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Multivalently Meromorphic Functions with two Fixed Points Defined by Jackson (i,j)-Derivative

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Abstract

In this paper, we define a new subclass of meromorphic p-valent functions on the punctured unit disk $U^* = \{z \in \mathbb{C}, 0 < |z| < 1\} = U/\{0\}$ *with two fixed points by making use of Jackson* (i,j)*-deravative. Coefficient estimate, distortion bounds, as well as radius of meromorphically p-valent starlikeness are obtained. We also establish some results concerning the convolution products and inclusion results.*

Keywords: *Meromorphic p-valent functions, Hadamard product(convolution), Integral operator, Jackson* (i,j)*-derivative, Arithmetic mean, Weighted mean.*
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1 Introduction

Let $\sum_P$ denote the class of functions of the form

$$f(z) = \frac{a_p}{z^p} + \sum_{n=1}^{\infty} a_n z^{n-p}, (a_n \geq 0; p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1}$$

which are meromorphic and p-valent in the puncture unit disk

$$U^* = \{z \in \mathbb{C}, 0 < |z| < 1\} = U/\{0\}.$$

Let

$$g(z) = \frac{b_p}{z^p} + \sum_{n=1}^{\infty} b_n z^{n-p}, (b_n \geq 0; p \in \mathbb{N} = \{1, 2, 3, ...\}),$$

then the Hadmard product (or convolution) of $f(z)$ and $g(z)$ is defined as

$$f(z)*g(z) = (f*g)(z) = \frac{a_p b_p}{z^p} + \sum_{n=1}^{\infty} a_n b_n z^{n-p}, (a_n, b_n \geq 0; p \in \mathbb{N} = \{1, 2, 3, ...\}).$$

Quantum calculus (q-calculus) has created many interests among the researchers, it has several applications in various branches of mathematics and physics. The application of q-calculus was initiated by Jackson [5, 6] at the beginning 0f 19^{th} century. Later, Chakrabarti and Jagannathan defined Jackson (i, j)-derivative as a generalization of q-derivative (see [4]).

Now, we introduce some definitions and conceptes of $(i, j)-$calculus that were used in this paper by assuming as i and j are fixed number such that $0 < i < j \leq 1$.

The $(i, j)-$derivative for the function $f(z)$ of the form (1) is defined as (see [4])

$$(\partial_{i,j} f)(z) = \frac{f(iz) - f(jz)}{(i-j)z}, z \in U. \tag{2}$$

It is clear that the $(i, j)-$derivative for a function $f(z)$ in $\sum_P$ of the form (1)

$$(\partial_{i,j})f(z) = \frac{-[p]_{i,j}}{i^p j^p} \frac{a_p}{z^{p+1}} + \sum_{n=1}^{\infty} [n-p]_{i,j} a_n z^{n-p-1} \tag{3}$$

where $[n-p]_{i,j}$ denotes twin-basic number defined in [17] by

$$[n-p]_{i,j} = \frac{i^{n-p} - j^{n-p}}{i-j},$$

and

$$\lim_{i\to 1}[n-p]_{i,j} = [n-p]_j = \frac{1-j^{n-p}}{1-j}, j \neq 1.$$

Definition 1 *For $0 \leq \vartheta < 1$, a function $f(z)$ of the form (1) is said to be in the class $\mathcal{M}_{i,j}S^*(\vartheta)$ of starlike function of order ϑ in U^* if it satisfies the condition*

$$\Re\left\{\frac{-z\partial_{i,j}f(z)}{f(z)}\right\} \geq \vartheta. \tag{4}$$

Definition 2 *For $0 \leq \vartheta < 1$, a function $f(z)$ of the form (1) is said to be in the class $\mathcal{M}_{i,j}C^*(\vartheta)$ of convex function of order ϑ in U^* if it satisfies the condition*

$$\Re\left\{-\left(1+\frac{z\partial_{i,j}(\partial_{i,j}f(z))}{\partial_{i,j}f(z)}\right)\right\} \geq \vartheta. \tag{5}$$

For p-valent meromorphic function $f(z) \in \sum_P$, the normalization

$$z^{1+p}f(z)\mid_{z=0}= 0 \qquad \text{and} \qquad z^p f(z)\mid_{z=0}= 1 \tag{6}$$

is classical. One can obtain interesting results by applying Montel's normalization [14] of the form

$$z^{1+p}f(z)\mid_{z=0}= 0 \qquad \text{and} \qquad z^p f(z)\mid_{z=\rho}= 1. \tag{7}$$

Where ρ is a fixed point of the unit disc U^*. We see that if $\rho = 0$ the normalization (7) is the classical normalization (6).

Many important properties of certain subclasses of meromorphic p-valent functions were studied by several authors as Uralegaddi and Ganigi [19], Uralegaddi and Somanatha [20], Mogra et al. [13], Aouf [1], Aouf and Hossen [2], Joshi and Aouf [7], Owa and Pascu [15], Joshi and Srivastava [8],Aouf et al. [3],Raina and Srivastava [16], Yang [23], Kulkarni et al. [9], Liu [10] and Liu and Srivastava [11] and [12].

We define the following new subclass $\mathcal{M}_{i,j}(p,\alpha,\beta)$ of meromorphically p-valent functions in $\sum_P$ by using the $(i,j)-$derivative with Montel's normalization, to study some special properties of $f(z) \in \mathcal{M}_{i,j}(p,\alpha,\beta)$ like coefficient estimate, distortion bounds and radius of meromorphically p-valent starlikeness. We also establish some results concerning the convolution products.

Definition 3 *For $\alpha \geq \frac{1}{2+\beta}$ and $0 \leq \beta < 1$, let $\mathcal{M}_{i,j}(p,\alpha,\beta)$ the multivalently meromorphic function $f(z) \in \sum_P$ with two fixed points (classical normalization) if it satisfies the following*

$$\left|\frac{i^p j^p z(\partial_{i,j} f(z))}{[p]_{i,j} f(z)} + \alpha + \alpha\beta\right| \leq \Re\left\{\frac{-i^p j^p z(\partial_{i,j} f(z))}{[p]_{i,j} f(z)} + \alpha - \alpha\beta\right\}. \quad (8)$$

Example 4 *If $i, j \to 1^-$, and $f(z)$ of the form (1), then we obtain the new class $\mathcal{M}(p,\alpha,\beta)$ defined by*

$$\left|\frac{zf'(z)}{pf(z)} + \alpha + \alpha\beta\right| \leq \Re\left\{\frac{-zf'(z)}{pf(z)} + \alpha - \alpha\beta\right\}$$

which defined in [22].

Further, we can state the subclass $\mathcal{M}_{i,j}(p,\alpha,\beta,\rho)$ satisfying the condition (8) with Montel's normalization (7).

In this section we obtain certain characterization properties for $f(z) \in \mathcal{M}_{i,j}(p,\alpha,\beta)$.

2 Properties of the class $\mathcal{M}_{i,j}(p,\alpha,\beta)$

Theorem 5 *Let $f(z) \in \sum_P$, then $f(z)$ is in the class $\mathcal{M}_{i,j}(p,\alpha,\beta)$ if and only if*

$$\sum_{n=1}^{\infty} d_n |a_n| \leq (1-\alpha\beta)[p]_{i,j} a_p \quad (9)$$

where $d_n = (i^p j^p [n-p]_{i,j} + \alpha\beta[p]_{i,j})$ and $\alpha > \frac{1}{2+\beta}$; $0 \leq \beta < 1$; $p \in \mathbb{N}$.

Proof. Suppose that $f(z) \in \mathcal{M}_{i,j}(p,\alpha,\beta)$, then by the inequality

$$\left|\frac{i^p j^p z(\partial_{i,j} f(z))}{[p]_{i,j} f(z)} + \alpha + \alpha\beta\right| \leq \Re\left\{\frac{-i^p j^p z(\partial_{i,j} f(z))}{[p]_{i,j} f(z)} + \alpha - \alpha\beta\right\}$$

that is,

$$\begin{aligned}\Re\left\{\frac{i^p j^p z(\partial_{i,j} f(z))}{[p]_{i,j} f(z)} + \alpha + \alpha\beta\right\} &\leq \left|\frac{i^p j^p z(\partial_{i,j} f(z))}{[p]_{i,j} f(z)} + \alpha + \alpha\beta\right| \\ &\leq \Re\left\{\frac{-i^p j^p z(\partial_{i,j} f(z))}{[p]_{i,j} f(z)} + \alpha - \alpha\beta\right\}.\end{aligned}$$

$$\Re\left\{\frac{i^p j^p z(\partial_{i,j} f(z))}{[p]_{i,j} f(z)} + \alpha\beta\right\} \leq 0.$$

Substituting for $\partial_{i,j} f(z)$ from (3) and $f(z)$, we get

$$\Re\left(\frac{\frac{-[p]_{i,j}a_p}{z^p} + i^p j^p \sum_{n=1}^{\infty}[n-p]_{i,j} a_n z^{n-p}}{[p]_{i,j}\frac{a_p}{z^p} + \sum_{n=1}^{\infty}[p]_{i,j} a_n z^{n-p}} + \alpha\beta\right) \leq 0.$$

Since $\Re(z) \leq |z|$, we have

$$\left| -[p]_{i,j}a_p + i^p j^p \sum_{n=1}^{\infty}[n-p]_{i,j} a_n z^n + [p]_{i,j}\alpha\beta a_p + [p]_{i,j}\alpha\beta \sum_{n=1}^{\infty} a_n z^n \right| \leq 0$$

and by letting $|z| \to 1^-$, we get

$$\sum_{n=1}^{\infty}(i^p j^p [n-p]_{i,j} + \alpha\beta[p]_{i,j})\,|a_n| \leq (1-\alpha\beta)[p]_{i,j} a_p.$$

Conversely, assume that (9) holds true and from (8), we have

$$\Re\left\{\frac{i^p j^p z(\partial_{i,j} f(z))}{[p]_{i,j} f(z)} + \alpha\beta\right\} \leq 0$$

and

$$\Re\left(\frac{\frac{-[p]_{i,j}a_p}{z^p} + i^p j^p \sum_{n=1}^{\infty}[n-p]_{i,j} a_n z^{n-p}}{[p]_{i,j}\frac{a_p}{z^p} + \sum_{n=1}^{\infty}[p]_{i,j} a_n z^{n-p}} + \alpha\beta\right) \leq 0.$$

Since $\Re(z) \leq |z|$, we have

$$\sum_{n=1}^{\infty}\frac{(i^p j^p [n-p]_{i,j} + \alpha\beta[p]_{i,j})\,|a_n|}{(1-\alpha\beta)[p]_{i,j} a_p} \leq 1,$$

which completes the proof .

For the sake of brevity throughout this paper, we let $d_n = (i^p j^p [n-p]_{i,j} + \alpha\beta[p]_{i,j})$ and $\alpha > \frac{1}{2+\beta}$; $0 \leq \beta < 1$; $p \in \mathbb{N}$, unless otherwise state. ■

Theorem 6 ***(Coefficient Estimate)*** *Let $f(z) \in \mathcal{M}_{i,j}(p, \alpha, \beta)$, then*

$$\sum_{n=1}^{\infty} |a_n| \leq \frac{(1-\alpha\beta)[p]_{i,j} a_p}{d_n}. \tag{10}$$

Remark 7 *If we put $i, j \to 1^-$in Theorem 5, we have the result of [21] when $s = 0$.*

Theorem 8 *Let $f(z) \in \mathcal{M}_{i,j}(p, \alpha, \beta, \rho)$, then*

$$\sum_{n=1}^{\infty} |a_n| \leq \frac{(1-\alpha\beta)[p]_{i,j}}{d_n + (1-\alpha\beta)[p]_{i,j}\rho^n}. \tag{11}$$

Proof. Let $f(z) \in \mathcal{M}_{i,j}(p, \alpha, \beta, \rho)$. Since $f(z) \in \mathcal{M}_{i,j}(p, \alpha, \beta)$ by Theorem 5, we have

$$\sum_{n=1}^{\infty} d_n |a_n| \leq (1-\alpha\beta)[p]_{i,j} a_p.$$

For $f(z) \in \sum_P$, by Montel's normalization (7), we have

$$z^p\left(\frac{a_p}{z^p} + \sum_{n=1}^{\infty} a_n z^{n-p}\right)\Bigg|_{z=\rho} = \left(a_p + \sum_{n=1}^{\infty} a_n z^n\right)\Bigg|_{z=\rho} = 1,$$

and then

$$a_p = 1 - \sum_{n=1}^{\infty} a_n \rho^n.$$

Therefore from (9), we have

$$\sum_{n=1}^{\infty} d_n |a_n| \leq (1-\alpha\beta)[p]_{i,j}\left(1 - \sum_{n=1}^{\infty} a_n \rho^n\right)$$

$$\sum_{n=1}^{\infty} \left[d_n + (1-\alpha\beta)[p]_{i,j}\rho^n\right] |a_n| \leq (1-\alpha\beta)[p]_{i,j}.$$

Hence

$$\sum_{n=1}^{\infty} |a_n| \leq \frac{(1-\alpha\beta)[p]_{i,j}}{[d_n + (1-\alpha\beta)[p]_{i,j}\rho^n]}.$$

■

Theorem 9 ***(Distortion Bounds***) *If* $f(z) \in \mathcal{M}_{i,j}(p,\alpha,\beta,\rho)$, *then*

$$\left(\frac{d_1-(1-\alpha\beta)[p]_{i,j}r}{d_1+(1-\alpha\beta)[p]_{i,j}\rho}\right)r^{-p} \le |f(z)| \le \left(\frac{d_1+(1-\alpha\beta)[p]_{i,j}r}{d_1+(1-\alpha\beta)[p]_{i,j}\rho}\right)r^{-p}, (0<|z|=r<1). \tag{12}$$

Proof. Let $f(z) \in \mathcal{M}_{i,j}(p,\alpha,\beta,\rho)$, then from Theorem 8 we have

$$\sum_{n=1}^{\infty}|a_n| \le \frac{(1-\alpha\beta)[p]_{i,j}}{[d_n+(1-\alpha\beta)[p]_{i,j}\rho^n]}$$

which yields

$$\sum_{n=1}^{\infty}|a_n| \le \frac{(1-\alpha\beta)[p]_{i,j}}{[d_1+(1-\alpha\beta)[p]_{i,j}\rho]}.$$

From (1), we have

$$|f(z)| =| a_p z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} | \tag{13}$$

by using Montel's normalization (7), we get

$$a_p = 1-\sum_{n=1}^{\infty} a_n\rho^n \tag{14}$$

from (13) and (14),

$$\begin{aligned} |f(z)| &\le \left(1-\sum_{n=1}^{\infty}|a_n|\rho^n+\sum_{n=1}^{\infty}|a_n|r^n\right)r^{-p} \\ &\le \left(1-(\rho-r)\sum_{n=1}^{\infty}|a_n|\right)r^{-p} \\ &\le \left(\frac{d_1+(1-\alpha\beta)[p]_{i,j}r}{d_1+(1-\alpha\beta)[p]_{i,j}\rho}\right)r^{-p}. \end{aligned}$$

On the other hand we have

$$|f(z)| \ge \left(\frac{d_1-(1-\alpha\beta)[p]_{i,j}r}{d_1+(1-\alpha\beta)[p]_{i,j}\rho}\right)r^{-p},$$

which completes the proof. ■

By using classical normalization, (that is by taking $\rho = 0$) we can state the following distortion result without proof.

Theorem 10 *If $f(z) \in \mathcal{M}_{i,j}(p, \alpha, \beta)$, then*

$$\left(1 - \frac{(1-\alpha\beta)[p]_{i,j}r}{d_1}\right) r^{-p} \le |f(z)| \le \left(1 + \frac{(1-\alpha\beta)[p]_{i,j}r}{d_1}\right) r^{-p},$$

$(0 \le |z| = r < 1)$.

3 Radius of Meromorphically p-valent Starlikeness

Theorem 11 *Let the function $f(z)$ defined by (1) be in the class $\mathcal{M}_{i,j}(p, \alpha, \beta)$, then we have $f(z)$ is meromorphically p-valent starlike of order $\gamma (0 \le \gamma < p)$ in the disk $|z| < r_1$, that is,*

$$\Re\left(\frac{-zf'(z)}{f(z)}\right) > \gamma, \qquad |z| < r_1; 0 \le \gamma < p; p \in \mathbb{N},$$

where

$$|z| \le \left\{\frac{d_n(p-\gamma)}{(n-p+\gamma)(1-\alpha\beta)[p]_{i,j}}\right\}^{\frac{1}{n}}. \tag{15}$$

Proof. Let $f(z) = \frac{a_p}{z^p} + \sum\limits_{n=1}^{\infty} a_n z^{n-p}$. Then we can get

$$\left|\frac{\frac{zf'(z)}{f(z)} + p}{\frac{zf'(z)}{f(z)} - p + 2\gamma}\right| \le \frac{\sum\limits_{n=1}^{\infty} n a_n |z|^n}{2(p-\gamma)a_p + \sum\limits_{n=1}^{\infty}(n-2p+2\gamma)a_n |z|^n}.$$

Thus, we have the desired inequality

$$\left|\frac{\frac{zf'(z)}{f(z)} + p}{\frac{zf'(z)}{f(z)} - p + 2\gamma}\right| \le 1, \quad \text{if,} \quad \sum_{n=1}^{\infty} \frac{(n-p+\gamma)}{|p-\gamma| a_p} |a_n| |z|^n \le 1. \tag{16}$$

Since $f(z) \in \mathcal{M}_{i,j}(p, \alpha, \beta)$ from Theorem 5, we have

$$\sum_{n=1}^{\infty} \frac{d_n}{(1-\alpha\beta)[p]_{i,j} a_p} |a_n| \le 1. \tag{17}$$

From (16) and (17)

$$\frac{(n-p+\gamma)}{|p-\gamma|\,a_p}\,|z|^n \leq \frac{d_n}{(1-\alpha\beta)[p]_{i,j}a_p}$$

$$|z| \leq \left\{\frac{d_n(p-\gamma)}{(n-p+\gamma)(1-\alpha\beta)[p]_{i,j}}\right\}^{\frac{1}{n}}.$$

Hence the proof. ■

4 Convolution Properties

For functions

$$f_k(z) = a_{p,k}z^{-p} + \sum_{n=1}^{\infty} |a_{n,k}|\, z^{n-p}, \qquad (k=1,2; p \in \mathbb{N}) \tag{18}$$

we define the Hadamard product or convolution of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = a_{p,1}a_{p,2}z^{-p} + \sum_{n=1}^{\infty} |a_{n,1}|\,|a_{n,2}|\, z^{n-p}. \tag{19}$$

Theorem 12 *For functions $f_k(k=1,2)$ defined by (18) be in the class $\mathcal{M}_{i,j}(p,\alpha,\beta)$. Then $(f_1 * f_2)(z) \in \mathcal{M}_{i,j}(p,\alpha,\delta)$ where*

$$\delta \leq \frac{1}{\alpha}\left(1 - \frac{[p]_{i,j}(1-\alpha\beta)^2\left\{[p]_{i,j} + i^p j^p[1-p]_{i,j}\right\}}{d_1^2 + [p]_{i,j}^2(1-\alpha\beta)^2}\right)$$

where $d_1 = (i^p j^p[1-p]_{i,j} + \alpha\beta[p]_{i,j})$.

Proof. Let $f_1(z) = a_{p,1}z^{-p} + \sum_{n=1}^{\infty} |a_{n,1}|\, z^{n-p}$ and $f_2(z) = a_{p,2}z^{-p} + \sum_{n=1}^{\infty} |a_{n,2}|\, z^{n-p}$ be in the class $\mathcal{M}_{i,j}(p,\alpha,\beta)$. Then by Theorem 5, we have

$$\sum_{n=1}^{\infty} \frac{d_n}{(1-\alpha\beta)[p]_{i,j}a_{p,1}}\,|a_{n,1}| \leq 1,$$

$$\sum_{n=1}^{\infty} \frac{d_n}{(1-\alpha\beta)[p]_{i,j}a_{p,2}}\,|a_{n,2}| \leq 1.$$

Employing the technique used earlier by Schild and Silverman [18], we need to find smallest δ such that

$$\sum_{n=1}^{\infty}\frac{(i^p j^p[n-p]_{i,j}+\alpha\delta[p]_{i,j})}{(1-\alpha\delta)[p]_{i,jq}a_{p,1}a_{p,2}}\left|a_{n,1}\right|\left|a_{n,2}\right|\leq 1. \tag{20}$$

By Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty}\frac{d_n}{(1-\alpha\beta)[p]_{i,j}\sqrt{a_{p,1}a_{p,2}}}\sqrt{\left|a_{n,1}\right|\left|a_{n,2}\right|}\leq 1 \tag{21}$$

then

$$\frac{(i^p j^p[n-p]_{i,j}+\alpha\delta[p]_{i,j})}{(1-\alpha\delta)[p]_{i,j}a_{p,1}a_{p,2}}\left|a_{n,1}\right|\left|a_{n,2}\right|\leq\frac{d_n}{(1-\alpha\beta)[p]_{i,j}\sqrt{a_{p,1}a_{p,2}}}\sqrt{\left|a_{n,1}\right|\left|a_{n,2}\right|}. \tag{22}$$

Hence

$$\sqrt{\left|a_{n,1}\right|\left|a_{n,2}\right|}\leq\frac{d_n(1-\alpha\delta)\sqrt{a_{p,1}a_{p,2}}}{(1-\alpha\beta)(i^p j^p[n-p]_{i,j}+\alpha\delta[p]_{i,j})}. \tag{23}$$

From (21) we know that

$$\sqrt{\left|a_{n,1}\right|\left|a_{n,2}\right|}\leq\frac{(1-\alpha\beta)[p]_{i,j}\sqrt{a_{p,1}a_{p,2}}}{d_n}. \tag{24}$$

From (23) and (24)

$$\frac{(1-\alpha\beta)[p]_{i,j}\sqrt{a_{p,1}a_{p,2}}}{d_n}\leq\frac{d_n(1-\alpha\delta)\sqrt{a_{p,1}a_{p,2}}}{(1-\alpha\beta)(i^p j^p[n-p]_{i,j}+\alpha\delta[p]_{i,j})}.$$

It follows that

$$\delta\leq\frac{1}{\alpha}\left(1-\frac{[p]_{i,j}(1-\alpha\beta)^2\left\{[p]_{i,j}+i^p j^p[n-p]_{i,j}\right\}}{d_n^2+[p]_{i,j}^2(1-\alpha\beta)^2}\right).$$

Now defining a function $\Psi(n)$ by

$$\Psi(n)=\frac{1}{\alpha}\left(1-\frac{[p]_{i,j}(1-\alpha\beta)^2\left\{[p]_{i,j}+i^p j^p[n-p]_{i,j}\right\}}{d_n^2+[p]_{i,j}^2(1-\alpha\beta)^2}\right),(n\geq 1),$$

we observe that $\Psi(n)$ is an increasing function of n. We thus, conclude that

$$\delta=\Psi(1)\leq\frac{1}{\alpha}\left(1-\frac{[p]_{i,j}(1-\alpha\beta)^2\left\{[p]_{i,j}+i^p j^p[1-p]_{i,j}\right\}}{d_1^2+[p]_{i,j}^2(1-\alpha\beta)^2}\right),$$

which completes the proof. ■

Theorem 13 *For functions* $f_1(z) \in \mathcal{M}_{i,j}(p,\alpha,\beta)$ *and* $f_2(z) \in \mathcal{M}_{i,j}(p,\alpha,\gamma)$, *then* $(f_1 * f_2)(z) \in \mathcal{M}_{i,j}(p,\alpha,\zeta)$, *where*

$$\zeta \leq \frac{1}{\alpha}\left[\frac{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma) - i^p j^p [p]_{i,j}[n-p]_{i,j}(1-\alpha\beta)(1-\alpha\gamma)}{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma) + [p]_{i,j}^2(1-\alpha\beta)(1-\alpha\gamma)}\right]$$

where

$d_n(p,\alpha,\beta) = (i^p j^p [n-p]_{i,j} + \alpha\beta[p]_{i,j})$,
$d_n(p,\alpha,\gamma) = (i^p j^p [n-p]_{i,j} + \alpha\gamma[p]_{i,j})$

and

$d_n(p,\alpha,\zeta) = (i^p j^p [n-p]_{i,j} + \alpha\zeta[p]_{i,j})$.

Proof. For the functions

$$f_1(z) = a_{p,1}z^{-p} + \sum_{n=1}^{\infty} |a_{n,1}|\, z^{n-p} \in \mathcal{M}_{i,j}(p,\alpha,\beta)$$

and

$$f_2(z) = a_{p,2}z^{-p} + \sum_{n=1}^{\infty} |a_{n,2}|\, z^{n-p} \in \mathcal{M}_{i,j}(p,\alpha,\gamma),$$

then by Theorem 5, we have

$$\sum_{n=1}^{\infty} \frac{d_n(p,\alpha,\beta)}{(1-\alpha\beta)[p]_{i,j}a_{p,1}}\, |a_{n,1}| \leq 1 \tag{25}$$

and

$$\sum_{n=1}^{\infty} \frac{d_n(p,\alpha,\gamma)}{(1-\alpha\gamma)[p]_{i,j}a_{p,2}}\, |a_{n,2}| \leq 1, \tag{26}$$

where

$d_n(p,\alpha,\beta) = (i^p j^p [n-p]_{i,j} + \alpha\beta[p]_{i,j})$ and $d_n(p,\alpha,\gamma) = (i^p j^p [n-p]_{i,j} + \alpha\gamma[p]_{i,j})$.

Since $(f_1 * f_2)(z) \in \mathcal{M}_{i,j}(p,\alpha,\zeta)$ and by Theorem 5, we have

$$\sum_{n=1}^{\infty} \frac{d_n(p,\alpha,\zeta)}{(1-\alpha\zeta)[p]_{i,j}a_{p,1}a_{p,2}}\, |a_{n,1}|\, |a_{n,2}| \leq 1. \tag{27}$$

Applying Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{\sqrt{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)}}{[p]_{i,j}\sqrt{(1-\alpha\beta)(1-\alpha\gamma)a_{p,1}a_{p,2}}}\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq 1. \tag{28}$$

From (27) and (28), we have

$$\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq \frac{\sqrt{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}} \frac{(1-\alpha\zeta)}{d_n(p,\alpha,\zeta)} \sqrt{a_{p,1}a_{p,2}}. \tag{29}$$

We know that

$$\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq \frac{[p]_{i,j}\sqrt{(1-\alpha\beta)(1-\alpha\gamma)a_{p,1}a_{p,2}}}{\sqrt{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)}}. \tag{30}$$

From (29) and (4.13), we get

$$\frac{[p]_{i,j}\sqrt{(1-\alpha\beta)(1-\alpha\gamma)a_{p,1}a_{p,2}}}{\sqrt{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)}} \leq \frac{\sqrt{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}} \frac{(1-\alpha\zeta)}{d_n(p,\alpha,\zeta)} \sqrt{a_{p,1}a_{p,2}}$$

$$\zeta \leq \frac{1}{\alpha}\left[\frac{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma) - i^p j^p [p]_{i,j}[n-p]_{i,j}(1-\alpha\beta)(1-\alpha\gamma)}{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma) + [p]_{i,j}^2(1-\alpha\beta)(1-\alpha\gamma)}\right].$$

Now defining a function $\Psi(n)$ by

$$\Psi(n) = \frac{1}{\alpha}\left[\frac{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma) - i^p j^p [p]_{i,j}[n-p]_{i,j}(1-\alpha\beta)(1-\alpha\gamma)}{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma) + [p]_{i,j}^2(1-\alpha\beta)(1-\alpha\gamma)}\right],$$

for $n \geq 1$, we observe that $\Psi(n)$ is an increasing function of n. We thus, conclude that

$$\zeta \leq \frac{1}{\alpha}\left[\frac{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma) - i^p j^p [p]_{i,j}[n-p]_{i,j}(1-\alpha\beta)(1-\alpha\gamma)}{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma) + [p]_{i,j}^2(1-\alpha\beta)(1-\alpha\gamma)}\right],$$

which completes the proof. ■

Theorem 14 *Let the functions $f_k(z)(k=1,2)$ defined by (18) be in the class $\mathcal{M}_{i,j}(p,\alpha,\beta,\rho)$. Then the function $H(z)$ defined by*

$$H(z) = (a_{p,1}+a_{p,2})z^{-p} + \sum_{n=1}^{\infty}(|a_{n,1}|^2 + |a_{n,2}|^2)z^{n-p}$$

is in the class $\mathcal{M}_{i,j}(p,\alpha,\gamma,\rho)$, where

$$\gamma \leq \frac{1}{\alpha}\left(\frac{c_1^2 - 2(1-\alpha\beta)^2[p]_{i,j}^2\rho^n - 2i^p j^p[1-p]_{i,j}(1-\alpha\beta)^2[p]_{i,j}}{c_1^2 + 2(1-\alpha\beta)^2[p]_{i,j}^2 - 2(1-\alpha\beta)^2[p]_{i,j}^2\rho}\right).$$

Proof. Note that

$$\sum_{n=1}^{\infty}\left[\frac{C_n}{(1-\alpha\beta)[p]_{i,j}}\right]^2|a_{n,k}|^2 \leq \sum_{n=1}^{\infty}\left[\frac{C_n}{(1-\alpha\beta)[p]_{i,j}}|a_{n,k}|\right]^2 \leq 1, (k=1,2),$$

where $C_n = d_n + (1-\alpha\beta)[p]_{i,j}\rho^n$.

For $f_k(z) \in \mathcal{M}_{i,j}(p,\alpha,\beta,\rho)(k=1,2)$, we have

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{C_n}{(1-\alpha\beta)[p]_{i,j}}\right]^2(|a_{n,1}|^2+|a_{n,2}|^2) \leq 1. \tag{31}$$

Therefore we have to find the largest γ such that

$$\sum_{n=1}^{\infty}\left[\frac{[n-p]_{i,j}i^pj^p+\alpha\gamma[p]_{i,j}+(1-\alpha\gamma)[p]_{i,j}\rho^n}{(1-\alpha\gamma)[p]_{i,j}}\right](|a_{n,1}|^2+|a_{n,2}|^2) \leq 1. \tag{32}$$

From (31) and (32) we get

$$\left[\frac{[n-p]_{i,j}i^pj^p+\alpha\gamma[p]_{i,j}+(1-\alpha\gamma)[p]_{i,j}\rho^n}{(1-\alpha\gamma)[p]_{i,j}}\right] \leq \frac{1}{2}\left[\frac{C_n}{(1-\alpha\beta)[p]_{i,j}}\right]^2, (n \geq 1).$$

$$\gamma \leq \frac{1}{\alpha}\left(\frac{c_n^2-2(1-\alpha\beta)^2[p]_{i,j}^2\rho^n-2i^pj^p[n-p]_{i,j}(1-\alpha\beta)^2[p]_{i,j}}{c_n^2+2(1-\alpha\beta)^2[p]_{i,j}^2-2(1-\alpha\beta)^2[p]_{i,j}^2\rho^n}\right), (n \geq 1).$$

Now defining a function $\Psi(n)$ by

$$\Psi(n) = \frac{1}{\alpha}\left(\frac{c_n^2-2(1-\alpha\beta)^2[p]_{i,j}([p]_{i,j}\rho^n+i^pj^p[n-p]_{i,j})}{c_n^2+2(1-\alpha\beta)^2[p]_{i,j}^2(1-\rho^n)}\right), (n \geq 1)$$

we observe that $\Psi(n)$ is an increasing function of n. We thus, conclude that

$$\gamma \leq \frac{1}{\alpha}\left(\frac{c_1^2-2(1-\alpha\beta)^2[p]_{i,j}^2\rho^n-2i^pj^p[1-p]_{i,j}(1-\alpha\beta)^2[p]_{i,j}}{c_1^2+2(1-\alpha\beta)^2[p]_{i,j}^2-2(1-\alpha\beta)^2[p]_{i,j}^2\rho}\right).$$

∎

5 Closure Properties

Theorem 15 *Let the function $f(z)$ given by (1) be in $\mathcal{M}_{i,j}(p,\alpha,\beta)$. Then the integral operator*

$$F(z) = c\int_0^1 u^{c+p-1} f(uz)du, (0 < u \leq 1, 0 < c < \infty),$$

is in $\mathcal{M}_{i,j}(p,\alpha,\delta)$, where

$$\delta = \frac{1}{\alpha}\left(\frac{(n+c)d_n - ci^p j^p [n-p]_{i,j}(1-\alpha\beta)}{(n+c)d_n + c[p]_{i,j}(1-\alpha\beta)}\right). \tag{33}$$

Proof. Let $f(z) \in \mathcal{M}_{i,j}(p,\alpha,\beta)$. Then

$$\begin{aligned} F(z) &= c\int_0^1 u^{c+p-1} f(uz)du \\ &= c\int_0^1 u^{c+p-1}\left(\frac{a_p}{(uz)^p} + \sum_{n=1}^{\infty} a_n (uz)^{n-p}\right)du \\ &= \frac{a_p}{z^p} + \sum_{n=1}^{\infty}(\frac{c}{n+c})a_n z^{n-p}. \end{aligned}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{c(i^p j^p [n-p]_{i,j} + \alpha\delta[p]_{i,j})}{(n+c)(1-\alpha\delta)[p]_{i,j}a_p} a_n \leq 1. \tag{34}$$

Since $f(z) \in \mathcal{M}_{i,j}(p,\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty}\frac{(i^p j^p [n-p]_{i,j} + \alpha\beta[p]_{i,j})}{(1-\alpha\beta)[p]_{i,j}a_p} a_n \leq 1.$$

Note that (34) is satisfied if

$$\frac{c(i^p j^p [n-p]_{i,j} + \alpha\delta[p]_{i,j})}{(n+c)(1-\alpha\delta)[p]_{i,j}a_p} \leq \frac{(i^p j^p [n-p]_{i,j} + \alpha\beta[p]_{i,j})}{(1-\alpha\beta)[p]_{i,j}a_p}$$

or

$$c(i^p j^p [n-p]_{i,j} + \alpha\delta[p]_{i,j})(1-\alpha\beta) \leq (n+c)(i^p j^p [n-p]_{i,j} + \alpha\beta[p]_{i,j})(1-\alpha\delta).$$

Solving for δ, we have

$$\delta \leq \frac{1}{\alpha}\left(\frac{(n+c)d_n - ci^p j^p[n-p]_{i,j}(1-\alpha\beta)}{(n+c)d_n + c[p]_{i,j}(1-\alpha\beta)}\right).$$

■

Theorem 16 *Let $f(z)$ given by (1), be in $\mathcal{M}_{i,j}(p,\alpha,\beta)$. Then*

$$F(z) = \frac{1}{c}\{(c+p)f(z) + zf'(z)\} = \frac{a_p}{z^p} + \sum_{n=1}^{\infty}\frac{c+n}{c}a_n z^{n-p}, \quad c > 0,$$

is in $\mathcal{M}_{i,j}(p,\alpha,\beta)$ for $|z| \leq r(p,\alpha,\beta,\delta)$, where

$$r(p,\alpha,\beta,\delta) = \inf_n \left(\frac{c(1-\alpha\delta)(i^p j^p[n-p]_{i,j} + \alpha\beta[p]_{i,j})}{(c+n)(1-\alpha\beta)(i^p j^p[n-p]_{i,j} + \alpha\delta[p]_{i,j})}\right)^{\frac{1}{n}}, n = 1,2,3,.... \tag{35}$$

Proof. Let $w = \left\{\frac{-i^p j^p z(\partial_{i,j} f(z))}{\alpha[p]_{i,j} f(z)}\right\}$. Then it is sufficient to show that

$$\left|\frac{w+p}{w-p+2\delta}\right| < 1.$$

A computation shows that this is satisfied if

$$\sum_{n=1}^{\infty}\left(\frac{c+n}{c}\right)\frac{(i^p j^p[n-p]_q + \alpha\delta[p]_q)}{(1-\alpha\delta)[p]_{i,j}a_p}a_n |z|^n \leq 1. \tag{36}$$

Since $f(z) \in \mathcal{M}_{i,j}(p,\alpha,\beta)$, by Theorem 5, we have

$$\sum_{n=1}^{\infty}\frac{(i^p j^p[n-p]_{i,j} + \alpha\beta[p]_{i,j})}{(1-\alpha\beta)[p]_{i,j}a_p}|a_n| \leq 1.$$

The equation (36) is satisfied if

$$\sum_{n=1}^{\infty}\left(\frac{c+n}{c}\right)\frac{(i^p j^p[n-p]_{i,j} + \alpha\delta[p]_{i,j})}{(1-\alpha\delta)[p]_{i,j}a_p}|z|^n \leq \sum_{n=1}^{\infty}\frac{(i^p j^p[n-p]_{i,j} + \alpha\beta[p]_{i,j})}{(1-\alpha\beta)[p]_{i,j}a_p}.$$

A simple computation yields, the inequality asserted in equation (35). ■

Theorem 17 ***(Arithmetic Mean)*** *Let $f_k(z)k = 1, 2, ..., \mu)$ defined by*

$$f_k(z) = \frac{a_{p,k}}{z^p} + \sum_{n=1}^{\infty} a_{n,k} z^{n-p}, \qquad (a_{n,k} \geq 0, k = 1, 2, ..., \mu, n \geq 1)$$

be in the class $\mathcal{M}_{i,j}(p, \alpha, \beta, \rho)$. Then the arithmetic mean of $f_k(z)(k = 1, 2, ..., \mu)$ defined by

$$g(z) = \frac{1}{\mu} \sum_{n=1}^{\mu} f_k(z)$$

is also in the class $\mathcal{M}_{i,j}(p, \alpha, \beta, \rho)$.

Proof. Since $f_k(z) \in \mathcal{M}_{i,j}(p, \alpha, \beta, \rho)(k = 1, 2, ..., \mu)$, then by using Theorem 8, we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} [d_n + (1 - \alpha\beta)[p]_{i,j} \rho^n] \left(\frac{1}{\mu} \sum_{k=1}^{\mu} a_{n,k} \right) \\
= & \frac{1}{\mu} \sum_{k=1}^{\mu} \left(\sum_{n=1}^{\infty} [d_n + (1 - \alpha\beta)[p]_{i,j} \rho^n] \, a_{n,k} \right) \\
\leq & \frac{1}{\mu} \sum_{k=1}^{\mu} (1 - \alpha\beta)[p]_{i,j} \\
\leq & (1 - \alpha\beta)[p]_{i,j}
\end{aligned}$$

which in view of Theorem 8, again implies that $g(z) \in \mathcal{M}_{i,j}(p, \alpha, \beta, \rho)$ and so the proof is complete. ■

Theorem 18 ***(Weighted Mean)*** *Let $f_k(z)(k = 1, 2)$ defined by*

$$f_k(z) = \frac{a_{p,k}}{z^p} + \sum_{n=1}^{\infty} a_{n,k} z^{n-p}, \qquad (a_{n,k} \geq 0, k = 1, 2)$$

be in the class $\mathcal{M}_{i,j}(p, \alpha, \beta, \rho)$. Then the weighted mean of $f_k(z)(k = 1, 2)$ defined by

$$W_c(z) = \frac{1}{2} [(1 - c) f_1(z) + (1 + c) f_2(z)] \tag{37}$$

is also in the class $\mathcal{M}_{i,j}(p, \alpha, \beta, \rho)$.

Proof. Since

$$f_k(z) = \frac{a_{p,k}}{z^p} + \sum_{n=1}^{\infty} a_{n,k} z^{n-p} \in \mathcal{M}_{i,j}(p, \alpha, \beta, \rho)$$

for $(a_{n,k} \geq 0, k = 1, 2)$ and by (37) we have,

$$W_c(z) = (a_{p,1} + a_{p,2}) z^{-p} + \sum_{n=1}^{\infty} \frac{1}{2} \left[(1-c) a_{n,1} + (1+c) a_{n,2}\right] z^{n-p}.$$

From Theorem 8,

$$\sum_{n=1}^{\infty} \frac{(d_n + (1-\alpha\beta)[p]_{i,j} \rho^n)}{(1-\alpha\beta)[p]_{i,j}} |a_{n,1}| \leq 1 \tag{38}$$

and

$$\sum_{n=1}^{\infty} \frac{(d_n + (1-\alpha\beta)[p]_{i,j} \rho^n)}{(1-\alpha\beta)[p]_{i,j}} |a_{n,2}| \leq 1. \tag{39}$$

By using (38) and (39) in (37), we have

$$\begin{aligned} W_c(z) &= \frac{1}{2}(1-c)(1-\alpha\beta)[p]_{i,j} + \frac{1}{2}(1+c)(1-\alpha\beta)[p]_{i,j} \\ &\leq (1-\alpha\beta)[p]_{i,j}. \end{aligned}$$

Therefore $W_c(z) \in \mathcal{M}_{i,j}(p, \alpha, \beta, \rho)$, which completes the proof. ∎

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