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Coefficient bounds for a general class of complex order Defined by q-analogue Salagean operator

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Abstract

In this paper, by using the q-Sălăgean operator we define a class of univalent functions with complex order and obtain some coefficient bounds for functions belonging to this class.

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1 Introduction

Let $\mathcal{A}$ be the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , (z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}) \tag{1}$$

and S be the subclass of $\mathcal{A}$ which are univalent. For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, $f(z)$ is subordinate to $g(z)$ $(f(z) \prec g(z))$, if there exists a function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $f(z) = g(\omega(z))$ and if $g(z)$ is univalent in $\mathbb{U}$, then (see for details [17] and [25]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\wp$ be the class of all analytic and univalent functions ϕ in $\mathbb{U}$ with $\phi(0) = 1$, $\phi'(0) > 0$.

In [24] for $f \in S$, Ma and Minda defined the classes $S^*(\phi)$ and $C(\phi)$ satisfying $\frac{zf'(z)}{f(z)} \prec \phi(z)$ and$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$, respectively, which for $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}$ reduce to the classes $S^*(\alpha)$ and $C(\alpha)$ (the classes of starlike and convex functions of order α, respectively $(0 \leq \alpha < 1)$).

For a function $f(z) \in S$, given by (1) and $0 < q < 1$, the Jackson's $q-$derivative is defined by [23] (also see [1], [7], [11], [15], [20], [33], [34], [36] and [37]):

$$\begin{aligned} D_q f(z) &= \frac{f(qz) - f(z)}{(q-1)\,z}, \left(z \in \mathbb{U}, 0 < q < 1, z \neq 0\right), \\ &= 1 + \sum_{k=2}^{\infty} [k]_q \, a_k z^{k-1}, \end{aligned} \tag{2}$$

$D_q f(0) = f'(0)$ and

$$[k]_q = \frac{1-q^k}{1-q} (0 < q < 1). \tag{3}$$

For $f \in \mathcal{A}$, Govindaraj and Sivasubramanian [21] defined and discussed the Sălăgean $q-$ difference operator by:

$$\begin{aligned} D_q^0 f(z) &= f(z) \\ D_q^1 f(z) &= z D_q f(z) \\ D_q^n f(z) &= z D_q (D_q^{n-1} f(z)) \\ D_q^n f(z) &= z + \sum_{k=2}^{\infty} [k]_q^n \, a_k z^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\}, 0 < q < 1, z \in \mathbb{U}). \end{aligned} \tag{4}$$

We note that

$$\lim_{q \to 1^-} D_q^n f(z) = D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0, z \in \mathbb{U}), \tag{5}$$

where $D^n f(z)$ is the Sălăgean operator [32] (see also [2], [3], [4], [5], [6], [8], [10], [13], [14] and [22]).

Making use of the $q-$Sălăgeăn operator D_q^n, we introduce a new class of analytic functions as following:

Definition 1. A function $f(z) \in S$ is said to be in the class $M^n(q,b,\lambda,\phi)$, if

$$1+\frac{1}{b}\left[\frac{(1-\lambda)D_q^{n+1}f(z)+\lambda D_q^{n+2}f(z)}{(1-\lambda)D_q^{n}f(z)+\lambda D_q^{n+1}f(z)}-1\right] \prec \phi(z)\ (b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, 0 \le \lambda \le 1, n \in \mathbb{N}_0).$$

For suitable choices of n, b, q, λ and $\phi(z)$, we obtain subclasses:

(i) $M^n(q,1,\lambda;\phi) = \mathcal{P}\Sigma_q^n(\lambda,\phi)$ (see [19]);

(ii) $\lim_{q\to 1^-} M^n(q,b,\lambda;\phi) = M_{\lambda,n}^b(\phi)$ (see [18]);

(iii) $\lim_{q\to 1^-} M^n(q,1,\lambda;\phi) = M_{\lambda,n}(\phi)$ (see [30]);

(iiii) $\lim_{q\to 1^-} M^n(q,b,0;\phi) = H^n(b,\phi)$ (see [12]);

(v)$\lim_{q\to 1^-} M^0(q,1,\lambda;\phi) = M_\lambda(\phi)$ (see [35]);

(vi)$\lim_{q\to 1^-} M^n(q,b,0;\frac{1+Az}{1+Bz}) = H_n^b(A,B)$ (see [16]);

(vii)$\lim_{q\to 1^-} M^n(q,b,0;\frac{1+z}{1-z}) = S^n(b)$ (see [7]);

(viii)$\lim_{q\to 1^-} M^0(q,b,0;\frac{1+z}{1-z}) = S(b)$ (see [28], [29] and [9]);

(viiii)$\lim_{q\to 1^-} M^1(q,b,0;\frac{1+z}{1-z}) = C(b)$ (see [26], [27] and [9]);

(x)$\lim_{q\to 1^-} M^0(q,b,0;\phi) = S^*(b,\phi)$ and $\lim_{q\to 1^-} M^1(q,b,0;\phi) = C(b,\phi)$ (see [31]).

In order to prove our results, we need the following lemmas.

Lemma 1[24]. If $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is a function with positive real part in $\mathbb{U}$ and μ is a complex number, then

$$\left|c_2 - \mu c_1^2\right| \le 2\max\{1; |2\mu - 1|\}.$$

The result is sharp for the function

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

Lemma 2 [24]. If $p_1(z) = 1 + c_1 z + c_2 z^2 + ...$ is an analytic function with a positive real part in $\mathbb{U}$, then

$$\left|c_2 - v c_1^2\right| \le \begin{cases} -4v+2, & \text{if } v \le 0, \\ 2, & \text{if } 0 \le v \le 1, \\ 4v-2, & \text{if } v \ge 1, \end{cases}$$

when $v < 0$ or $v > 1$, the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z} \quad (0 \le \lambda \le 1),$$

or one of its rotations. If $v = 1$, the equality holds if and only if

$$\frac{1}{p_1(z)} = \left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z} \quad (0 \le \lambda \le 1),$$

or one of its rotations. Also the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$:

$$\left|c_2 - vc_1^2\right| + v\left|c_1\right|^2 \leq 2 \quad \left(0 < v \leq \frac{1}{2}\right)$$

and

$$\left|c_2 - vc_1^2\right| + (1-v)\left|c_1\right|^2 \leq 2 \quad \left(\frac{1}{2} < v < 1\right).$$

2 Main Results

Unless otherwise mentioned, we assume throughout this paper that $\lambda \geq 0, b \in \mathbb{C}^*, n \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

Using lemma 1, we have the following theorem:

Theorem 1. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $\phi(z) \in \wp$ and $B_1 \neq 0$. If $f(z) \in M^n(q, b, \lambda, \phi)$, and μ is a complex number, then

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| \leq & \frac{B_1|b|}{[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)]} \max\{1; \\ & \left|\frac{B_2}{B_1} + B_1 b(1 - \frac{[(1-\lambda)[3]_q^n([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)]}{[(1-\lambda)[2]_q^n([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)]^2}\mu)\right|\right\}. \end{aligned} \quad (6)$$

The result is sharp.

Proof. If $f(z) \in M^n(q, b, \lambda, \phi)$, then there is a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$1 + \frac{1}{b}\left(\frac{(1-\lambda)D_q^{n+1}f(z)+\lambda D_q^{n+2}f(z)}{(1-\lambda)D_q^n f(z)+\lambda D_q^{n+1}f(z)} - 1\right) = \phi(\omega(z)). \quad (7)$$

Define a function $p_1(z)$ by

$$p_1(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + ... \quad . \quad (8)$$

Since $\omega(z)$ is a Schwarz function, we see that $\operatorname{Re}\{p_1(z)\} > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by:

$$p(z) = 1 + \frac{1}{b}\left(\frac{(1-\lambda)D_q^{n+1}f(z)+\lambda D_q^{n+2}f(z)}{(1-\lambda)D_q^n f(z)+\lambda D_q^{n+1}f(z)} - 1\right) \quad (9)$$

In view of (7), (8) and (9), we have

$$\begin{aligned} \phi(\omega(z)) &= p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = \phi\left(\frac{c_1 z + c_2 z^2 + ...}{2 + c_1 z + c_2 z^2 + ...}\right) \\ &= \phi\left\{\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + ...\right]\right\} \\ &= 1 + \frac{1}{2}c_1 B_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}c_1^2 B_2\right]z^2 + ... \quad . \end{aligned} \quad (10)$$

Now by substituting (10) in (7), we have

$$\begin{aligned} & 1+\frac{1}{b}\left(\tfrac{(1-\lambda)D_q^{n+1}f(z)+\lambda D_q^{n+2}f(z)}{(1-\lambda)D_q^{n}f(z)+\lambda D_q^{n+1}f(z)}-1\right) \\ = & 1+\tfrac{1}{2}c_1B_1z+\left[\tfrac{1}{2}B_1\left(c_2-\tfrac{c_1^2}{2}\right)+\tfrac{1}{4}c_1^2B_2\right]z^2+\ldots \ . \end{aligned}$$

So, comparing the coefficients we obtain

$$\left[(1-\lambda)[2]_q^n([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]a_2=\frac{B_1c_1b}{2},$$

$$\begin{aligned} & \left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]a_3- \\ & \left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2a_2^2 \\ = & \tfrac{1}{2}bB_1\left(c_2-\tfrac{c_1^2}{2}\right)+\tfrac{1}{4}bB_2c_1^2, \end{aligned}$$

or, equivalenty,

$$a_2=\frac{B_1c_1b}{2\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]}, \tag{11}$$

$$a_3=\frac{bB_1}{2\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}\left\{c_2-\tfrac{1}{2}[1-\tfrac{B_2}{B_1}-B_1b]c_1^2\right\}. \tag{12}$$

Therefore,

$$a_3-\mu a_2^2=\frac{bB_1}{2\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}\{c_2-vc_1^2\}, \tag{13}$$

where

$$v=\tfrac{1}{2}\left[1-\tfrac{B_2}{B_1}-B_1b\left(1-\frac{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}\mu\right)\right]. \tag{14}$$

Our result now follows by using Lemma 1. The result is sharp for the functions

$$1+\frac{1}{b}\left(\tfrac{(1-\lambda)D_q^{n+1}f(z)+\lambda D_q^{n+2}f(z)}{(1-\lambda)D_q^{n}f(z)+\lambda D_q^{n+1}f(z)}-1\right)=\phi(z^2)$$

and

$$1+\frac{1}{b}\left(\tfrac{(1-\lambda)D_q^{n+1}f(z)+\lambda D_q^{n+2}f(z)}{(1-\lambda)D_q^{n}f(z)+\lambda D_q^{n+1}f(z)}-1\right)=\phi(z).$$

This completes the proof of Theorem 1. ■

Remark 1.

(i) Putting $b=1$ in Theorem 1, we get the result obtained by [19, Theorem 1];

(ii) When $q\to 1^-$ in Theorem 1, we get the result obtained by [18, Theorem 1].

3 Fekete-Szegö inequalities for the function class $M^n(q,b,\lambda,\phi)$

Theorem 2. Let $\phi(z)=1+B_1z+B_2z^2+...(B_i>0, i=1,2)$. Also let

$$\sigma_1=\frac{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2(bB_1^2+B_2-B_1)}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]bB_1^2}, \tag{15}$$

$$\sigma_2=\frac{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2(bB_1^2+B_2+B_1)}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]bB_1^2}, \tag{16}$$

$$\sigma_3=\frac{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2(bB_1^2+B_2)}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]bB_1^2}. \tag{17}$$

If $f(z)\in M^n(q,b,\lambda;\phi)$, then

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{B_2|b|}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}+ & \\ |b|^2B_1^2\left(1-\frac{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}\mu\right) & \mu\leq\sigma_1, \\ \frac{|b|}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]} & \sigma_1\leq\mu\leq\sigma_2, \\ \frac{-B_2|b|}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}- & \\ |b|^2B_1^2\left(1-\frac{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}\mu\right) & \mu\geq\sigma_2.\end{cases} \tag{18}$$

Further, if $\sigma_3\leq\mu\leq\sigma_2$, then

$$\begin{aligned}&|a_3-\mu a_2^2|+\frac{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]bB_1^2}\left[B_1+B_2\right.\\&\left.-\frac{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]\mu-\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}bB_1^2\right]|a_2|^2\\\leq\ &\frac{|b|B_1}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}.\end{aligned} \tag{19}$$

If $\sigma_1\leq\mu\leq\sigma_3$, then

$$\begin{aligned}&|a_3-\mu a_2^2|+\frac{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]|b|B_1^2}\left[B_1-B_2\right.\\&\left.+\frac{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]\mu-\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}bB_1^2\right]|a_2|^2\\\leq\ &\frac{|b|B_1}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}.\end{aligned} \tag{20}$$

The result is sharp.

Proof. For $f(z) \in M^n(q, b, \lambda, \phi)$, $p(z)$ given by (9) and $p_1(z)$ given by (8), then a_2 and a_3 are given as same as in Theorem 1. From (13) and (14), we have:

(1) if $\mu \leq \sigma_1$, then $v \leq 0$. By applying Lemma 2 to (13), then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{B_1|b|}{2\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}(-4v+2) \\ &\leq \frac{B_2|b|}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]} \\ &\quad + \frac{|b|^2 B_1^2}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}\Bigg[1- \\ &\quad \frac{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}\mu\Bigg], \end{aligned} \tag{21}$$

which is evidently inequality (18) of Theorem 2.

If $\mu = \sigma_1$, then $v = 0$, then equality holds if and only if

$$p_1(z) = (\frac{1}{2} + \frac{1}{2}\gamma)\frac{1+z}{1-z} + (\frac{1}{2} - \frac{1}{2}\gamma)\frac{1-z}{1+z} \qquad (0 \leq \gamma \leq 1).$$

(2) if $\sigma_1 \leq \mu \leq \sigma_2$, we note that

$$\max\left\{\frac{1}{2}\left[1 - \frac{B_2}{B_1} - B_1 b\left(1 - \frac{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}\mu\right)\right]\right\} \leq 1,$$

then applying Lemma 2 to equality (13), we have

$$|a_3 - \mu a_2^2| \leq \frac{|b|}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]},$$

which is evidently inequality (18) of Theorem 2. If $\sigma_1 < \mu < \sigma_2$, then

$$p_1(z) = \frac{1+z^2}{1-z^2}.$$

(3) if $\mu \geq \sigma_2$, then $v \geq 1$. By applying Lemma 2 to (13), then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{-B_2|b|}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]} - \\ &\quad |b|^2 B_1^2\left(1 - \frac{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}\mu\right), \end{aligned}$$

which is evidently inequality (18) of Theorem 2 . If $\mu = \sigma_2$, then we have $v = 1$ therefore equality holds if and only if

$$\frac{1}{p_1(z)} = (\frac{1}{2} + \frac{1}{2}\gamma)\frac{1+z}{1-z} + (\frac{1}{2} - \frac{1}{2}\gamma)\frac{1-z}{1+z} \qquad (0 \leq \gamma \leq 1, z \in \mathbb{U}).$$

For the values $\sigma_1 < \mu < \sigma_3$, we have

$$\begin{aligned} & |a_3-\mu a_2^2|+(\mu-\sigma_1)|a_2|^2 \qquad (22) \\ = & \tfrac{|b|B_1}{2\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}|c_2-vc_1^2|+[\mu- \\ & \tfrac{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2(bB_1^2+B_2-B_1)}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]bB_1^2}\Bigg] \tfrac{|b|^2B_1^2}{4\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]}|c_1|^2 \\ = & \tfrac{|b|B_1}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}\left\{\frac{1}{2}(|c_2-vc_1^2|+v|c_1|^2)\right\}. \qquad (23) \end{aligned}$$

Now by applying Lemma 2 to equality (13), we have

$$|a_3-\mu a_2^2|+(\mu-\sigma_1)|a_2|^2 \leq \tfrac{|b|B_1}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]},$$

which is evidently inequality (20) of Theorem 2.

Next for the values of $\sigma_3 < \mu < \sigma_2$, we have

$$\begin{aligned} & |a_3-\mu a_2^2|+(\sigma_2-\mu)|a_2|^2 \\ = & \tfrac{|b|B_1}{2\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}|c_2-vc_1^2|+ \\ & \Bigg(\tfrac{\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2(bB_1^2+B_2+B_1)}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]bB_1^2} \\ & -\mu)\,\tfrac{|b|^2B_1^2}{4\left[[2]_q^n(1-\lambda)([2]_q-1)+\lambda[2]_q^{n+1}([2]_q-1)\right]^2}|c_1|^2= \qquad (24) \\ & \tfrac{|b|B_1}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]}\left\{\frac{1}{2}\left(|c_2-vc_1^2|+(1-v)|c_1|^2\right)\right\}. \end{aligned}$$

Now applying Lemma 2 to equality (13), we have

$$|a_3-\mu a_2^2|+(\sigma_2-\mu)|a_2|^2 \leq \tfrac{|b|B_1}{\left[[3]_q^n(1-\lambda)([3]_q-1)+\lambda[3]_q^{n+1}([3]_q-1)\right]},$$

which is inequality (19).

To show that the bounds are sharp, we define the functions $\chi_{\phi n}\,(n=2,3,4,...)$, $\digamma_\beta$ and ξ_β $(0 \leq \beta \leq 1)$, respectively, by

$$1+\frac{1}{b}\left[\tfrac{(1-\lambda)D_q^{n+1}\chi_{\phi n}(z)+\lambda D_q^{n+2}\chi_{\phi n}(z)}{(1-\lambda)D_q^{n}\chi_{\phi n}(z)+\lambda D_q^{n+1}\chi_{\phi n}(z)}-1\right]=\phi\left(z^{n-1}\right),$$

$$\chi_{\phi n}(0)=0=\chi_{\phi n}^{'}(0)-1,$$

$$1+\frac{1}{b}\left[\tfrac{(1-\lambda)D_q^{n+1}\digamma_\beta(z)+\lambda D_q^{n+2}\digamma_\beta(z)}{(1-\lambda)D_q^{n}\digamma_\beta(z)+\lambda D_q^{n+1}\digamma_\beta(z)}-1\right]=\phi\left(\frac{z(z+\beta)}{1+\beta z}\right),$$

$$\digamma_\beta(0)=0=\digamma_\beta^{'}(0)-1$$

and

$$1+\frac{1}{b}\left[\frac{(1-\lambda)D_q^{n+1}\xi_\beta(z)+\lambda D_q^{n+2}\xi_\beta(z)}{(1-\lambda)D_q^{n}\xi_\beta(z)+\lambda D_q^{n+1}\xi_\beta(z)}-1\right]=\phi\left(-\frac{1+\beta z}{z\,(z+\beta)}\right),$$

$$\xi_\beta\,(0)=0=\xi_\beta^{'}\,(0)-1.$$

Clearly, the functions $\chi_{\phi n}, F_\beta$ and $\xi_\beta \in M^n(q,b,\lambda,\phi)$. If $\mu<\sigma_1$ or $\mu>\sigma_2$, then the equality holds if and only if $f(z)$ is $\chi_{\phi 2}$, or one of its rotations. When $\sigma_1<\mu<\sigma_2$, the equality holds if and only if $f(z)$ is $\chi_{\phi 3}$, or one of its rotations. If $\mu=\sigma_1$, then the equality holds if and only if $f(z)$ is F_β, or one of its rotations. If $\mu=\sigma_2$, then the equality holds if and only if $f(z)$ is ξ_β, or one of its rotations. This completes the proof of Theorem 2. ■

Remark 2.

(i) Putting $b=1$ in Theorem 2, we get the result obtained by [19, Theorem 2];

(ii) When $q\to 1^-$ in Theorem 2, we get the result obtained by [18, Theorem 2].

For different values of q,b,λ,ϕ in Theorems 1 and 2, we obtain results corresponding to the classes mentioned in the introduction.

4 Open Problem

The authors suggest to study the class

$$1+\frac{1}{b}\left(\frac{(1-\lambda)R_q^{\mu+1}f(z)+\lambda R_q^{\mu+2}f(z)}{(1-\lambda)R_q^{\mu}f(z)+\lambda R_q^{\mu+1}f(z)}-1\right)\prec\varphi(z),$$

where $R_q^\mu f(z)(\mu>-1)$ is the $q-$analogue of Ruscheweyh differential operator .

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References

[1] M. H. Annby and Z. S. Mansour, $q-$Fractional Calculas Equations, Lecture Noes in Math. 2056, Springer-Verlag Berlin Heidelberg, 2012.

[2] M. K. Aouf, Neighborhoods of certain classes of analytic functions with negative coefficients, Internet. J. Math. Sci., 2006 (2006), Article ID38258, 1-6.

[3] M. K. Aouf, Subordination properties for a certain class of analytic functions defined by Sălăgeăn operator, Appl. Math. Letters, 22 (2009), 1581-1585.

[4] M. K. Aouf, A subclass of uniformly convex functions with negative coefficients, Math. Tome 52 (75) (2010), no. 2, 99-111.

[5] M. K. Aouf, F. M. AL-Oboudi and M. M. Haidan, On a certain class of analytic functions with complex order and defined by Sălăgeăn operator, Math. Tome 47 (70) (2005), no. 1, 3-18.

[6] M. K. Aouf, H. E. Darwish, and A. A. Attiya, On a class of certain analytic functions of complex order. Indian J. Pure Appl. Math., 32 (2001), no. 10, 1443-1452.

[7] M. K. Aouf, H. E. Darwish and G. S. Salagean, On a generalaization of starlike functions with negative coefficients, Math. Tome 43, 66 (2001), no. 1, 3-10.

[8] M. K. Aouf, H. M. Hossen and Y. Lashin, On certain families of analytic functions with negative coefficients, Indian J. Pure Appl. Math., 31 (2000), no. 8, 999-1015.

[9] M. K. Aouf, S. Owa, and M. Obradovic, Certain classes of analytic functions of complex order and type beta. Rend Mat., 11(1991), 691-714.

[10] M. K. Aouf and G. S. Sălăgeăn, Generalization of certain subclass of convex functions and a corresponding subclass of starlike functions with negative coefficients, Math. Tome 50 (73) (2008), no. 2, 119-138.

[11] M. K. Aouf and T. M. Seoudy, Convolution properties for classes of bounded analytic functions with complex order defined by q-derivative operator, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math., 113 (2019), no. 2, 1279-1288.

[12] M. K. Aouf and H. Silverman, Fekete-Szegö inequality for n-starlike functions of complex order. Adv. Math. Sci. J., (2008), 1-12.

[13] M. K. Aouf and H. Silverman, Class of uniformaly convex functions with two fixed points, PanAmer. Math. J., 19 (2009), no. 2, 1-15.

[14] M. K. Aouf and H. M. Srivastava, Some families of starlike functions with negative coefficients, J. Math. Anal. Appl., 203 (1996), 762-790.

[15] A. Aral, V. Gupta, and R. P. Agarwal, Applications of q-Calculus in Operator Theory, Springer, New York, USA, 2013.

[16] A. A. Attiya, On a generalization class of bounded starlike functions of complex order. Appl. Math. Comput., 187 (2007), 62-67.

[17] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Science Book Publ. Cluj-Napoca, 2005.

[18] R. M. El-Ashwah and A. H. Hassan, Fekete-Szegö inequalities for certain subclaases of analytic functions, defined by using Sălăgean operator, Miskolc Univ. Press, 17 (2017), no. 2, 827-836.

[19] M. Gangadharan, S. Yalcin, and Şahsene Altinkaya, Fekete-Szegö inequalities for subclasses of bi-univalent functions, associated with Sălăgean type q-difference operator, Afr. Mat. 30 (2019), 979-987.

[20] G. Gasper and M. Rahman, Basic Hypergeometric Series, Combridge Univ. Press, Cambrididge, U. K., 1990.

[21] M. Govindaraj, and S. Sivasubramanian, On a class of analytic function related to conic domains involving q-calculus, Analysis Math., 43 (3) (2017), no. 5, 475-487.

[22] H. M. Hossen, G. S. Sălăgeăn and M. K. Aouf, Notes on certain classes of analytic functions with negative coefficients, Math. Tome 39 (62) (1997), no. 2, 165-179.

[23] F. H. Jackson, On q-functions and a certain difference operator, Transactions of the Royal Society of Edinburgh, 46 (1908), 253-281.

[24] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in. Proceedings of the Conference on Complex Analysis, Z. Li, F. Ren, L. Lang and S. Zhang(Eds), Int. Press, (1994), 157-169.

[25] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York and Basel, 225 (2000).

[26] M. A. Nasr and M. K. Aouf, On convex functions of complex order, Mansoura Bull. Sci., 8 (1982), 565-582.

[27] M. A. Nasr and M. K. Aouf, Bounded convex functions of complex order. Bull. Fac. Sci. Mansoura Univ., 10 (1983), 513-527.

[28] M. A. Nasr and M. K. Aouf, Bounded starlike functions of complex order. Proc. Indian Acad. Sci., 92 (1983), 97-102.

[29] M. A.Nasr and M. K. Aouf, Starlike function of complex order, Natur. Sci. Math., 25 (1985), 1-12.

[30] H. Orhan and E. Gunes, Fekete-Szegö inequality for certain subclasses of analytic functions. General Math., 14 (2006), no. 1, 41-54.

[31] V. Ravichandran, Y. Polatoglu, M. Bolcal and A. Sen, Certain subclasses of starlike and convex functions of complex order, Hacettepe, J. Math. Stat., 34 (2005), 9-15.

[32] G. Sălăgean, Subclasses of univalent functions, Lecture note in Math., Springer-Verlag, 1013 (1983), 362-372.

[33] T. M. Seoudy and M. K. Aouf, Convolution properties for certain classes of analytic functions defined by q-derivative operator, Abstr. Appl. Anal. 2014 (2014), Article ID 846719, 1-7.

[34] T. M. Seoudy and M. K. Aouf, Coefficient estimates of new classes of q-starlike and q-convex functions of complex order, J. Math. Ineq., 10 (2016), no. 1, 135-145.

[35] T. N. Shanmugam and S. Sivasubramanian, On the Fekete-Szegö problem for some subclasses of analytic functions. J. Inequal. Pure Appl. Math., 6 (2005), no. 2, 1-6.

[36] H. M. Srivastava and A. O. Mostafa, M. K. Aouf and H. M. Zayed, Basic and fractional q-calculus and associated Fekete-Szegö problem for p-valently q-starlike functions and p-valently q-convex functions of complex order, Miskolc Math. Notes, 20 (2019), no. 1, 489-509.

[37] H. M. Zayed and M. K. Aouf, Subclasses of analytic functions of complex order associated with q-Mittag Leffler function, J. Egyption Math. Soc., 26 (2018), no. 2, 278-286.