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Some preserving sandwich results for a class of meromorphic multivalent functions associated with an integral operator

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In this paper,we obtain some subordination, superordination and sandwich-preserving results for certain class of $p-$*valent meromorphic functions, which is defined by integral operator.*

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1 Introduction

Let $H(\mathbb{U})$ be the class of functions analytic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, n]$ be the subclass of $H(\mathbb{U})$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...,$$

with $H_0 = H[0, 1]$ and $H = H[1, 1]$. Let Σ_p denote the class of all $p-$valent meromorphic functions of the form:

$$f(z)=z^{-p}+\sum_{k=1-p}^{\infty}a_kz^k\ \ (p\in\mathbb{N}=\{1,2,...\};z\in\mathbb{U}^*=\mathbb{U}\backslash\{0\}). \tag{1}$$

For two functions $f(z)$, $F(z)\in H(\mathbb{U})$, $f(z)$ is subordinate to $F(z)$ or $F(z)$ is superordinate to $f(z)\,(f(z)\prec F(z))$ in $\mathbb{U}$, if there exists a function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$, $f(z)=F(\omega(z))$ $(z\in\mathbb{U})$ and if $F(z)$ is univalent in $\mathbb{U}$, then (see [1] and [4]):

$$f(z)\prec F(z)\Longleftrightarrow f(0)=F(0) \text{ and } f(\mathbb{U})\subset F(\mathbb{U}).$$

Let $\phi:\mathbb{C}^2\times\mathbb{U}\to\mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the first-order differential subordination:

$$\phi(p(z),zp^{'}(z);z)\prec h(z), \tag{2}$$

then $p(z)$ is a solution of the differential subordination (2). The univalent function $q(z)$ is called a dominant of the solution of the differential subordination (2) if $p(z)\prec q(z)$ for all $p(z)$ satisfying (2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q}\prec q$ for all dominants of (2) is called the best dominant. If $p(z)$ and $\phi(p(z),zp^{'}(z);z)$ are univalent in $\mathbb{U}$ and $p(z)$ satisfies first-order differential superordination:

$$h(z)\prec\phi(p(z),zp^{'}(z);z), \tag{3}$$

then $p(z)$ is a solution of the differential superordination (3). An analytic function $q(z)$ is called a subordinant of solutions of the differential superordination (3) if $q(z)\prec p(z)$ for all $p(z)$ satisfying (3). A univalent subordinant $\tilde{q}$ satisfies $q\prec\tilde{q}$ for all subordinants of (3) is called the best subordinant (see [4] and [5]).

For $f(z)\in\Sigma_p, 0\le\mu<1, 0\le\delta\le 1$ and $p\in\mathbb{N}$, we define the following operator:

$$\begin{aligned} I_{p,\mu}^{\delta}f(z) &= \frac{1}{(1-\mu)^{\delta+1}\,\Gamma(\delta+1)}\int_0^{\infty}t^{\delta+p}e^{-\left(\frac{t}{1-\mu}\right)}f(zt)dt \\ &= z^{-p}+\sum_{k=1-p}^{\infty}\frac{\Gamma(\delta+k+1+p)}{\Gamma(\delta+1)}(1-\mu)^{k+p}a_kz^k. \end{aligned} \tag{4}$$

From (4), we can easily obtain the following identities:

$$z\left(I_{p,\mu}^{\delta}f(z)\right)^{'}=(\delta+1)\,I_{p,\mu}^{\delta+1}f(z)-(\delta+1+p)\,I_{p,\mu}^{\delta}f(z). \tag{5}$$

We note that: $I_{1,\mu}^{\delta}f(z)=I_{\mu}^{\delta}f(z)$.

To prove our results, we need the following definitions and Lemmas.

Definition 1 [4]. Denote by $\mathcal{F}$ the set of all functions $q(z)$ which are analytic and injective on $\bar{U}\backslash E(q)$, where

$$E(q)=\left\{\xi\in\partial\mathbb{U}:\lim_{z\to\xi}q\left(z\right)=\infty\right\},$$

and such that $q^{'}(\xi)\neq 0$ for $\xi\in\bar{U}\backslash E(q)$. Further let the subclass of $\mathcal{F}$ for which $q(0)=a$ be denoted by $\mathcal{F}(a),\mathcal{F}(0)\approxeq\mathcal{F}_0$ and $\mathcal{F}(1)\approxeq\mathcal{F}_1$.

Definition 2 [5]. A function $L(z,t)(z\in\mathbb{U},t\geq 0)$ is said to be a subordination chain if $L(0,t)$ is analytic and univalent in $\mathbb{U}$ for all $t\geq 0, L(z,0)$ is continuously differentiable on $[0,1)$ for all $z\in\mathbb{U}$ and $L(z,t_1)\prec L(z,t_2)$ for all $0\leq t_1\leq t_2$.

Lemma 1 [7] . Let $L(z,t)=a_1(t)z+a_2(t)z^2+...$, with $a_1(t)\neq 0$ for all $t\geq 0$ and $\lim_{t\to\infty}|a_1(t)|=\infty$. Suppose that $L(.;t)$ is analytic in $\mathbb{U}$ for all $t\geq 0, L(z;.)$ is continuously differentiable on $[0;+\infty)$ for all $z\in\mathbb{U}$. If $L(z,t)$ satisfies

$$\operatorname{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\}>0\quad(z\in\mathbb{U},t\geq 0)$$

and

$$|L(z,t)|\leq K_0\left|a_1(t)\right|,|z|<r_0<1,t\geq 0,$$

for some positive constants K_0 and r_0, then $L(z,t)$ is a subordination chain.

Lemma 2 [3] . Suppose that the function $H:\mathbb{C}^2\to\mathbb{C}$ satisfies the condition

$$\operatorname{Re}\{H(is;t)\}\leq 0$$

for all real s and for all $t\leq\frac{-k(1+s^2)}{2}, k\in\mathbb{N}$. If the function $p(z)=1+p_kz^k+p_{k+1}z^{k+1}+...$ is analytic in $\mathbb{U}$ and

$$\operatorname{Re}\{H(p(z);zp^{'}(z))\}>0\quad(z\in\mathbb{U}),$$

then $\operatorname{Re}\{p(z)\}>0$ for $z\in\mathbb{U}$.

Lemma 3 [6] . Let $\kappa,\epsilon\in\mathbb{C}$ with $\kappa\neq 0$ and let $h\in H(\mathbb{U})$ with $h(0)=0$ if $\operatorname{Re}\{\kappa h(z)+\epsilon\}>0(z\in\mathbb{U})$, then the solution of the following differential equation:

$$q(z)+\frac{zq^{'}(z)}{\kappa q(z)+\epsilon}=h(z)\quad(z\in\mathbb{U};q(0)=0)$$

is analytic in $\mathbb{U}$ and satisfies $\operatorname{Re}\{\kappa q(z)+\epsilon\}>0$ for $z\in\mathbb{U}$.

Lemma 4 [4] . Let $p\in\mathcal{F}(a)$ and let $q(z)=a+a_kz^k+a_{k+1}z^{k+1}+...$ be analytic in $\mathbb{U}$ with $q(z)\neq a$ and $k\geq 1$. If q is not subordinate to p, then there exists two points $z_0=r_0e^{i\theta}\in\mathbb{U}$ and $\xi_0\in\partial\bar{U}\backslash E(q)$ such that

$$q(\mathbb{U}_{r_0})\subset p(\mathbb{U});q(z_0)=p(\xi_0)\text{ and }z_0p^{'}(z_0)=m\xi_0p^{'}(\xi_0)(m\geq k).$$

Lemma 5 [7] . Let $q \in H[a,1]$ and $\varphi : \mathbb{C}^2 \to \mathbb{C}$. Also set $\varphi\left(q(z), zq^{'}(z)\right) = h(z)$. If $L(z,t) = \varphi\left(q(z), tzq^{'}(z)\right)$ is a subordination chain and $q \in H[a,1] \cap \mathcal{F}(a)$, then

$$h(z) \prec \varphi(q(z), zq^{'}(z)),$$

implies that $q(z) \prec p(z)$. Furthermore, if $\varphi(q(z), zq^{'}(z)) = h(z)$ has a univalent solution $q \in \mathcal{F}(a)$, then q is the best subordinant.

In this paper, we investigate several properties for the class defined by the operator $I^{\delta}_{p,\mu} f(z)$.

2 Main Results

Unless otherwise mentioned, we assume throughout this paper that $0 \le \mu < 1, 0 \le \delta \le 1$, $0 < \lambda \le 1, 0 < \gamma \le 1$, $p \in \mathbb{N}$ and $z \in \mathbb{U}^*$.

Theorem 1. Let $f, g \in \Sigma_p$ and let

$$\left(\phi(z) = (1-\lambda)\left(z^p I^{\delta}_{p,\mu} g(z)\right)^{\gamma} + \lambda \left(\frac{I^{\delta+1}_{p,\mu} g(z)}{I^{\delta}_{p,\mu} g(z)}\right)\left(z^p I^{\delta}_{p,\mu} g(z)\right)^{\gamma}\right), \tag{6}$$

where

$$\operatorname{Re}\left\{1 + \frac{z\phi^{''}(z)}{\phi^{'}(z)}\right\} > -\eta$$

and η is given by

$$\eta = \frac{1 + \left[\frac{\gamma(\delta+1)}{\lambda}\right]^2 - \left|1 - \left[\frac{\gamma(\delta+1)}{\lambda}\right]^2\right|}{4\left[\frac{\gamma(\delta+1)}{\lambda}\right]}. \tag{7}$$

Then

$$\begin{aligned} &(1-\lambda)\left(z^p I^{\delta}_{p,\mu} f(z)\right)^{\gamma} + \lambda \left(\frac{I^{\delta+1}_{p,\mu} f(z)}{I^{\delta}_{p,\mu} f(z)}\right)\left(z^p I^{\delta}_{p,\mu} f(z)\right)^{\gamma} \\ \prec\ &(1-\lambda)\left(z^p I^{\delta}_{p,\mu} g(z)\right)^{\gamma} + \lambda \left(\frac{I^{\delta+1}_{p,\mu} g(z)}{I^{\delta}_{p,\mu} g(z)}\right)\left(z^p I^{\delta}_{p,\mu} g(z)\right)^{\gamma} \end{aligned} \tag{8}$$

implies that

$$\left(z^p I^{\delta}_{p,\mu} f(z)\right)^{\gamma} \prec \left(z^p I^{\delta}_{p,\mu} g(z)\right)^{\gamma} \tag{9}$$

and the function $\left(z^p I^{\delta}_{p,\mu} g(z)\right)^{\gamma}$ is the best dominant.

Proof. Define the functions $F(z)$ and $G(z)$ in $\mathbb{U}$ by

$$F(z) = \left(z^p I^{\delta}_{p,\mu} f(z)\right)^{\gamma} \text{ and } G(z) = \left(z^p I^{\delta}_{p,\mu} g(z)\right)^{\gamma} \tag{10}$$

and assume, without loss of generality, that $G(z)$ is analytic, univalent on $\bar{U}$ and $G^{'}(\xi) \neq 0$ $(|\xi| = 1)$. If not, then we replace $F(z)$ and $G(z)$ by $F(\nu z)$ and $G(\nu z)$, respectively, with $0 < \nu < 1$. These new functions have the desired properties on $\bar{U}$, so we can use them in the proof of our theorem, the results would follow by letting $\nu \to 1$. We first show that, if

$$q(z) = 1 + \frac{zG^{''}(z)}{G^{'}(z)}, \tag{11}$$

then

$$\operatorname{Re}\{q(z)\} > 0.$$

From (5) and the definition of G, ϕ, we obtain that

$$\phi(z) = G(z) + \frac{\lambda}{\gamma(1+\delta)} zG^{'}(z). \tag{12}$$

Differentiating (12), then

$$\phi^{'}(z) = \left(1 + \frac{\lambda}{\gamma(1+\delta)}\right) G^{'}(z) + \frac{\lambda}{\gamma(1+\delta)} zG^{''}(z). \tag{13}$$

Combining (11) and (13), we have

$$1 + \frac{z\phi^{''}(z)}{\phi^{'}(z)} = q(z) + \frac{zq^{'}(z)}{q(z) + \frac{\gamma(1+\delta)}{\lambda}} = h(z). \tag{14}$$

It follows from (6) and (14) that

$$\operatorname{Re}\left\{h(z) + \frac{\gamma(1+\delta)}{\lambda}\right\} > 0. \tag{15}$$

Moreover, by using Lemma 3, we conclude that the differential equation (14) has a solution $q(z) \in H(\mathbb{U})$ with $h(0) = q(0) = 1$. Let

$$H(u,v) = u + \frac{v}{u + \frac{\gamma(1+\delta)}{\lambda}} + \eta,$$

where η is given by (7). From (14) and (15), we obtain

$$\operatorname{Re}\left\{H(q(z), zq^{'}(z))\right\} > 0.$$

To verify the condition

$$\operatorname{Re}\{H(iu,v)\} \leq 0 \quad \left(u \in \mathbb{R}; v \leq -\frac{1+u^2}{2}\right), \tag{16}$$

we have

$$\begin{aligned}\operatorname{Re}\{H(iu,v)\} &= \operatorname{Re}\left\{iu + \frac{v}{iu+\frac{\gamma(1+\delta)}{\lambda}} + \eta\right\} = \\ \frac{\frac{\gamma(1+\delta)}{\lambda}v}{u^2+\left(\frac{\gamma(1+\delta)}{\lambda}\right)^2} + \eta &\leq -\frac{\sigma(u,\eta,\gamma,\lambda,\delta)}{2\left[u^2+\left(\frac{\gamma(1+\delta)}{\lambda}\right)^2\right]},\end{aligned}$$

where

$$\sigma(u,\eta,\gamma,\lambda,\delta) = \left[\frac{\gamma(1+\delta)}{\lambda} - 2\eta\right]u^2 - 2\eta\left[\frac{\gamma(1+\delta)}{\lambda}\right]^2 + \frac{\gamma(1+\delta)}{\lambda}. \quad (17)$$

For η given by (7), we have $\sigma(u,\eta,\gamma,\lambda,\delta)$ in (17) is positive, which implies that (16) holds. Thus, by using Lemma 2, we have

$$\operatorname{Re}\{q(z)\} > 0.$$

That is, that $G(z)$ defined by (10) is convex in $\mathbb{U}$. Next, we prove that (8) implies that

$$F(z) \prec G(z),$$

for F and G defined by (10). Consider $L(z,t)$ given by

$$L(z,t) = G(z) + \frac{\lambda(1+t)}{\gamma(1+\delta)}zG'(z) \quad (0 \leq t < \infty). \quad (18)$$

We note that

$$\frac{\partial L(z,t)}{\partial z}\Big|_{z=0} = G'(0)\left(1+\frac{\lambda(1+t)}{\gamma(1+\delta)}\right) \neq 0 \quad (0 \leq t < \infty).$$

This show that

$$L(z,t) = a_1(t)z + ...,$$

satisfies $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t\to\infty}|a_1(t)| = +\infty$. From (18) and for all $t \geq 0$, we have

$$\frac{|L(z,t)|}{|a_1(t)|} = \frac{\left|G(z)+\frac{\lambda(1+t)}{\gamma(1+\delta)}zG'(z)\right|}{\left|1+\frac{\lambda(1+t)}{\gamma(1+\delta)}\right|} \leq \frac{|G(z)|+\frac{\lambda(1+t)}{\gamma(1+\delta)}\left|zG'(z)\right|}{1+\frac{\lambda(1+t)}{\gamma(1+\delta)}}. \quad (19)$$

Since G is convex and normalized in $\mathbb{U}$, the following well-known growth and distortion sharp inequalities (see [2]) are true:

$$\begin{aligned}\frac{r}{1+r} &\leq |G(z)| \leq \frac{r}{1-r} \quad \text{if } |z| \leq r < 1, \\ \frac{1}{(1+r)^2} &\leq \left|G'(z)\right| \leq \frac{1}{(1-r)^2} \quad \text{if } |z| \leq r < 1.\end{aligned}$$

Using the right-hand sides of these inequalities in (19), we have

$$\frac{|L(z,t)|}{|a_1(t)|} = \frac{r}{(1-r)^2}\left[\frac{(1-r)\,\gamma\,(1+\delta)+\lambda\,(1+t)}{\gamma\,(1+\delta)+\lambda\,(1+t)}\right] \leq \frac{r}{(1-r)^2} \quad (|z|\leq r, t\geq 0)$$

and thus, the second assumption of Lemma 1 holds. Furthermore,

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \mathrm{Re}\left\{\frac{\gamma\,(1+\delta)}{\lambda}+(1+t)\,q(z)\right\} > 0.$$

Therefore, by using Lemma 1, we deduce that $L(z,t)$ is a subordination chain. So

$$\phi(z) = G(z) + \frac{\lambda}{\gamma\,(1+\delta)} zG^{'}(z) = L(z,0)$$

and

$$L(z,0) \prec L(z,t),$$

which implies that

$$L(\xi,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}) \quad (\xi \in \partial U)\,. \tag{20}$$

If F is not subordinate to G, by using Lemma 4, we know that there exists two points $z_0 \in \mathbb{U}$ and $\xi_0 \in \partial U$ such that

$$F(z_0) = G(\xi_0) \text{ and } z_0F^{'}(z_0) = (1+t)\,\xi_0 G^{'}(\xi_0) \tag{21}$$

Hence, by virtue of (8), (10), (18) and (21), we have

$$\begin{aligned} L(\xi_0,t) &= G(\xi_0) + \frac{\lambda\,(1+t)\,\xi_0 G^{'}(\xi_0)}{\gamma\,(1+\delta)} = F(z_0) + \frac{\lambda z_0 F^{'}(z_0)}{\gamma\,(1+\delta)} \\ &= (1-\lambda)\left(z_0^p I_{p,\mu}^{\delta} f(z_0)\right)^{\gamma} + \lambda\left(\frac{I_{p,\mu}^{\delta+1} f(z_0)}{I_{p,\mu}^{\delta} f(z_0)}\right)\left(z_0^p I_{p,\mu}^{\delta} f(z_0)\right)^{\gamma} \in \phi(\mathbb{U}). \end{aligned}$$

This contradicts (20). Thus, we deduce that $F \prec G$. Considering $F = G$, we see that G is the best dominant. This completes the proof of Theorem 1. ■

We now derive the following theorem.

Theorem 2. Let $f, g \in \Sigma_p$ and $\phi(z)$ as in (6), and η is given by (7), If

$$(1-\lambda)\left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma} + \lambda\left(\frac{I_{p,\mu}^{\delta+1} f(z)}{I_{p,\mu}^{\delta} f(z)}\right)\left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma}$$

is univalent in $\mathbb{U}$ and $\left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma} \in F$, then

$$\begin{aligned} &(1-\lambda)\left(z^p I_{p,\mu}^{\delta} g(z)\right)^{\gamma} + \lambda\left(\frac{I_{p,\mu}^{\delta+1} g(z)}{I_{p,\mu}^{\delta} g(z)}\right)\left(z^p I_{p,\mu}^{\delta} g(z)\right)^{\gamma} \\ \prec\ &(1-\lambda)\left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma} + \lambda\left(\frac{I_{p,\mu}^{\delta+1} f(z)}{I_{p,\mu}^{\delta} f(z)}\right)\left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma} \end{aligned}$$

implies that

$$\left(z^p I_{p,\mu}^{\delta} g(z)\right)^{\gamma} \prec \left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma}$$

and the function $\left(z^p I_{p,\mu}^{\delta} g(z)\right)^{\gamma}$ is the best subordinant.

Proof. Suppose that F, G and q are defined by (10) and (11), respectively. By applying the similar method as in the proof of Theorem 1, we get

$$\operatorname{Re}\{q(z)\} > 0.$$

Next, to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L(z,t)$ be defined by (18). Since G is convex, by applying a similar method as in Theorem 1, we deduce that $L(z,t)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G \prec F$. Moreover, since

$$\phi(z) = G(z) + \frac{\lambda}{\gamma(1+\delta)} zG'(z) = \varphi(G(z), zG'(z)),$$

has a univalent solution G, it is the best subordinant. This completes the proof of Theorem 2. ■

Combining Theorem 1 and Theorem 2, we get the following "sandwich-type result".

Theorem 3. Let $f, g_i \in \Sigma_p \, (i = 1, 2)$ and let

$$\operatorname{Re}\left\{1 + \frac{z\phi_i''(z)}{\phi_i'(z)}\right\} > -\delta g_i(z)$$

$$\left(\phi_i(z) = (1-\lambda)\left(z^p I_{p,\mu}^{\delta} g_i(z)\right)^{\gamma} + \lambda \left(\frac{I_{p,\mu}^{\delta+1} g_i(z)}{I_{p,\mu}^{\delta} g_i(z)}\right)\left(z^p I_{p,\mu}^{\delta} g_i(z)\right)^{\gamma} (i = 1, 2)\right),$$

where η is given by (7). If

$$(1-\lambda)\left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma} + \lambda \left(\frac{I_{p,\mu}^{\delta+1} f(z)}{I_{p,\mu}^{\delta} f(z)}\right)\left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma}$$

is univalent in $\mathbb{U}$ and $\left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma} \in F$, then

$$\begin{aligned}
&(1-\lambda)\left(z^p I_{p,\mu}^{\delta} g_1(z)\right)^{\gamma} + \lambda \left(\frac{I_{p,\mu}^{\delta+1} g_1(z)}{I_{p,\mu}^{\delta} g_1(z)}\right)\left(z^p I_{p,\mu}^{\delta} g_1(z)\right)^{\gamma} \\
\prec\ &(1-\lambda)\left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma} + \lambda \left(\frac{I_{p,\mu}^{\delta+1} f(z)}{I_{p,\mu}^{\delta} f(z)}\right)\left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma} \\
\prec\ &(1-\lambda)\left(z^p I_{p,\mu}^{\delta} g_2(z)\right)^{\gamma} + \lambda \left(\frac{I_{p,\mu}^{\delta+1} g_2(z)}{I_{p,\mu}^{\delta} g_2(z)}\right)\left(z^p I_{p,\mu}^{\delta} g_2(z)\right)^{\gamma}
\end{aligned}$$

implies that

$$\left(z^p I_{p,\mu}^{\delta} g_1(z)\right)^{\gamma} \prec \left(z^p I_{p,\mu}^{\delta} f(z)\right)^{\gamma} \prec \left(z^p I_{p,\mu}^{\delta} g_2(z)\right)^{\gamma}$$

and the functions $\left(z^p I_{p,\mu}^{\delta} g_1(z)\right)^{\gamma}$ and $\left(z^p I_{p,\mu}^{\delta} g_2(z)\right)^{\gamma}$ are, respectively, the best subordinant and the best dominant.

3 Open Problem

The authors suggest to study this class defined by the operator

$$\begin{aligned} I_p^{\alpha,\gamma} f(z) &= \frac{1}{z^{p+1}\Gamma(\alpha-\gamma+1)} \int_0^z \left(\log\frac{z}{t}\right)^{\alpha-\gamma} t^p f(t)dt \\ &= \frac{1}{z^p} + \sum_{k=m}^{\infty} \left(\frac{1}{k+p+1}\right)^{\alpha-\gamma+1} a_k z^k \quad (\alpha,\gamma>0)\,. \end{aligned}$$

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References

[1] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Science Book Publ. Cluj-Napoca, 2005.

[2] T. H. Gronwall, Some remarks on conformal representation, Ann. Math., 16(1915), 72-76.

[3] S. S. Miller and P. T. Mocanu, Differential Subordinations and univalent functions, Michigan Math. J., 28(1981), no. 2, 157-172.

[4] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.

[5] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, Complex Variables. Theory Appl., 48(2003), no. 10, 815-826.

[6] S. S. Miller and P. T. Mocanu, Univalent solutions of Briot-Bouquet differential equations, J. Differential Equations, 56(1985), no. 3, 297-309.

[7] Ch. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Gottingen, (1975).