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A Subclass of Univalent Functions Defined by a Generalized Differential Operator

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Abstract

In this current paper, we establish a new class of univalent function $\mathcal{G}R_{q,\zeta}^{\mu_1,\mu_2}(h,t,l,\beta)$ ***defined by comprehensive differential operator. We investigate the adequate condition for a function*** $f(z)$ ***to be in this class.***

Keywords: *Starlike function, univalent function, Ruscheweyh derivative, convex function, bazilevic function.*
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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{\tau=2}^{\infty} a_\tau z^\tau = z + a_2 z^2 + \cdots \tag{1}$$

which are holomorphic in the open unit disk

$$\mathcal{U} = \{z \in \mathcal{C} : |z| < 1\}.$$

Let $\mathcal{W}(\mathcal{U})$ be the space of analytic functions in $\mathcal{U}$. For $e \in \mathbb{C}$ and $q \in \mathcal{N}$ we indicate by

$$\mathcal{W}[e,q] := \left\{ f \in \mathcal{W}(\mathcal{U}) : f(z) = e + e_q z^q + e_{q+1} z^{q+1} + \cdots \right\} \tag{2}$$

and
$$\mathcal{A}_q = \left\{ f \in \mathcal{W}(\mathcal{U}) : f(z) = z + e_{q+1} z^{q+1} + e_{q+2} z^{q+2} + \cdots \right\} \tag{3}$$
with $\mathcal{A}_1 = \mathcal{A}$ and $z \in \mathcal{U}$.
Let $\mathcal{Q} \subset \mathcal{A}$ indicate the class of univalent functions in $\mathcal{U}$ and by $S^*(\zeta) \subset \mathcal{Q}$ indicate the class of starlike functions of order ζ, $0 \leq \zeta < 1$ which gratify the condition
$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \zeta. \tag{4}$$
Also, a function $f(z)$ essential to $\mathcal{K}(\zeta) \subset \mathcal{Q}$ is evident to be convex functions of order ζ, $0 \leq \zeta < 1$ in $\mathcal{U}$, supposing
$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \zeta, \tag{5}$$
and signify $\mathcal{R}(\zeta) \subset \mathcal{Q}$ as the class of bounded turning functions which gratify the condition
$$\Re\{f'(z)\} > \zeta,$$
where $z \in \mathcal{U}$.

Definition 1.1 [10] *For $f \in \mathcal{A}$, $h, q, l \in \mathcal{N}_0 = \{0, 1, 2, 3, ...\}$, $\mu_1 \geq \mu_2 \geq 0$, $t \geq 0$ and $t + l > 0$, the operator $D^{q,h}_{\mu_1,\mu_2,t,l}$ is defined by $D^{q,h}_{\mu_1,\mu_2,t,l} : \mathcal{A} \longrightarrow \mathcal{A}$*
$$D^{q,h}_{\mu_1,\mu_2,t,l} f(z) = \mathcal{M}^h_{\mu_1,\mu_2,t,l}(z) * R^q f(z) \quad (z \in \mathcal{U}). \tag{6}$$
We define the holomorphic function
$$\mathcal{M}^h_{\mu_1,\mu_2,t,l}(z) = z + \sum_{\tau=2}^{\infty} \left[\frac{t(1 + (\mu_1 + \mu_2)(\tau - 1)) + l}{t(1 + \mu_2(\tau - 1)) + l}\right]^h z^\tau,$$
where $h, l \in \mathcal{N}_0 = \{0, 1, 2, 3, \cdots\}$, $\mu_1 \geq \mu_2 \geq 0$, $t \geq 0$ and $t + l > 0$. $\mathcal{R}^q f(z)$ denote Ruscheweyh derivative[11] *and given by*
$$\mathcal{R}^q f(z) = z + \sum_{\tau=2}^{\infty} \frac{(q + \tau - 1)!}{q!(\tau - 1)!} a_\tau z^\tau,$$
where $q \in \mathcal{N}_0$ and $z \in \mathcal{U}$.
If $f(z) \in \mathcal{A}$, $f(z) = z + \sum_{\tau=2}^{\infty} a_\tau z^\tau$, then we have
$$D^{q,h}_{\mu_1,\mu_2,t,l} f(z) = z + \sum_{\tau=2}^{\infty} \left[\frac{t(1 + (\mu_1 + \mu_2)(\tau - 1)) + l}{t(1 + \mu_2(\tau - 1)) + l}\right]^h \frac{(q + j - 1)!}{q!(j - 1)!} a_\tau z^\tau \tag{7}$$
where $h, q, l \in \mathcal{N}_0 = \{0, 1, 2, 3, \cdots\}$, $\mu_1 \geq \mu_2 \geq 0$, $t \geq 0$ and $t + l > 0$.

Remark 1.2 . *It follows from the* (7) *that*

$$(t(1+\mu_2(\tau-1))+l)D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)=(t(1+\mu_2(\tau-1)-\mu_1)+l)D^{q,h}_{\mu_1,\mu_2,t,l}f(z) + t\mu_1 z(D^{q,h}_{\mu_1,\mu_2,t,l}f(z))'$$

for $z\in\mathbb{U}$.

Remark 1.3 .

- $D^{q,0}_{\mu_1,\mu_2,t,l}f(z)=D^q f(z)$ *which exactly is the Ruscheweyh derivative* [11],
- $D^{0,h}_{1,0,1,0}f(z)=D^h f(z)$ *which exactly is the Sălăgean derivative* [12],
- $D^{0,h}_{0,\mu_2,1,0}f(z)=D^h_{\mu_2}f(z)$ *which exactly is the Al-Oboudi operator* [1],
- $D^{q,h}_{\mu_1,0,1,0}f(z)=D^{q,h}_{\mu_1}f(z)$ *defined by Al-Shaqsi and Darus* [2],
- $D^{q,h}_{1,0,1,1}f(z)=D^{q,h}f(z)$ *defined by Uralegaddi and Somanatha* [15],
- $D^{0,h}_{1,0,1,l}f(z)=D^h_l f(z)$ *defined by Cho and Srivastava* [5],
- $D^{q,h}_{\mu_1,\mu_2,1,0}f(z)=D^{q,h}_{\mu_1,\mu_2}f(z)$ *defined by Eljamal and Darus* [7],
- $D^{q,h}_{\mu_1,\mu_2,0,l}f(z)=D^{q,h}_{\mu_1,\mu_2,l}f(z)$ *defined by El-Yagubi and Darus* [6],
- $D^{0,h}_{\mu_1,0,1,l}f(z)=D^h_{\mu_1,l}f(z)$ *defined by Cătas*[4],
- $D^{0,h}_{\mu_1,0,1,l}f(z)=D^h_{\mu_1 l}f(z)$ *defined by Swamy* [14].

Lemma 1.4 [9] *Let* ϑ *be holomorphic in* $\mathcal{U}$ *with* $\vartheta(0)=1$, *if*

$$\Re\left(1+\frac{z\vartheta'(z)}{\vartheta(z)}\right)>\frac{3\zeta-1}{2\zeta},$$

then $\Re(\vartheta(z))>\zeta$ *in* $\mathcal{U}$, $z\in\mathcal{U}$ *and* $\frac{1}{2}\le\zeta<1$.

2 Main Results

Definition 2.1 *A function* $f\in\mathcal{A}$, $h,q,l\in\mathcal{N}_0$, $\mu_1\ge\mu_2\ge0$, $t\ge0$, $t+l>0$, $\beta\ge-2$ *and* $0\le\zeta<1$ *is in the class* $\mathcal{G}R^{\mu_1,\mu_2}_{q,\zeta}(h,t,l,\beta)$ *if*

$$\left|\frac{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}{z}\left(\frac{D^{q,h}_{\mu_1,\mu_2,t,l}f(z)}{z}\right)^{\beta}-1\right|<1-\zeta. \tag{8}$$

Note that inequality (8) *implies that*

$$\Re\left(\frac{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}{z}\left(\frac{D^{q,h}_{\mu_1,\mu_2,t,l}f(z)}{z}\right)^{\beta}\right) > \zeta \quad (0 \leq \zeta < 1),$$

where $z \in \mathcal{U}$.

Remark 2.2 *The family* $\mathcal{G}R^{\mu_1,\mu_2}_{q,\zeta}(h,t,l,\beta)$ *have various classes of holomorphic univalent functions which includes:*

- *Given* $\mu_1 = t = 1$, $\mu_2 = l = q = 0$, $h = 0$ *and* $\beta = -1$, *then*

$$\mathcal{G}R^{1,0}_{0,\zeta}(0,1,0,-1) \equiv S^*(\zeta)$$

- *Given* $\mu_1 = t = 1$, $\mu_2 = l = q = 0$, $h = 1$ *and* $\beta = -1$, *then*

$$\mathcal{G}R^{1,0}_{0,\zeta}(1,1,0,-1) \equiv \mathcal{K}(\zeta)$$

- *Given* $\mu_1 = t = 1$, $\mu_2 = l = q = 0$, $h = 0$ *and* $\beta = 0$, *then*

$$\mathcal{G}R^{1,0}_{0,\zeta}(0,1,0,0) \equiv \mathcal{R}(\zeta)$$

- *Given* $\mu_1 = t = 1$, $\mu_2 = l = q = 0$, $h = 0$ *and* $\beta = -2$, *then we have the class*

$$\mathcal{G}R^{1,0}_{0,\zeta}(0,1,0,-2) \equiv \mathcal{B}(\zeta)$$

  investigated by Frasin and Darus [8].

- *Given* $\mu_1 = t = 1$, $\mu_2 = l = q = 0$ *and* $h = 0$ *then we have the class*

$$\mathcal{G}R^{1,0}_{0,\zeta}(0,1,0,\beta) \equiv \mathcal{B}(\beta,\zeta)$$

  inroduced by Singh [13] *and studied by Babalola* [3].

Theorem 2.3 *Let* $f \in \mathcal{A}$ *be of the form* $f(z) = z + \sum_{\tau=2}^{\infty} a_\tau z^\tau$, $h,q,l \in \mathcal{N}_0$, $\mu_1 \geq \mu_2 \geq 0$, $t \geq 0$, $t+l > 0$, $\beta \geq -2$ *and* $\frac{1}{2} \leq \zeta < 1$, *if*

$$\Re\Bigg(\frac{(t(1+\mu_2(\tau-1))+l)}{t\mu_1}\frac{D^{q,h+2}_{\mu_1,\mu_2,t,l}f(z)}{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)} + \frac{\beta(t(1+\mu_2(\tau-1))+l)}{t\mu_1}\frac{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}{D^{q,h}_{\mu_1,\mu_2,t,l}f(z)} - \frac{(1+\beta)(t(1+\mu_2(\tau-1))+l)}{t\mu_1} + 1\Bigg) > \frac{3\zeta-1}{2\zeta}, \quad (9)$$

then $f(z) \in \mathcal{G}R^{\mu_1,\mu_2}_{q,\zeta}(h,t,l,\beta)$.

Proof. If we denote by

$$\vartheta(z)=\frac{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}{z}\left(\frac{D^{q,h}_{\mu_1,\mu_2,t,l}f(z)}{z}\right)^{\beta}$$

where $\vartheta(z)=1+\vartheta_1 z+\vartheta_2 z+\cdots$, $\vartheta(z)\in\mathcal{W}[1,1]$, by simplification

$$\ln(\vartheta(z))=\ln(D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z))-\ln(z)+\beta\ln(D^{q,h}_{\mu_1,\mu_2,t,l}f(z))-\beta\ln(z)$$

and by simple differentiation it implies that

$$\frac{\vartheta'(z)}{\vartheta(z)}=\frac{(D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z))'}{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}+\frac{\beta(D^{q,h}_{\mu_1,\mu_2,t,l}f(z))'}{D^{q,h}_{\mu_1,\mu_2,t,l}f(z)}-\frac{1}{z}-\frac{\beta}{z}$$

$$\frac{\vartheta'(z)}{\vartheta(z)}=\frac{(t(1+\mu_2(\tau-1))+l)}{zt\mu_1}\frac{D^{q,h+2}_{\mu_1,\mu_2,t,l}f(z)}{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}+\frac{\beta(t(1+\mu_2(\tau-1))+l)}{zt\mu_1}\frac{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}{D^{q,h}_{\mu_1,\mu_2,t,l}f(z)}$$
$$-\frac{\beta(t(1+\mu_2(\tau-1)-\mu_1)+l)}{zt\mu_1}-\frac{(t(1+\mu_2(\tau-1)-\mu_1)+l)}{zt\mu_1}-\frac{1}{z}-\frac{\beta}{z}$$

Multiply through by z,

$$\frac{z\vartheta'(z)}{\vartheta(z)}=\frac{(t(1+\mu_2(\tau-1))+l)}{t\mu_1}\frac{D^{q,h+2}_{\mu_1,\mu_2,t,l}f(z)}{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}+\frac{\beta(t(1+\mu_2(\tau-1))+l)}{t\mu_1}\frac{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}{D^{q,h}_{\mu_1,\mu_2,t,l}f(z)}$$
$$-\frac{\beta(t(1+\mu_2(\tau-1)-\mu_1)+l)}{t\mu_1}-\frac{(t(1+\mu_2(\tau-1)-\mu_1)+l)}{t\mu_1}-1-\beta$$

$$\frac{z\vartheta'(z)}{\vartheta(z)}=\frac{(t(1+\mu_2(\tau-1))+l)}{t\mu_1}\frac{D^{q,h+2}_{\mu_1,\mu_2,t,l}f(z)}{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}+\frac{\beta(t(1+\mu_2(\tau-1))+l)}{t\mu_1}\frac{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}{D^{q,h}_{\mu_1,\mu_2,t,l}f(z)}$$
$$-\frac{(1+\beta)(t(1+\mu_2(\tau-1))+l)}{t\mu_1}$$

$$\frac{z\vartheta'(z)}{\vartheta(z)}=\frac{(t(1+\mu_2(\tau-1))+d)}{t\mu_1}\frac{D^{q,h+2}_{\mu_1,\mu_2,t,l}f(z)}{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}+\frac{\beta(t(1+\mu_2(\tau-1))+l)}{t\mu_1}\frac{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}{D^{q,h}_{\mu_1,\mu_2,t,l}f(z)}$$
$$-\frac{(1+\beta)(t(1+\mu_2(\tau-1))+l)}{t\mu_1}\quad(10)$$

In the interpretation of the Theorem 2.3, it implies that

$$\Re\left(1+\frac{z\vartheta'(z)}{\vartheta(z)}\right)>\frac{3\zeta-1}{2\zeta}$$

Hence, by Lemma 1.4, we have

$$\Re\left(\frac{D^{q,h+1}_{\mu_1,\mu_2,t,l}f(z)}{z}\left(\frac{D^{q,h}_{\mu_1,\mu_2,t,l}f(z)}{z}\right)^{\beta}\right)>\zeta$$

therefore, $f(z) \in \mathcal{G}R^{\mu_1,\mu_2}_{q,\zeta}(h,t,l,\beta)$ by Definition 2.1.

We have the subsequent corollaries as a result of the above theorem.

Choosing $\mu_1 = t = 1$, $\mu_2 = l = q = 0$, $h = 1$, $\beta = -1$ and $\zeta = \frac{1}{2}$, we have

Corollary 2.4 *Suppose $f(z) \in \mathcal{A}$ also*

$$\Re\left(\frac{z^2 f'''(z) + 3zf''(z) + f'(z)}{zf''(z) + f'(z)} - \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2},$$

then

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2},$$

hence $f(z) \in \mathcal{G}R^{1,0}_{0,\frac{1}{2}}(1,1,0,-1) \equiv \mathcal{K}(\frac{1}{2})$, where $z \in \mathcal{U}$.

Choosing $\mu_1 = t = 1$, $\mu_2 = l = q = 0$, $h = 1$, $\beta = 0$ and $\zeta = \frac{1}{2}$, we have

Corollary 2.5 *Suppose $f(z) \in \mathcal{A}$ also*

$$\Re\left(\frac{z^3 f'''(z) + 3z^2 f''(z) + zf'(z)}{z^2 f''(z) + zf'(z)}\right) > \frac{1}{2},$$

then

$$\Re\left[f'(z) + zf''(z)\right] > \frac{1}{2},$$

where $z \in \mathcal{U}$.

Choosing $\mu_1 = t = 1$, $\mu_2 = l = q = 0$, $h = 0$, $\beta = -1$ and $\zeta = \frac{1}{2}$, we have

Corollary 2.6 *Suppose $f(z) \in \mathcal{A}$ also*

$$\Re\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) > -\frac{3}{2},$$

hence $f(z) \in \mathcal{G}R^{1,0}_{0,\frac{1}{2}}(0,1,0,-1) \equiv S^*(\frac{1}{2})$, where $z \in \mathcal{U}$.

Choosing $\mu_1 = t = 1$, $\mu_2 = l = q = 0$, $h = 0$, $\beta = 0$ and $\zeta = \frac{1}{2}$, we have

Corollary 2.7 *Suppose $f(z) \in \mathcal{A}$ also*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2},$$

then

$$\Re\left(f'(z)\right) > \frac{1}{2},$$

hence $f \in \mathcal{G}R^{1,0}_{0,\frac{1}{2}}(0,1,0,0) \equiv \mathcal{R}(\frac{1}{2})$, where $z \in \mathcal{U}$.

Choosing $\mu_1 = t = 1$, $\mu_2 = d = n = 0$, $h = 0$, $\beta = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$, we have

Corollary 2.8 *Suppose $f(z) \in \mathcal{A}$ also*

$$\Re\left(2\left(\frac{zf''(z)}{f'(z)} + 1\right) - \frac{zf'(z)}{f(z)}\right) > 0$$

then

$$\Re\left(\frac{z^{\frac{1}{2}}f'(z)}{f^{\frac{1}{2}}(z)}\right) > \frac{1}{2},$$

hence $f(z)$ is Bazilevic of order $\frac{1}{2}$, type $\frac{1}{2}$ in $\mathcal{U}$, where $z \in \mathcal{U}$.

Choosing $\mu_1 = t = 1$, $\mu_2 = d = n = 0$, $h = 0$, $\beta = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$, we have

Corollary 2.9 *Suppose $f(z) \in \mathcal{A}$ also*

$$\Re\left(\frac{zf'(z)}{f(z)} + 2\left(\frac{zf''(z)}{f'(z)} + 1\right)\right) > 1,$$

then

$$\Re\left(\frac{f^{\frac{1}{2}}(z)f'(z)}{z^{\frac{1}{2}}}\right) > \frac{1}{2},$$

hence $f(z)$ is Bazilevic of order $\frac{1}{2}$, type $\frac{3}{2}$ in $\mathcal{U}$, where $z \in \mathcal{U}$.

3 Open Problem

The open problem is to determine a generic class of univalent functions such has $\mathcal{G}R^{\mu_1,\mu_2}_{q,\zeta}(h,t,l,\beta)$, $h,q,l \in \mathcal{N}_0$, $\mu_1 \geq \mu_2 \geq 0$, $t \geq 0$, $t + l > 0$, $\beta \geq -2$ and $\frac{1}{2} \leq \zeta < 1$ is contained inside and possible to obtain. Compare the new results with the results given by [3].

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