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On a subclass of bi-univalent functions associated to Horadam polynomials

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Abstract

In the present article, our goal is finding estimates on the Taylor-Maclaurin coefficients $|a_2|$ ***and*** $|a_3|$ ***for a new class of bi-univalent functions defined by means of the Horadam polynomials. Fekete-Szegö inequalities of functions belonging to this subclass are also founded.***

Keywords: *Analytic functions, Univalent and bi-univalent functions, Fekete-Szegö problem, Horadam polynomials, Coefficient bounds; Subordination.*

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1 Introduction and Preliminaries

Let A be the family of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit open disk $U = \{z : z \in C, |z| < 1\}$. Let S be class of all functions in A which are univalent and normalized by the conditions

$$f(0) = 0 = f^{'}(0) - 1$$

in U. Two of the most famous subclasses of univalent functions class S are the class $S^*(\alpha)(0 \leq \alpha < 1)$ of starlike functions of order α and the class $K(\alpha)(0 \leq \alpha < 1)$ of convex functions of order α. For two functions f and g, are analytic in U, we say that the function f is subordinate to g in U, written as $f(z) \prec g(z), z \in U$, if there exists an analytic function $w(z)$ defined on U with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \text{ for all } z \in U,$$

such that $f(z) = g(w(z))$ for all $z \in U$. Also, it is known that

$$f(z) \prec g(z) \ (z \in U) \ \Rightarrow \ f(0) = g(0) \ \text{ and } \ f(U) \subset g(U).$$

For every univalent function $f \in S$ with inverse f^{-1}, is defined as

$$f^{-1}(f(z)) = z \quad (z \in U),$$

and

$$f^{-1}(f(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(w) = w + a_2 w^2 + (2a_2^2 - 3a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots . \qquad (2)$$

A function $f \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Earlier, Lewin [1] investigated the bi-univalent functions and derived that $|a_2| < 1.51$. Later, Brannan and Taha [2], Srivastava et al. [9] and many researchers (for examples, see [3]-[8]) proved some results within these coefficient for different classes.

Moreover, Brannan and Taha [2] introduced certain subclasses of the bi-univalent function class Σ' (see [10]), and they found non-sharp estimates for the initial for the classes $S^*_{\Sigma'}(\alpha)$ and $K_{\Sigma'}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α in U, respectively, corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$.

Up to now, for the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|(n \in N \setminus \{1, 2\})$ for each $f \in \Sigma'$ given by (1) is still an open problem.

Recently, Horcum and Kocer [11] considered Horadam polynomials $h_n(x)$, which are given by the following recurrence relation

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad (n \in N \geq 2), \qquad (3)$$

with $h_1 = a,\ h_2 = bx$, and $h_3 = pbx^2 + aq$ where $(a, b, p, q$ are some real constants) .
The characteristic equation of recurrence relation (3) is

$$t^2 - pxt - q = 0. \tag{4}$$

This equation has two real roots;

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2},$$

and

$$\beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

Some special cases of the Horadam polynomials are:

- If $a = b = p = q = 1$, the Fibonacci polynomials sequence is obtained

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ F_1(x) = 1,\ F_2(x) = x.$$

- If $a = 2, b = p = q = 1$, the Lucas polynomials sequence is obtained

$$L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x),\ L_0(x) = 2,\ L_1(x) = x.$$

- If $a = q = 1, b = p = 2$, the Pell polynomials sequence is obtained

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x),\ p_1(x) = 1,\ P_2(x) = 2x.$$

- If $a = b = p = 2, q = 1$, the Pell-Lucas polynomials sequence is obtained

$$Q_{n-1}(x) = 2xQ_{n-2}(x) + Q_{n-3}(x),\ Q_0(x) = 2,\ Q_1(x) = 2x.$$

- If $a = 1, b = p = 2, q = -1$, the Chebyshev polynomials of second kind sequence is obtained

$$U_{n-1}(x) = 2xU_{n-2}(x) + U_{n-3}(x),\ U_0(x) = 1,\ U_1(x) = 2x.$$

- If $a = b = 1, p = 2, q = -1$, the Chebyshev polynomials of First kind sequence is obtained

$$T_{n-1}(x) = 2xT_{n-2}(x) + T_{n-3}(x),\ T_0(x) = 1,\ T_1(x) = x.$$

- If $x = 1$, The Horadam numbers sequence is obtained

$$h_{n-1}(1) = ph_{n-2}(1) + qh_{n-3}(1),\ h_0(1) = a,\ h_1(1) = b.$$

More details associated with these polynomials sequences in ([12], [13], [14], [15]).

Remark 1.1 *[12] Let $\Omega(x,z)$ be the generating function of the Horadam polynomials $h_n(x)$. Then*

$$\Omega(x,z) = \frac{a+(b-ap)xt}{1-pxt-qt^2} = \sum_{n=1}^{\infty} h_n(x) z^{n-1}. \tag{5}$$

In this paper, Our goal is using the Horadam polynomials $h_n(x)$ and the generating function $\Omega(x,z)$ which are given by the recurrence relation (3) and (5), respectively, to introduce a new subclass of the bi-univalent function class Σ'. Also, we provide the initial coefficients and the Fekete-Szegö inequality for functions belonging to the class $\Sigma'(x)$.

2 Coefficient Bounds for the Class $\Sigma'(x)$

We begin by introducing the class $\Sigma'(x)$ of bi-univalent functions in the following definition.

Definition 2.1 *A function $f \in \Sigma'$ given by (1) is said to be in the class $\Sigma'(x)$, if the following conditions are satisfied:*

$$f^{'}(z) \prec \Omega(x,z) + 1 - \alpha \tag{6}$$

and

$$g^{'}(w) \prec \Omega(x,w) + 1 - \alpha \tag{7}$$

where the real constants a, b and q are as in (3) and $g = f^{-1}$ is given by (2).

Theorem 2.2 *Let the function $f \in \Sigma'$ given by (1) be in the class $\Sigma'(x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|bx^2(3b-4p)-4aq|}} \tag{8}$$

$$|a_3| \leq \frac{|bx|}{3} + \frac{(bx)^2}{4}, \tag{9}$$

and for some $\eta \in R$,

$$|a_3 - \eta a_2^2| \leq \left\{ \begin{array}{ll} \frac{|2bx|}{3} & , \quad |\eta - 1| \leq 1 - \frac{|4(pbx^2+qa)|}{3b^2x^2} \\ \frac{|bx|^3|1-\eta|}{3b^2x^2-4(pbx^2+qa)} & , \quad |\eta - 1| \geq 1 - \frac{|4(pbx^2+qa)|}{3b^2x^2} \end{array} \right\}. \tag{10}$$

Proof

Let $f \in \Sigma'$ be given by the Taylor-Maclaurin expansion (1). Then, for some analytic functions Ψ and Φ such that $\Psi(0) = \Phi(0) = 0$, $|\psi(z)| < 1$ and $|\Phi(w)| < 1,\ z, w \in U$ and using Definition 1, we can write

$$f^{'}(z) = \omega(x, \Phi(z)) + 1 - \alpha$$

and

$$g^{'}(w) = \omega(x, \psi(w)) + 1 - \alpha$$

or, equivalently,

$$f^{'}(z) = 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \cdots \tag{11}$$

and

$$g^{'}(z) = 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^3 + \cdots . \tag{12}$$

From (11) and (12), we obtain

$$f^{'}(z) = 1 + h_2(x)p_1 z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \cdots \tag{13}$$

and

$$f^{'}(z) = 1 + h_2(x)p_1 w + [h_2(x)q_2 + h_3(x)q_1^2]w^2 + \cdots . \tag{14}$$

Notice that if

$$|\Phi(z)| = |p_1 z + p_2 z^2 + p_3 z^3 + ...| < 1 \quad (z \in U)$$

and

$$|\psi(w)| = |q_1 w + q_2 w^2 + q_3 w^3 + ...| < 1 \quad (w \in U),$$

then

$$|p_i| \leq 1 \quad and \quad |q_i| \leq 1 \quad (i \in N).$$

Thus, upon comparing the corresponding coefficients in (13) and (14), we have

$$2a_2 = h_2(x)p_1 \tag{15}$$

$$3a_3 = h_2(x)p_2 + h_3(x)p_1^2 \tag{16}$$

$$-2a_2 = h_2(x)q_1 \tag{17}$$

$$3\left[2a_2^2 - a_3\right] = h_2(x)q_2 + h_3(x)q_1^2. \tag{18}$$

From (15) and (17), we find that

$$p_1 = -q_1 \tag{19}$$

and

$$8a_2^2 = h_2^2(x)(p_1^2 + q_1^2). \tag{20}$$

Also, by using (18) and (16), we obtain

$$6a_2^2 = h_2(x)(p_2 + q_2) + h_3(x)(p_1^2 + q_1^2). \tag{21}$$

By using (19) in (21), we get

$$\left[6 - \frac{8h_3(x)}{[h_2(x)]^2}\right] a_2^2 = h_2(x)(p_2 + q_2). \tag{22}$$

From (3), and (22), we have the desired inequality (8).

Next, by subtracting (18) from (16), we have

$$6\left[a_3 - a_2^2\right] = h_2(x)(p_2 - q_2) + h_3(x)(p_1^2 - q_1^2). \tag{23}$$

In view of (19) and (20), Equation (23) becomes

$$a_3 = a_2^2 + \frac{h_2(x)(p_2 - q_2)}{6}. \tag{24}$$

Hence using (19) and applying (3), we get desired inequality (9).
Now, by using (22) and (24) for some $\eta \in R$, we get

$$a_3 - \eta a_2^2 = \frac{[h_2(x)]^3(1-\eta)(p_2+q_2)}{6[h_2(x)]^2 - 8h_3(x)} + \frac{h_2(x)(p_2-q_2)}{6}$$

$$= h_2(x)\left[\left(\Theta(\eta, x) + \frac{1}{6}\right) p_2 + \left(\Theta(\eta, x) - \frac{1}{6}\right) q_2\right],$$

where

$$\Theta(\eta, x) = \frac{[h_2(x)]^2(1-\eta)}{6[h_2(x)]^2 - 8h_3(x)}.$$

So, we conclude that

$$|a_3 - \eta a_2^2| \leq \left\{ \begin{array}{cc} \frac{h_2(x)}{3} & , \ |\Theta(\eta, x)| \leq \frac{1}{6)} \\ 4h_2(x)|\Theta(\eta, x)| & , \ |\Theta(\eta, x)| \geq \frac{1}{6} \end{array} \right\}.$$

This proves Theorem 2.2.

In light of Remark 1.1, Theorem 2.2 would yield the following known result.

Corollary 2.3 *For $t \in (1/2, 1)$, let the function $f \in \Sigma'$ given by (1) be in the class $\Sigma'(t)$. Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|1-t^2|}} \tag{25}$$

$$|a_3| \leq \frac{2t}{3} + t^2, \tag{26}$$

and for some $\eta \in R$,

$$|a_3 - \eta a_2^2| \leq \left\{ \begin{array}{cc} \frac{4t}{3} & , \ |\eta - 1| \leq \frac{1-t^2}{3t^2} \\ \frac{2|\eta-1|}{1-t^2} & , \ |\eta - 1| \geq \frac{1-t^2}{3t^2} \end{array} \right\}. \tag{27}$$

Taking $\eta = 1$ in Corollary 1, we get the following consequence.

Corollary 2.4 *For $t \in (1/2, 1)$, let the function $f \in \Sigma'$ given by (1) be in the class $\Sigma'(t)$. Then*

$$|a_3 - a_2^2| \leq \frac{4t}{3}. \tag{28}$$

3 Future Work

This section suggests obtaining Hankel determinants of second and third order for the class $\Sigma'(x)$ of bi-univalent functions.

Another investigation to consider, Magesh et al. [16] obtained initial estimates for certain subclasses of bi-univalent functions. Obtaining Hankel determinants of second and third order are issues to be investigated.

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