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On The Univalence Criteria For Analytic Functions Defined By Differential Operator

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Abstract

In this paper we obtain sufficient condition for univalence of analytic functions defined by differential operator.

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1 Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$.

Let S denote the subclass of A, which consists of functions of the form (1) that are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in E.

In geometric function theory, the univalence of complex functions is an important property, but it is difficult, and in many cases impossible, to show directly that a certain complex function is univalent. For this reason, many authors found different types of sufficient conditions of univalence. One of the most important of these conditions of univalence in the domains E and the exterior of a closed unit disk is the well-known criterion of Becker [5]. Becker's work depends upon a clever use of the theory of Loewner chains and the generalized Loewner differential equation. Extensions of this criterion were given by Deniz and Orhan [7], Ali et al. [1] and Nehari [9].

Let f be a function in the class A. We define the following differential operator introduced by Raducanu and Orhan [12]

$$\begin{aligned} D^0_{\lambda,\mu}f(z) &= f(z)\\ D^1_{\lambda,\mu}f(z) &= \lambda\mu z^2 f''(z) + (\lambda-\mu)zf'(z) + (1-\lambda-\mu)f(z)\\ &\vdots\\ D^m_{\lambda,\mu}f(z) &= D(D^{m-1}_{\lambda,\mu}f(z)), \end{aligned} \tag{2}$$

where $0 \le \mu \le \lambda \le 1$ and $m \in N = \{1, 2, \cdots\}$.

If f is given by (1) then by the definition of the operator $D^m_{\lambda,\mu}f$ if is easy to see that

$$\begin{aligned} D^m_{\lambda,\mu}f(z) &= z + \sum_{n=2}^{\infty}[1 + (\lambda\mu n + \lambda - \mu)(n-1)]^m a_n z^n\\ &= z + \sum_{n=2}^{\infty} B_n(\lambda,\mu,m)a_n z^n \end{aligned} \tag{3}$$

when $\lambda = 1$ and $\mu = 0$, we get the Salagean differential operator [13] and when $\mu = 0$, we obtain the differential operator defined by Al-Oboudi [4]. Further study on generalized differential operators can be found in the literature(see for instant [2, 3]).

In this paper we derive sufficient conditions of univalence for the generalized operator $D^m_{\lambda,\mu}f(z)$. Also, a number of known univalent conditions would follow upon specializing the parameters involved. In order to prove our results we need the following Lemmas.

Lemma 1.1 *[5] Let $f \in A$. If for all $z \in E$*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{4}$$

then the function f is univalent in E.

Lemma 1.2 *[10] Let $f \in A$. If for all $z \in E$*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1, \tag{5}$$

then the function f is univalent in E.

Lemma 1.3 *[14] Let μ be a real number $\mu > \frac{1}{2}$ and $f \in A$. If for all $z \in E$*

$$(1 - |z|^{2\mu}) \left| \frac{zf''(z)}{f'(z)} + 1 - \mu \right| \leq \mu, \tag{6}$$

then the function f is univalent in E.

Lemma 1.4 *[8] If $f \in S$ (the class of univalent functions) and*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \tag{7}$$

then $\sum\limits_{n=1}^{\infty} (n-1)|b_n|^2 \leq 1$.

Lemma 1.5 *[11] Let $\nu \in C, Re\{\nu\} \geq 0$ and $f \in A$. If for all $z \in E$*

$$\tfrac{1-|z|^{2Re(\nu)}}{Re(\nu)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{8}$$

then the function

$$F_\nu(z) = \left(\nu \int_0^z u^{\nu-1} f'(u) du \right)^{\frac{1}{\nu}}$$

is univalent in E.

2 Main Results

In this section, we establish the sufficient conditions to obtain a univalence for analytic functions involving the differential operator.

Theorem 2.1 *Let $f \in A$. If for all $z \in E$*

$$\sum_{n=1}^{\infty} B_n(\lambda, \mu, m)[n(2n-1)]|a_n| \leq 1, \tag{9}$$

then $D^m_{\lambda\mu} f(z)$ is univalent in E.

Proof. Let $f \in A$. Then for all $z \in E$, we have

$$(1-|z|^2)\left|\frac{z(D^m_{\lambda\mu}f(z))''}{(D^m_{\lambda\mu}f(z))'}\right| \le (1+|z|^2)\left|\frac{z(D^m_{\lambda\mu}f(z))''}{(D^m_{\lambda\mu}f(z))'}\right|$$

$$\le \frac{2\sum\limits_{n=2}^{\infty} n(n-1)B_n(\lambda,\mu,m)|a_n|}{1-\sum\limits_{n=2}^{\infty} nB_n(\lambda,\mu,m)|a_n|}$$

the last inequality is less than 1 if the assertion (9) is hold. Thus in view of Lemma 1.1, $D^m_{\lambda\mu}f(z)$ is univalent in E.

Theorem 2.2 *Let $f \in A$. If for all $z \in E$*

$$B_n(\lambda,\mu,m)|a_n| \le \frac{1}{\sqrt{7}}, \tag{10}$$

then $D^m_{\lambda\mu}f(z)$ is univalent in E.

Let $f \in A$. It suffices to show that

$$\left|\frac{z^2(D^m_{\lambda\mu}f(z))'}{2(D^m_{\lambda\mu}f(z))^2}\right| \le 1.$$

Now

$$\left|\frac{z^2(D^m_{\lambda\mu}f(z))'}{2(D^m_{\lambda\mu}f(z))^2}\right| \le \frac{1+\sum\limits_{n=2}^{\infty} nB_n(\lambda,\mu,m)|a_n|}{2(1-2\sum\limits_{n=2}^{\infty}[B_n(\lambda,\mu,m)])^m|a_n| - (\sum\limits_{n=2}^{\infty} B_n(\lambda,\mu,m)|a_n|^2)}.$$

The last inequality is less than 1 if the assertion (10) is hold. Thus in view of Lemma 1.2, $D^m_{\lambda\mu}f(z)$ is univalent in E.

Theorem 2.3 *Let $f \in A$. If for all $z \in E$*

$$\sum_{n=1}^{\infty} n[2(n-1)+(2\mu-1)]B_n(\lambda,\mu,m)|a_n| \le 2\mu-1,\ \ \mu > \frac{1}{2}, \tag{11}$$

then $D^m_{\lambda\mu}f(z)$ is univalent in E.

Proof. Let $f \in A$. Then for all $z \in E$, we have

$$(1-|z|^{2\mu})\left|\frac{z(D^m_{\lambda\mu}f(z))''}{(D^m_{\lambda\mu}f(z))'}+1-\mu\right| \le (1+|z|^2)\left|\frac{z(D^m_{\lambda\mu}f(z))''}{(D^m_{\lambda\mu}f(z))'}\right| + |1-\mu|$$

$$\le \frac{2\sum\limits_{n=2}^{\infty} B_n(\lambda,\mu,m)[n(n-1)]|a_n|}{1-\sum\limits_{n=2}^{\infty} nB_n(\lambda,\mu,m)|a_n|} + |1-\mu|$$

the last inequality is less than μ if the assertion (11) is hold. Thus in view of Lemma 1.3, $D^m_{\lambda\mu}f(z)$ is univalent in E.

As applications of Theorems 2.1, 2.2 and 2.3, we have the following Theorem.

Theorem 2.4 *Let $f \in A$. If for all $z \in E$ one of the inequality (9-11) holds then*

$$\sum_{n=1}^{\infty}(n-1)|b_n|^2 \leq 1, \tag{12}$$

where $\frac{z}{D^m_{\lambda\mu}f(z)} = 1 + \sum\limits_{n=1}^{\infty} b_n z^n$.

Proof. Let $f \in A$. Then in view of Theorems 2.1, 2.2 or 2.3, $D^m_{\lambda\mu}f(z)$ is univalent in E.
Hence by Lemma 1.4, we obtain the result.

Theorem 2.5 *Let $f \in A$. If for all $z \in E$*

$$\sum_{n=1}^{\infty} n[2(n-1) + Re(v)]B_m(\lambda,\mu,m)|a_n| \leq Re(v), \ \ Re(v) > 0, \tag{13}$$

then

$$G_v(z) = \left(v \int_0^z u^{v-1}[D^m_{\lambda\mu}f(z)]'du \right)^{\frac{1}{v}}$$

is univalent in E.

Let $f \in A$. Then for all $z \in E$,

$$\begin{aligned} \frac{1-|z|^{2Re(v)}}{Re(v)} \left| \frac{z(D^m_{\lambda\mu}f(z))''}{(D^m_{\lambda\mu}f(z))'} \right| &\leq \frac{1+|z|^{2Re(v)}}{Re(v)} \left| \frac{z(D^m_{\lambda\mu}f(z))''}{(D^m_{\lambda\mu}f(z))'} \right| \\ &\leq \frac{2\sum\limits_{n=2}^{\infty} n(n-1)B_n(\lambda,\mu,m)|a_n|}{1-\sum\limits_{n=2}^{\infty} nB_n(\lambda,\mu,m)|a_n|} \end{aligned}$$

the last inequality is less than 1 if the assertion (13) is hold. Thus in view of Lemma 1.5, $G_v(z)$ is univalent in E.

3 Open problems

Problem: One can define another class by using another linear operator or an integral operator the same way as in this paper and hence new results can be obtained.

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