2, A, B) \subset \mathcal{M}_{p,q}(\alpha_1, A, B).$$

Proof. Let $0 \leq \alpha_1 < \alpha_2$ and suppose that

$$\phi(z) = \frac{1}{\delta(p,q)} \frac{f^{(q)}(z)}{z^{(p-q)}} \quad (z \in \Delta). \tag{43}$$

Then the function $\phi(z)$ is analytic in Δ with $\phi(0) = 1$. Differentiating both sides of (43) with respect to z and using (2), we have

$$\begin{aligned} &(1-\alpha_2)\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} + \alpha_2 \frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}} \\ = \; & \phi(z) + \left(\frac{\alpha_2}{p-q}\right) z\phi'(z) \prec h(z). \end{aligned} \tag{44}$$

Hence an application of Lemma 1, yields

$$\phi(z) \prec h(z). \tag{45}$$

Noting that $0 \leq \frac{\alpha_1}{\alpha_2} < 1$ and that $h(z)$ is convex univalent in Δ, it follows from (43), (44) and (45) that

$$\begin{aligned} &(1-\alpha_1)\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} + \alpha_1 \frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}} \\ = \; & \frac{\alpha_1}{\alpha_2}\left((1-\alpha_2)\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} + \alpha_2 \frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}}\right) + \left(1 - \frac{\alpha_1}{\alpha_2}\right)\phi(z) \\ \prec \; & h(z). \end{aligned}$$

Thus $f(z) \in \mathcal{M}_{p,q}(\alpha_1, A, B)$ and the proof of Theorem 15 is completed. ■

Theorem 16 *Let* $0 \leq \alpha_1 < \alpha_2$. *Then*

$$\mathcal{N}_{p,q}(\alpha_2, A, B) \subset \mathcal{N}_{p,q}(\alpha_1, A, B).$$

Proof. Let $0 \leq \alpha_1 < \alpha_2$ and suppose that

$$\varphi(z) = \frac{1}{p-q} \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \quad (z \in \Delta). \tag{46}$$

Then the function $\varphi(z)$ is analytic in Δ with $\varphi(0) = 1$. Differentiating both sides of (46) logarithmically with respect to z and using (5), we have

$$\begin{aligned} &\frac{1}{p-q}\left[(1-\alpha_2)\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + \alpha_2\left(1 + \frac{zf^{(q+2)}}{f^{(q+1)}(z)}\right)\right] \\ = \; & \varphi(z) + \frac{\alpha_2 z\varphi'(z)}{(p-q)\varphi(z)} \prec h(z). \end{aligned} \tag{47}$$

Hence an application of Lemma 1 yields

$$\phi(z) \prec h(z). \tag{48}$$

Noting that $0 \leq \frac{\alpha_1}{\alpha_2} < 1$ and that $h(z)$ is starlike univalent in Δ, it follows from (46), (47) and (48) that

$$\begin{aligned} & \frac{1}{p-q}\left[(1-\alpha_1)\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + \alpha_1\left(1+\frac{zf^{(q+2)}}{f^{(q+1)}(z)}\right)\right] \\ = \; & \frac{\alpha_1}{\alpha_2}\left(\frac{1}{p-q}\left[(1-\alpha_2)\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + \alpha_2\left(1+\frac{zf^{(q+2)}}{f^{(q+1)}(z)}\right)\right]\right) + \left(1-\frac{\alpha_1}{\alpha_2}\right)\phi(z) \\ \prec \; & h(z). \end{aligned}$$

Thus $f(z) \in \mathcal{N}_{p,q}(\alpha_1, A, B)$ and the proof of Theorem 16 is completed. ■

5 Open Problem

We can effect by some linear operators and solve some problems.

References

[1] M. K. Aouf, Certain subclass of multivalent functions with negative coefficients defined by using a differential operator, J. Math and Applications 30(2008), 5-21.

[2] M. K. Aouf, Certain classes of p-valent functions with negative coefficients, Acta Universalities Apulensis 30(2012), 271-291.

[3] M. K. Aouf and A. O. Mostafa, Certain classes of p-valent functions defined by convolution, General Math. Vol. 20, No. 1 (2012), 85-98.

[4] D. J. Hallenbeck and St. Ruscheweyh, Subordination by convex functions, Proc. Amer. Math. Soc. 52(1975) 191-195.

[5] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of certain family of integral operators, J. Math. Anal. Appl. 276 (2002) 432-445.

[6] E. A. Elrifai, H. E. Darwish and A. R. Ahmed, On subordination for Higher-Order derivatives of multivalent functions associated with the generalized Srivastava-Attiya operator, Demonstratio Math. Vol. XLV, No. 1 (2012), 39-50.

[7] T. H. MacGregor, Functions whose derivative has a positive real part, Trans. Amer. Math. Soc. 104 (1962) 532-537.

[8] S. S. Miller and P. T. Mocanu, Differential subordination and univalent functions, Michigan Math. J. 28 (1981) 157-171.

[9] S. S. Miller and P. T. Mocanu, Univalent solutions of Briot-Bouquet differential Subordinations, J. differential Equations 58 (1985) 297-309.

[10] S. S. Miller and P. T. Mocanu, Differential subordination : Theory and Applications, in: Series on Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker, New York and Basel, 2000.

[11] P. T. Mocanu, Une propriete de convexite generalisee dans la representation conforme, Mathematica (Cluj) 11 (1969), 127-133.

[12] P. T. Mocanu and M. O. Reade, The order of starlikeness of certain univalent functions, Notices Amer. Math. Soc. 18 (1971), 815.

[13] M. O. Reade and P. T. Mocanu, The radius of $\alpha-$convexity of starlike functions, Notices Amer. Math. Soc. 19 (1972), A-110.

[14] J. Patel, Nak Eun Cho and H. M. Srivastava, Certain subclasses of multivalent functions associated with a family of linear operators, Mathematical and Computer Modelling 43 (2006) 320-338.

[15] St. Ruscheweyh and T. Sheil-Small, Hadamard products of schicht functions and the Polya-Schoenberg conjecture, Comment. Math. Helv., 48 (1973) 119-35.

[16] J. Stankiewicz and Z. Stankiewicz, Some applications of the Hadamard convolution in the theory of functions, Ann. Univ. Mariae Curie-Sklodowska Sect. A 40 (1986) 251-265.

[17] E. T. Whittaker and G. N. Watson, A Course on Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions, Fourth edn., Cambridg University Press, Cambridge, 1927 (Reprinted).

[18] D. R. Wilken and J. Feng, A remark on convex and starlike functions, J. London Math. Soc. (2) 21 (1980) 287-290.

[19] T. Zeng, C. Y. Gao, Z. G. Wang and R. Aghalary, Certain subclasses of multivalent functions involving the Cho-Kwon-Srivastava operator, J. Math and Applications 30(2008), 161-170.