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# Subclass of meromorphically convex univalent functions with negative and fixed second coefficients

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**Abstract**

***In this paper we consider the subclass*** $\Lambda_k^c(\alpha, \beta, A, B)$ ***consisting of meromorphic functions analytic in*** $U^* = \{z : z \in \mathbb{C} : 0 < |z| < 1\}$ ***with negative and fixed second coefficients and obtain coefficient estimates, distortion theorems and closure theorems. Finally we obtain radius of convexity for functions in this class.***

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## 1 Introduction

Let $\Sigma$ be the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \; z \in \mathbb{U}^* = \{z : z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \backslash \{0\}. \qquad (1)$$

Let $\Sigma_k$ be the subclass of $\Sigma$ consisting of convex functons which satisfies the condition

$$-\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \quad (z\in\mathbb{U}^*), \tag{2}$$

and $\Sigma_k(\alpha)$ be the subclass of $\Sigma$ consisting of convex functons of order $\alpha$ satisfying the condition

$$-\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \quad (z\in\mathbb{U}^*, 0\le\alpha<1). \tag{3}$$

Let $\Lambda$ denote the subclass of $\Sigma$ consisting of functions of the form

$$f(z)=\frac{1}{z}-\sum_{k=1}^{\infty}|a_k|\,z^k. \tag{4}$$

A function $f(z)\in\Sigma$ is in the class $\Sigma_k(\alpha,\beta,A,B)$ if it satisfies the condition

$$\left|\frac{\frac{zf''(z)}{f'(z)}+2}{B\left(1+\frac{zf''(z)}{f'(z)}\right)+[B+(A-B)(1-\alpha)]}\right|<\beta \quad (z\in\mathbb{U}^*), \tag{5}$$

where $0\le\alpha<1$, $0<\beta\le1$, $-1\le A<B\le1$ and $0<B\le1$.

The class $\Sigma_k(\alpha,\beta,A,B)$ was defined and studied by Srivastava et al. [8] and Aouf and Shammaky [5, with p=1]. Let

$$\Lambda_k(\alpha,\beta,A,B)=\Sigma_k(\alpha,\beta,A,B)\cap\Lambda.$$

We note that

$$\Lambda_k(\alpha,1,-1,1)=\Lambda_k(\alpha),$$

which was defined and studied by Uralegaddi and Ganigi [10]. Furhermore,

$$\Lambda_k(0,1,A,B)=\Lambda_k(A,B)$$

is the class of $f(z)\in\Lambda$, satisfying the condition (see Srivastava et al. [8])

$$\left|\frac{\frac{zf''(z)}{f'(z)}+2}{B\left(1+\frac{zf''(z)}{f'(z)}\right)+A}\right|<1 \quad (z\in\mathbb{U}^*, -1\le A<B\le1, 0<B\le1). \tag{6}$$

We begin by recalling the following lemma due to Srivastava et al. [8] (see also Aouf and Shammaky [5, with p=1].

**Lemma 1.** Let the function $f(z)$ defined by (4) be analytic in $\mathbb{U}^*$. Then $f(z)\in\Lambda_k(\alpha,\beta,A,B)$ if and only if

$$\sum_{k=1}^{\infty}k\left\{(k+1)+\beta\left[Bk+(B-A)\alpha+A\right]\right\}|a_k|\le(B-A)\beta\,(1-\alpha). \tag{7}$$

In view of Lemma 1, we can see that $f(z) \in \Lambda_k(\alpha, \beta, A, B)$ satisfies the coefficient inequality

$$|a_1| \leq \frac{(B-A)\beta\,(1-\alpha)}{2+\beta\,[B+A+(B-A)\,\alpha]}. \tag{8}$$

For $0 < c < 1$, let

$$|a_1| = \frac{(B-A)\beta\,(1-\alpha)\,c}{2+\beta\,[B+A+(B-A)\,\alpha]}. \tag{9}$$

Let $\Lambda_k^c(\alpha, \beta, A, B)$ denote the subclass of $\Lambda_k(\alpha, \beta, A, B)$ consisting of functions of form

$$f(z) = \frac{1}{z} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}z - \sum_{k=2}^{\infty} |a_k|\, z^k \quad (\,0 < c < 1). \tag{10}$$

We note that:

(*i*) $\Lambda_k^c(\alpha, 1, -1, 1) = \Lambda_k^c(\alpha)$,

(*ii*) $\Lambda_k^c(0, 1, A, B) = \Lambda_k^c(A, B)$.

In this paper the techniques used are similar to those of Aouf and Darwish [1], Aouf and Joshi [3], Aouf et al. [2, 4], Owa et al. [6], Silverman and Silvia [7] and Uralegaddi [9].

# 2 Coefficient Estimates

Unless indicated, we assume that $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $-1 \leq A < B \leq 1$ and $0 < B \leq 1, 0 < c < 1$, $f(z)$ defined by (10) and $z \in \mathbb{U}^*$.

**Theorem 1.** A function $f(z) \in \Lambda_k^c(\alpha, \beta, A, B)$ if and only if

$$\sum_{k=2}^{\infty} k\,\{(k+1) + \beta\,[Bk + A + (B-A)\alpha]\}\,|a_k| \leq (B-A)\beta\,(1-\alpha)\,(1-c). \tag{11}$$

**Proof.** Substituting (9) in (7) and simplifying we get the result. ■

**Corollary 2.** Let $f(z) \in \Lambda_k^c(\alpha, \beta, A, B)$. Then

$$|a_k| \leq \frac{(B-A)\beta\,(1-\alpha)\,(1-c)}{k\,\{(k+1) + \beta\,[Bk + A + (B-A)\alpha]\}}\ (k \geq 2). \tag{12}$$

The result is sharp for

$$f(z) = \frac{1}{z} - \frac{(B-A)\beta\,(1-\alpha)\,c}{2+\beta\,[B+A+(B-A)\,\alpha]}z - \frac{(B-A)\beta\,(1-\alpha)\,(1-c)}{k\,\{(k+1) + \beta\,[Bk + A + (B-A)\alpha]\}}z^k\ (k \geq 2). \tag{13}$$

# 3 Distortion Theorems

**Theorem 2.** Let $f(z) \in \Lambda_k^c(\alpha, \beta, A, B)$, then for $|z| = r < 1$, we have

$$\begin{aligned} & \frac{1}{r} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}r - \frac{(B-A)\beta(1-\alpha)(1-c)}{2\{3+\beta[2B+A+(B-A)\alpha]\}}r^2 \\ \leq & \ |f(z)| \leq \\ & \frac{1}{r} + \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}r + \frac{(B-A)\beta(1-\alpha)(1-c)}{2\{3+\beta[2B+A+(B-A)\alpha]\}}r^2. \end{aligned} \tag{14}$$

The result is sharp for

$$f(z) = \frac{1}{z} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}z - \frac{(B-A)\beta(1-\alpha)(1-c)}{2\{3+\beta[2B+A+(B-A)\alpha]\}}z^2. \tag{15}$$

**Proof.** Since $f(z) \in \Lambda_k^c(\alpha, \beta, A, B)$, then Theorem 1 yields

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{(B-A)\beta(1-\alpha)(1-c)}{2\{3+\beta[2B+A+(B-A)\alpha]\}}. \tag{16}$$

Thus, for $|z| = r < 1$, we have

$$\begin{aligned} |f(z)| & \leq \frac{1}{|z|} + \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}|z| + \sum_{k=2}^{\infty} |a_k| \, |z|^k \\ & \leq \frac{1}{r} + \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}r + r^2 \sum_{k=2}^{\infty} |a_k| \\ & \leq \frac{1}{r} + \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}r + \frac{(B-A)\beta(1-\alpha)(1-c)}{2\{3+\beta[2B+A+(B-A)\alpha]\}}r^2. \end{aligned}$$

Similarly we have

$$\begin{aligned} |f(z)| & \geq \frac{1}{|z|} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}|z| - \sum_{k=2}^{\infty} |a_k| \, |z|^k \\ & \geq \frac{1}{r} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}r - \frac{(B-A)\beta(1-\alpha)(1-c)}{2\{3+\beta[2B+A+(B-A)\alpha]\}}r^2. \end{aligned}$$

This completes the proof. ■

**Theorem 3.** Let $f(z) \in \Lambda_k^c(\alpha, \beta, A, B)$, then for $|z| = r < 1$, we have

$$\begin{aligned} & \frac{1}{r^2} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]} - \frac{(B-A)\beta(1-\alpha)(1-c)}{\{3+\beta[2B+A+(B-A)\alpha]\}}r \\ \leq & \ \left|f'(z)\right| \leq \\ & \frac{1}{r^2} + \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]} + \frac{(B-A)\beta(1-\alpha)(1-c)}{\{3+\beta[2B+A+(B-A)\alpha]\}}r. \end{aligned} \tag{17}$$

The result is sharp for $f(z)$ given by (15).

**Proof.** In view of Theorem 1, it follows that

$$\sum_{k=2}^{\infty} k\,|a_k| \leq \frac{(B-A)\beta(1-\alpha)(1-c)}{\{3+\beta[2B+A+(B-A)\alpha]\}}. \tag{18}$$

Thus, for $|z| = r < 1$, and making use of (18), we obtain

$$\begin{aligned}
\left|f'(z)\right| &\leq \frac{1}{|z|^2} + \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]} + \sum_{k=2}^{\infty} k\,|a_k|\,|z|^{k-1} \\
&\leq \frac{1}{r^2} + \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]} + r\sum_{k=2}^{\infty} k\,|a_k| \\
&\leq \frac{1}{r^2} + \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]} + \frac{(B-A)\beta(1-\alpha)(1-c)}{\{3+\beta[2B+A+(B-A)\alpha]\}}r.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\left|f'(z)\right| &\geq \frac{1}{|z|^2} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]} - \sum_{k=2}^{\infty} k\,|a_k|\,|z|^{k-1} \\
&\geq \frac{1}{r^2} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]} - \frac{(B-A)\beta(1-\alpha)(1-c)}{\{3+\beta[2B+A+(B-A)\alpha]\}}r.
\end{aligned}$$

This completes the proof. ■

# 4 Closure Theorems

Let $f_v(z)$ be defined, for $v = 1, 2, ..., m$, by

$$f_v(z) = \frac{1}{z} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}z - \sum_{k=2}^{\infty} |a_{k,v}|\,z^k. \tag{19}$$

**Theorem 4.** Let $f_v(z) \in \Lambda_k^c(\alpha, \beta, A, B)$ for $v = 1, 2, ..., m$. Then

$$g(z) = \frac{1}{z} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}z - \sum_{k=2}^{\infty} b_k z^k \quad (b_k \geq 0), \tag{20}$$

is also in the same class, where

$$b_k = \frac{1}{m}\sum_{v=1}^{m} |a_{k,v}|\,. \tag{21}$$

**Proof.** Since $f_v(z) \in \Lambda_k^c(\alpha,\beta,A,B)$ for $v = 1,2,...,m,$ it follows from Theorem 1 that

$$\sum_{k=2}^{\infty} k\left\{(k+1)+\beta\left[Bk+A+(B-A)\alpha\right]\right\}|a_{k,v}| \leq (B-A)\beta\,(1-\alpha)\,(1-c). \tag{22}$$

Hence

$$\begin{aligned}
&\sum_{k=2}^{\infty} k\left\{(k+1)+\beta\left[Bk+A+(B-A)\alpha\right]\right\} b_k \\
=\;& \sum_{k=2}^{\infty} k\left\{(k+1)+\beta\left[Bk+A+(B-A)\alpha\right]\right\}\left(\frac{1}{m}\sum_{v=1}^{m}|a_{k,v}|\right) \\
=\;& \frac{1}{m}\sum_{v=1}^{m}\sum_{k=2}^{\infty} k\left\{(k+1)+\beta\left[Bk+A+(B-A)\alpha\right]\right\}|a_{k,v}| \\
\leq\;& (B-A)\beta\,(1-\alpha)\,(1-c),
\end{aligned} \tag{23}$$

and the result follows. ■

**Theorem 5.** Let

$$f_1(z) = \frac{1}{z} - \frac{(B-A)\beta\,(1-\alpha)\,c}{2+\beta\left[B+A+(B-A)\,\alpha\right]}z \tag{24}$$

and

$$f_k(z) = \frac{1}{z} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}z - \frac{(B-A)\beta(1-\alpha)(1-c)}{k\{(k+1)+\beta[Bk+A+(B-A)\alpha]\}}z^k \quad (k \geq 2). \tag{25}$$

Then $f(z) \in \Lambda_k^c(\alpha,\beta,A,B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{26}$$

where $\mu_k \geq 0$ $(k \geq 1)$ and $\sum\limits_{k=1}^{\infty}\mu_k = 1$.

**Proof.** Suppose that

$$\begin{aligned}
f(z) &= \sum_{k=1}^{\infty} \mu_k f_k(z) \\
&= \frac{1}{z} - \frac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}z - \sum_{k=2}^{\infty} \frac{(B-A)\beta(1-\alpha)(1-c)}{k\{(k+1)+\beta[Bk+A+(B-A)\alpha]\}}\mu_k z^k.
\end{aligned} \tag{27}$$

Then it follows that

$$\begin{aligned} &\sum_{k=2}^{\infty} \tfrac{k\{(k+1)+\beta[Bk+A+(B-A)\alpha]\}}{\beta(B-A)(1-\alpha)(1-c)} \cdot \tfrac{(B-A)\beta(1-\alpha)(1-c)}{k\{(k+1)+\beta[Bk+A+(B-A)\alpha]\}} \mu_k \\ = &\sum_{k=2}^{\infty} \mu_k = 1-\mu_1 \leq 1. \end{aligned} \tag{28}$$

So, by Theorem 1, $f(z) \in \Lambda_k^c(\alpha,\beta,A,B)$.

Conversely, assume that $f(z) \in \Lambda_k^c(\alpha,\beta,A,B)$. Then

$$|a_k| \leq \tfrac{(B-A)\beta(1-\alpha)(1-c)}{k\{(k+1)+\beta[Bk+A+(B-A)\alpha]\}} \quad (k \geq 2). \tag{29}$$

Putting

$$\mu_k = \tfrac{k\{(k+1)+\beta[Bk+A+(B-A)\alpha]\}}{(B-A)\beta(1-\alpha)(1-c)} |a_k| \quad (k \geq 2), \tag{30}$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k, \tag{31}$$

we see that $f(z)$ can be expressed in the form (26). This completes the proof. ■

# 5 Radius of Convexity

**Theorem 6**. Let $f(z) \in \Lambda_k^c(\alpha,\beta,A,B)$. Then $f(z)$ is meromorphically convex in $|z| < r = r(\alpha,\beta,A,B,c)$, where $r(\alpha,\beta,A,B,c)$ is the largest value for which

$$\tfrac{3(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]} r^2 + \tfrac{k(k+2)(B-A)\beta(1-\alpha)(1-c)}{k\{(k+1)+\beta[Bk+A+(B-A)\alpha]\}} r^{k+1} = 1. \tag{32}$$

The result is sharp for

$$f_k(z) = \frac{1}{z} - \tfrac{3(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]} z - \tfrac{(B-A)\beta(1-\alpha)(1-c)}{k\{(k+1)+\beta[Bk+A+(B-A)\alpha]\}} z^k \quad \text{for some } k. \tag{33}$$

**Proof.** We must show that

$$\left| \frac{\left(zf'(z)\right)' + f'(z)}{f'(z)} \right| \leq 1 \quad for \quad |z| < r = r(\alpha,\beta,A,B,c).$$

Note that

$$\left| \frac{\left(zf'(z)\right)' + f'(z)}{f'(z)} \right| \leq \frac{\tfrac{2(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]} r^2 + \sum_{k=2}^{\infty} k(k+1)|a_k| r^{k+1}}{1 - \tfrac{(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]} r^2 - \sum_{k=2}^{\infty} k|a_k| r^{k+1}} \leq 1, \tag{34}$$

for $|z| < r$ if and only if

$$\frac{3(B-A)\beta\,(1-\alpha)\,c}{2+\beta\,[B+A+(B-A)\,\alpha]}r^2 + \sum_{k=2}^{\infty} k(k+2)\,|a_k|\,r^{k+1} \le 1. \qquad (35)$$

Since $f(z) \in \Lambda_k^c(\alpha, \beta, A, B)$, from (11) we may take

$$|a_k| = \frac{(B-A)\beta(1-\alpha)\mu_k}{k\{(k+1)+\beta[Bk+A+(B-A)\alpha]\}}, \quad \sum_{k=2}^{\infty} \mu_k \le 1. \qquad (36)$$

For each fixed $r$, we choose the positive integer $k_0 = k_0(r)$ for which $\frac{k(k+2)}{k\{(k+1)+\beta[Bk+A+(B-A)\alpha]\}}r^{k+1}$ is maximal. Then it follows that

$$\sum_{k=2}^{\infty} k(k+2)\,|a_k|\,r^{k+1} \le \frac{k_0(k_0+2)(B-A)\beta(1-\alpha)(1-c)}{k_0\{(k_0+1)+\beta[Bk_0+A+(B-A)\alpha]\}}r^{k_0+1}. \qquad (37)$$

Hence $f(z)$ is meromorphically convex in $|z| < r(\alpha, \beta, A, B, c)$ provided that

$$\frac{3(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}r^2 + \frac{k_0(k_0+2)(B-A)\beta(1-\alpha)(1-c)}{k_0\{(k_0+1)+\beta[Bk_0+A+(B-A)\alpha]\}}r^{k_0+1} \le 1. \qquad (38)$$

We find the value $r_0 = r_0(\alpha, \beta, A, B, c)$ and the corresponding integer $k_0(r_0)$ so that

$$\frac{3(B-A)\beta(1-\alpha)c}{2+\beta[B+A+(B-A)\alpha]}r_0^2 + \frac{k_0(k_0+2)(B-A)\beta(1-\alpha)(1-c)}{k_0\{(k_0+1)+\beta[Bk_0+A+(B-A)\alpha]\}}r_0^{k_0+1} = 1.$$

Then this value $r_0$ is the radius of meromorphically convex for $f(z) \in \Lambda_k^c(\alpha, \beta, A, B)$. ■

We note that:

$(i)$ Patting $\beta = B = 1$ and $A = -1$ in our results we obtain corresponding results for the class $\Lambda_k^c(\alpha)$,

$(ii)$ Patting $\alpha = 0$ and $\beta = 1$ in our results we obtain corresponding results for the class $\Lambda_k^c(A, B)$.

# 6 Open Problem

The authors suggest to study the properties of the subclass of the class $\Lambda_k(\alpha, \beta, A, B)$ when the first $n$ coefficients are fixed.

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