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New uncertainty principles for the Dunkl wavelet transform

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Abstract

After reviewing the Dunkl Pitt and the Dunkl Beckner inequalities we connect both the inequalities to show a generalization of uncertainty principles for the Dunkl wavelet transform. Next we present two concentration uncertainty principles such as Benedick-Amrein-Berthier's uncertainty principle and local uncertainty principle. Finally, we study the Dunkl logarithmic Sobolev inequalities. Obtaining best possible constants of inequalities, we connect the inequalities to show a generalization of the uncertainty principles of Heisenberg type.

Keywords: *Dunkl transform, Dunkl wavelet transform, Dunkl Pitt's inequality, Dunkl Beckner's inequality, Dunkl logarithm Sobolev inequality, Dunkl Benedick-Amrein-Berthier's uncertainty principle, Local uncertainty principle*
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1 Introduction

We consider the differential-difference operators T_j, $j = 1, 2, ..., d$, associated with a root system $\mathcal{R}$ and a multiplicity function k, introduced by Dunkl in [15], and called the Dunkl operators in the literature.

The Dunkl theory is based on the Dunkl kernel $K(\lambda, .), \lambda \in \mathbb{C}^d$, which is the unique analytic solution of the system

$$T_j u(x) = \lambda_j u(x), \quad j = 1, 2, ..., d,$$

satisfying the normalizing condition $u(0) = 1$.

With the kernel $K(\lambda, .)$, Dunkl have defined in [16] the Dunkl transform $\mathcal{F}_D$. For a family of weighted functions, ω_k, invariant under a finite reflection group W, Dunkl transform is an extension of the Fourier transform that defines an isometry of $L^2(\mathbb{R}^d, \omega_k(x)dx)$ onto itself. The basic properties of the Dunkl transforms have been studied by several authors, see [14, 15, 16, 60] and the references therein.

Very recently, many authors have been investigating the behavior of the Dunkl transform to several problems already studied for the Fourier transform; for instance, Babenko inequality [7], uncertainty principles [8, 31], real Paley-Wiener theorems [37], heat equation [49], Dunkl Gabor transform [34, 38, 40], Dunkl wavelet transform [61], and so on.

In the classical setting, the notion of wavelets was first introduced by Morlet, a French petroleum engineer at ELF-Aquitaine, in connection with his study of seismic traces. The mathematical foundations were given by Grossmann and Morlet in [24]. The harmonic analyst Meyer and many other mathematicians became aware of this theory and they recognized many classical results inside it (see [9, 32, 44, 59]). Classical wavelets have wide applications, ranging from signal analysis in geophysics and acoustics to quantum theory and pure mathematics (see [12, 26] and the references therein).

Next, the theory of wavelets and continuous wavelet transform has been extended in the context of the Dunkl seeting (see [61]).

This paper is a continuation of the papers [34, 39] in the study of the quantitative uncertainty principles for the Dunkl wavelet transform on $\mathbb{R}^d$. In the classical setting, the notion of the quantitative uncertainty principles for the wavelet transform was first introduced by Wilczok [63]. Next, this subject has been extended for the generalized wavelet transforms (see [4, 5, 34, 48] and others).

Very recently, many authors have been investigating the behavior of the Dunkl wavelet transform to several problems already studied for the classical wavelet transform; for instance, Uncertainty principles [22, 34], Localization theory [39], Reproducing kernel theory [56], and so on.

We recall that the classical quantitative uncertainty principles is just another name for some special inequalities. These inequalities give us information about how a function and its Fourier transform relate. They are called uncertainty principles since they are similar to the classical Heisenberg uncertainty principle, which has had a big part to play in the development and understanding of quantum physics.

The quantitative uncertainty principles have been studied by many authors for various Fourier transforms, for examples (cf. [3, 31, 33, 62]) and others.

To date, several generalizations, modifications and variations of the harmonic based uncertainty principles have appeared in the open literature, for instance, the logarithmic uncertainty principles, Benedick's uncertainty principle, Amrein's and Berthier's uncertainty principles, local uncertainty principles and much more [2, 6, 17, 18, 19, 28, 45, 46, 47, 53, 54, 55]. Thus, it is therefore interesting and worthwhile to investigate these kinds of uncertainty principles for the Dunkl wavelet transforms in arbitrary space dimensions.

The aim of this article is to formulate some novel uncertainty principles for the Dunkl wavelet transform. Firstly, we derive an analogue of Pitt's inequality for the Dunkl wavelet transform, then we formulate Beckner's uncertainty principle for this transform via two approaches: one based on a sharp estimate from Dunkl Pitt's inequality and the other from Dunkl Beckner's inequality. Secondly, we consider the logarithmic Sobolev inequalities for the Dunkl wavelet transforms which has a dual relation with Beckner's inequality. Thirdly, we derive Benedick-Amrein-Berthier's uncertainty principle for the Dunkl wavelet transforms which shows that it is impossible for a non-trivial function and its Dunkl wavelet transform to be both supported on sets of finite measure. Towards the culmination, we formulate local uncertainty principles for the continuous Dunkl wavelet transforms in arbitrary space dimensions.

The remaining part of the paper is organized as follows. In §2 we recall the main results about the harmonic analysis associated with the Dunkl operators. The §3 is devoted to proving an analogue of the Pitt inequality for the Dunkl wavelet transform. In §4 we derive the Beckner uncertainty principle for this transform. In §5 we present two concentration uncertainty principles for the Dunkl wavelet transform such as Benedick-Amrein-Berthier's uncertainty principle and local uncertainty principle. The last Section is devoted to proving the Dunkl logarithm Sobolev uncertainty principles for the Dunkl wavelet transform.

2 Preliminaries

This section gives an introduction to the Dunkl theory. Main references are [14, 15, 16, 50, 58, 60].

2.1 The Dunkl operators

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle , \rangle$ for which the basis $\{e_i,\ i = 1, ..., d\}$ is orthogonal and $||x|| = \sqrt{\langle x, x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{||\alpha||^2}\alpha. \tag{2.1}$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $\sigma_\alpha(R) = R$ for all $\alpha \in R$. For a given root system R the reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group associated with R.

We fix a positive root system $R_+ = \Big\{\alpha \in R\ : \langle \alpha, \beta\rangle > 0\Big\}$ for some $\beta \in \mathbb{R}^d\backslash \bigcup_{\alpha\in R} H_\alpha$.

We will assume that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R_+$. A function $k : \mathcal{R} \longrightarrow [0, \infty)$ is called a multiplicity function if it is invariant under the action of the associated reflection group W. For abbreviation, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha\in R_+} k(\alpha). \tag{2.2}$$

Moreover, let ω_k denotes the weight function

$$\omega_k(x) = \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}, \tag{2.3}$$

which is W-invariant and homogeneous of degree 2γ. We introduce the Mehta-type constant

$$c_k = \int_{\mathbb{R}^d} e^{-\frac{||x||^2}{2}} \omega_k(x)\, dx. \tag{2.4}$$

In the following we denote by

$C^p(\mathbb{R}^d)$ the space of functions of class C^p on $\mathbb{R}^d$.

$\mathcal{E}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$.

$\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.

$D(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$ which are of compact support.

$\mathcal{S}'(\mathbb{R}^d)$ the topological dual of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

The Dunkl operators $T_j,\ j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given by

$$T_j f(x) := \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad f \in C^1(\mathbb{R}^d), \tag{2.5}$$

where $\alpha_j = \langle \alpha, e_j\rangle$.

We define the Dunkl-Laplacian operator $\triangle_k$ on $\mathbb{R}^d$ by

$$\triangle_k f(x) := \sum_{j=1}^d T_j^2 f(x) = \triangle f(x) + 2 \sum_{\alpha \in R^+} k(\alpha)\Big(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}\Big),$$

where $\triangle$ and ∇ are the usual Euclidean Laplacian and the gardient operators on $\mathbb{R}^d$ respectively.

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x,y) &= y_j u(x,y), \quad j = 1, ..., d, \\ u(0,y) &= 1, \end{cases} \tag{2.6}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $K(x,y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

The function $K(x,z)$ admits for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ the following Laplace type integral representation

$$K(x,z) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle} d\mu_x(y), \tag{2.7}$$

where μ_x is the positive probability measure on $\mathbb{R}^d$, with support in the closed ball $B_d(0, ||x||)$ of center 0 and radius $||x||$.

2.2 The Dunkl transform

Notation. We denote by $L^p_k(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$\begin{aligned} ||f||_{L^p_k(\mathbb{R}^d)} &:= \Big(\int_{\mathbb{R}^d} |f(x)|^p \, d\gamma_k(x)\Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \le p < \infty, \\ ||f||_{L^\infty_k(\mathbb{R}^d)} &:= \operatorname{ess}\sup_{x \in \mathbb{R}^d} |f(x)| < \infty, \end{aligned}$$

where

$$d\gamma_k(x) := \omega_k(x)dx.$$

For $p = 2$, we provide this space with the scalar product

$$\langle f, g\rangle_{L_k^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\overline{g(x)}d\gamma_k(x).$$

If $\mathcal{F}$ is a space of a $\mathbb{C}$-valued functions on $\mathbb{R}^d$, denote by

$$\mathcal{F}_{rad} := \Big\{ f \in \mathcal{F} : f \circ A = f \text{ for all } A \in O(d, \mathbb{R}) \Big\}$$

the subspace of those $f \in \mathcal{F}$ which are radial. For $f \in \mathcal{F}_{rad}$ there exists a unique function $F : \mathbb{R}_+ \to \mathbb{C}$ such that $f(x) = F(||x||)$ for all $x \in \mathbb{R}^d$.

Remark 2.1. *By using the homogeneity of ω_k it is shown in [50] that for a radial function $f \in L_k^1(\mathbb{R}^d)$ the function F defined on $[0, \infty)$ by $f(x) = F(||x||)$, for all $x \in \mathbb{R}^d$ is integrable with respect to the measure $r^{2\gamma+d-1}dr$. More precisely,*

$$\int_{\mathbb{R}^d} f(x)d\gamma_k(x) = d_k \int_0^\infty F(r)r^{2\gamma+d-1}dr, \tag{2.8}$$

where

$$d_k := \frac{c_k}{2^{\gamma+\frac{d}{2}}\Gamma(\gamma + \frac{d}{2})}. \tag{2.9}$$

The Dunkl transform of a function f in $L_k^1(\mathbb{R}^d)$ is given by

$$\mathcal{F}_D(f)(y) = \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)K(-ix, y)d\gamma_k(x), \quad \text{for all } y \in \mathbb{R}^d. \tag{2.10}$$

In the following we give some properties of this transform (cf. [14, 16]).

i) For f in $L_k^1(\mathbb{R}^d)$ we have

$$||\mathcal{F}_D(f)||_{L_k^\infty(\mathbb{R}^d)} \leq \frac{1}{c_k}||f||_{L_k^1(\mathbb{R}^d)}. \tag{2.11}$$

ii) Inversion formula: Let f be a function in $L_k^1(\mathbb{R}^d)$, such that $\mathcal{F}_D(f) \in L_k^1(\mathbb{R}^d)$. Then

$$\mathcal{F}_D^{-1}(f)(x) = \mathcal{F}_D(f)(-x), \quad a.e.\ x \in \mathbb{R}^d. \tag{2.12}$$

Proposition 2.1. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^d)$ onto itself. If we put for f in $\mathcal{S}(\mathbb{R}^d)$*

$$\overline{\mathcal{F}_D}(f)(y) = \mathcal{F}_D(f)(-y), \quad y \in \mathbb{R}^d, \tag{2.13}$$

we have

$$\mathcal{F}_D\overline{\mathcal{F}_D} = \overline{\mathcal{F}_D}\mathcal{F}_D = Id.$$

Proposition 2.2. i*) Plancherel's formula for $\mathcal{F}_D$.*
For all f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |f(x)|^2 d\gamma_k(x) = \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi). \tag{2.14}$$

ii*) Plancherel's theorem for $\mathcal{F}_D$.*
The Dunkl transform $f \mapsto \mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L^2_k(\mathbb{R}^d)$.

iii*) Parseval's formula for $\mathcal{F}_D$.*
For all f, g in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)} d\gamma_k(x) = \int_{\mathbb{R}^d} \mathcal{F}_D(f)(\xi)\overline{\mathcal{F}_D(g)(\xi)} d\gamma_k(\xi). \tag{2.15}$$

Definition 2.1. *([50]) Let $x \in \mathbb{R}^d$. The Dunkl translation operator $f \mapsto \tau_x f$ is defined on $L^2_k(\mathbb{R}^d)$ by*

$$\mathcal{F}_D(\tau_x f) = K(ix, .)\mathcal{F}_D(f). \tag{2.16}$$

Using the Dunkl translation operator, we define the Dunkl convolution product of functions as follows (see [58, 60]).

Definition 2.2. *For f, g in $D(\mathbb{R}^d)$, we define the Dunkl convolution product by*

$$\forall\, x \in \mathbb{R}^d, \quad f *_D g(x) = \frac{1}{c_k} \int_{\mathbb{R}^d} \tau_x f(-y) g(y) d\gamma_k(y). \tag{2.17}$$

2.3 Basic Dunkl wavelet theory

In this subsection we recall some results introduced and proved by Trimèche in [61].

Let $a > 0$. The dilation operator Δ_a of a measurable function h, is defined by

$$\forall\, x \in \mathbb{R}^d,\ \Delta_a(h)(x) := \frac{1}{a^{\gamma+\frac{d}{2}}} h(\frac{x}{a}). \tag{2.18}$$

This operator satisfies.

Proposition 2.3. (i*) For all a, b in $(0, \infty)$, we have*

$$\Delta_a \Delta_b = \Delta_{ab}. \tag{2.19}$$

(ii*) Let $a > 0$. For all h in $L^2_k(\mathbb{R}^d)$, the function $\Delta_a(h)$ belongs to $L^2_k(\mathbb{R}^d)$ and we have*

$$||\Delta_a h||_{L^2_k(\mathbb{R}^d)} = ||h||_{L^2_k(\mathbb{R}^d)}, \tag{2.20}$$

and

$$\mathcal{F}_D(\Delta_a h)(y) = a^{\gamma+\frac{d}{2}} \mathcal{F}_D(h)(ay), \quad y \in \mathbb{R}^d. \tag{2.21}$$

(iii*) Let $a > 0$. For all h, g in $L^2_k(\mathbb{R}^d)$, we have*

$$\langle \Delta_a h, g\rangle_{L^2_k(\mathbb{R}^d)} = \langle h, \Delta_{\frac{1}{a}} g\rangle_{L^2_k(\mathbb{R}^d)}. \tag{2.22}$$

(iv*) Let $a > 0$ and $x \in \mathbb{R}^d$. We have*

$$\Delta_a \tau_x = \tau_{ax}\, \Delta_a. \tag{2.23}$$

Definition 2.3. *A Dunkl wavelet on $\mathbb{R}^d$ is a measurable function h on $\mathbb{R}^d$ satisfying for almost all $x \in \mathbb{R}^d \backslash \{0\}$, the condition*

$$0 < C_h = \int_0^\infty |\mathcal{F}_D(h)(\lambda x)|^2 \frac{d\lambda}{\lambda} < \infty. \tag{2.24}$$

Example 2.1. *The function α_t, $t > 0$, defined on $\mathbb{R}^d$ by*

$$\alpha_t(x) = \frac{1}{(2t)^{\gamma+\frac{d}{2}}} e^{-\frac{||x||^2}{4t}}, \tag{2.25}$$

satisfies

$$\forall\, y \in \mathbb{R}^d,\ \mathcal{F}_D(\alpha_t)(y) = e^{-t||y||^2}. \tag{2.26}$$

The function $h(x) = -\dfrac{d}{dt}\alpha_t(x)$ is a Dunkl wavelet on $\mathbb{R}^d$ in $\mathcal{S}(\mathbb{R}^d)$, and we have $C_h = \dfrac{1}{8t^2}$.

Let $a > 0$ and h be a Dunkl wavelet in $L^2_k(\mathbb{R}^d)$. We consider the family $h_{a,x}$, $x \in \mathbb{R}^d$, of functions on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$ defined by

$$h_{a,x}(y) := \tau_x(\Delta_a h)(y),\ y \in \mathbb{R}^d, \tag{2.27}$$

where τ_x, $x \in \mathbb{R}^d$, are the Dunkl translation operators given by (2.16).

We note that we have

$$\forall\, a > 0, \forall\, x \in \mathbb{R}^d, \quad ||h_{a,x}||_{L^2_k(\mathbb{R}^d)} \le ||h||_{L^2_k(\mathbb{R}^d)}. \tag{2.28}$$

Notation. We denote by

- $\mathbb{R}^{d+1}_+ = \Big\{(a,x) = (a, x_1, ..., x_d) \in \mathbb{R}^{d+1},\ a > 0\Big\}$.
- $L^p_{\mu_k}(\mathbb{R}^{d+1}_+)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}^{d+1}_+$ such that

$$\begin{aligned} \|f\|_{L^p_{\mu_k}(\mathbb{R}^{d+1}_+)} &:= \left(\int_{\mathbb{R}^{d+1}_+} |f(a,x)|^p d\mu_k(a,x)\right)^{\frac{1}{p}} < \infty,\ \ 1 \le p < \infty, \\ \|f\|_{L^\infty_{\mu_k}(\mathbb{R}^{d+1}_+)} &:= \operatorname*{ess\,sup}_{(a,x)\in\mathbb{R}^{d+1}_+} |f(a,x)| < \infty, \end{aligned}$$

where the measure μ_k is defined by

$$\forall\, (a,x) \in \mathbb{R}^{d+1}_+, \quad d\mu_k(a,x) = \frac{d\gamma_k(x)da}{a^{2\gamma+d+1}}.$$

Definition 2.4. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. The Dunkl continuous wavelet transform Φ^D_h on $\mathbb{R}^d$ is defined for regular functions f on $\mathbb{R}^d$ by*

$$\Phi^D_h(f)(a,x) = \frac{1}{c_k}\int_{\mathbb{R}^d} f(y)\overline{h_{a,x}(y)}\omega_k(y)dy = \frac{1}{c_k}\langle f, \tau_x\Delta_a h\rangle_{L^2_k(\mathbb{R}^d)},\ a > 0,\ x \in \mathbb{R}^d. \tag{2.29}$$

This transform can also be written in the form

$$\Phi^D_h(f)(a,x) = \breve{f} *_D \overline{\Delta_a h}(x), \tag{2.30}$$

where $\breve{f}(y) := f(-y)$, and $*_D$ is the Dunkl convolution product given by (2.17).

Remark 2.2. *Let h be a Dunkl wavelet in $L^2_k(\mathbb{R}^d)$. Then from the relation (2.30), for all f in $L^2_k(\mathbb{R}^d)$ we have*

$$\|\Phi_h^D(f)\|_{L^\infty_{\mu_k}(\mathbb{R}^{d+1}_+)} \leq \frac{1}{c_k}\|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)}. \tag{2.31}$$

Theorem 2.1. *(Plancherel's formula for Φ_h^D). Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. For all f in $L^2_k(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d}|f(x)|^2\omega_k(x)dx = \frac{1}{C_h}\int_0^\infty\int_{\mathbb{R}^d}|\Phi_h^D(f)(a,x)|^2 d\mu_k(a,x). \tag{2.32}$$

Corollary 2.1. *(Parseval's formula for Φ_h^D). Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$ and f_1, f_2 in $L^2_k(\mathbb{R}^d)$. Then, we have*

$$\int_{\mathbb{R}^d} f_1(x)\overline{f_2(x)}\omega_k(x)dx = \frac{1}{C_h}\int_0^\infty\int_{\mathbb{R}^d}\Phi_h^D(f_1)(a,x)\overline{\Phi_h^D(f_2)(a,x)}d\mu_k(a,x). \tag{2.33}$$

By Riesz-Thorin's interpolation theorem we obtain the following.

Proposition 2.4. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$, $f \in L^2_k(\mathbb{R}^d)$ and p belongs in $[2,\infty]$. We have*

$$\|\Phi_h^D(f)\|_{L^p_{\mu_k}(\mathbb{R}^{d+1}_+)} \leq (C_h)^{\frac{1}{p}}\Big(\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k}\Big)^{\frac{p-2}{p}}\|f\|_{L^2_k(\mathbb{R}^d)}. \tag{2.34}$$

Theorem 2.2. *(Inversion formula for Φ_h^D). Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. For all f in $L^1_k(\mathbb{R}^d)$ (resp. $L^2_k(\mathbb{R}^d)$) such that $\mathcal{F}_D(f)$ belongs to $L^1_k(\mathbb{R}^d)$ (resp. $L^1_k(\mathbb{R}^d)\bigcap L^\infty_k(\mathbb{R}^d)$) we have*

$$f(y) = \frac{1}{c_k\, C_h}\int_0^\infty\int_{\mathbb{R}^d}\Phi_h^D(f)(a,x)h_{a,y}(x)d\mu_k(a,x), \ a.e., \tag{2.35}$$

where for each $y \in \mathbb{R}^d$, both the inner integral and the outer integral are absolutely convergent, but possible not the double integral.

3 Pitt's inequality for the Dunkl wavelet transform

The Pitt inequality in the Dunkl setting expresses a fundamental relationship between a sufficiently smooth function and the corresponding Dunkl transform. This subject was firstly studied by Soltani [57]. Next Gorbachev et all in [25] have improved the result of Soltani and have given the Sharp Pitt inequality and logarithmic uncertainty principle for Dunkl transform on $\mathbb{R}^d$. More precisely they proved that, for every $f \in S(\mathbb{R}^d) \subseteq L^2_k(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d}||\xi||^{-2\lambda}|\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \leq C_k(\lambda)\int_{\mathbb{R}^d}||x||^{2\lambda}|f(x)|^2 d\gamma_k(x), \ 0 \leq \lambda < \frac{2\gamma+d}{2}, \tag{3.36}$$

where

$$C_k(\lambda) := 2^{-2\lambda}\Big[\frac{\Gamma(\frac{2\gamma+d-2\lambda}{4})}{\Gamma(\frac{2\gamma+d+2\lambda}{4})}\Big]^2 \tag{3.37}$$

and Γ denotes the well known Euler's Gamma function.

The main objective of this section is to formulate an analogue of Pitt's inequality (3.36) for the Dunkl wavelet transform. Formally, we start our investigation with the following lemma.

Lemma 3.1. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$, then for any $f \in L^2_k(\mathbb{R}^d)$, we have*
$$\mathcal{F}_D\Big(\Phi^D_h(f)(a,.)\Big)(\xi) = a^{\gamma+\frac{d}{2}}\overline{\mathcal{F}_D(h)(a\xi)}\mathcal{F}_D(f)(-\xi). \tag{3.38}$$

We are now in a position to establish the Pitt inequality for the Dunkl wavelet transforms.

Theorem 3.1. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. For any arbitrary $f \in S(\mathbb{R}^d) \subseteq L^2_k(\mathbb{R}^d)$, the Pitt inequality for the Dunkl wavelet transform is given by:*
$$C_h \int_{\mathbb{R}^d} ||\xi||^{-2\lambda}\big|\mathcal{F}_D(f)(\xi)\big|^2 d\gamma_k(\xi) \leq C_k(\lambda)\int_{\mathbb{R}^{d+1}_+} ||t||^{2\lambda}\big|\Phi^D_h(f)(a,t)\big|^2 d\mu_k(a,t),\ 0 \leq \lambda < \frac{2\gamma+d}{2}, \tag{3.39}$$
where $C_k(\lambda)$ is given by (3.37).

Proof. As a consequence of the inequality (3.36), we can write
$$\int_{\mathbb{R}^d} ||\xi||^{-2\lambda}\big|\mathcal{F}_D[\Phi^D_h(f)(a,.)](\xi)\big|^2 d\gamma_k(\xi) \leq C_k(\lambda)\int_{\mathbb{R}^d} ||t||^{2\lambda}\big|\Phi^D_h(f)(a,t)\big|^2 d\gamma_k(t),\ \text{for all } a \in (0,\infty) \tag{3.40}$$
which upon integration with respect to the Haar measure $\frac{da}{a^{2\gamma+d+1}}$ yields
$$\int_0^\infty \int_{\mathbb{R}^d} ||\xi||^{-2\lambda}\big|\mathcal{F}_D[\Phi^D_h(f)(a,.)](\xi)\big|^2 d\mu_k(a,\xi) \leq C_k(\lambda)\int_{\mathbb{R}^{d+1}_+} ||t||^{2\lambda}\big|\Phi^D_h(f)(a,t)\big|^2 d\mu_k(a,t). \tag{3.41}$$
Invoking Lemma 3.1, we can express the inequality (3.41) in the following manner:
$$\int_0^\infty \int_{\mathbb{R}^d} ||\xi||^{-2\lambda}|\mathcal{F}_D(f)(\xi)|^2 a^{2\gamma+d}|\mathcal{F}_D(h)(-a\xi)|^2 d\mu_k(a,\xi) \leq C_k(\lambda)\int_{\mathbb{R}^{d+1}_+} ||t||^{2\lambda}\big|\Phi^D_h(f)(a,t)\big|^2 d\mu_k(a,t).$$
Equivalently, we have
$$\int_{\mathbb{R}^d} ||\xi||^{-2\lambda}\big|\mathcal{F}_D(f)(\xi)\big|^2\Big\{\int_0^\infty |\mathcal{F}_D(h)(-a\xi)|^2\frac{da}{a}\Big\}d\gamma_k(\xi) \leq C_k(\lambda)\int_{\mathbb{R}^{d+1}_+} ||t||^{2\lambda}\big|\Phi^D_h(f)(a,t)\big|^2 d\mu_k(a,t).$$
Using the hypothesis on h, we obtain
$$C_h \int_{\mathbb{R}^d} ||\xi||^{-2\lambda}\big|\mathcal{F}_D(f)(\xi)\big|^2 d\gamma_k(\xi) \leq C_k(\lambda)\int_{\mathbb{R}^{d+1}_+} ||t||^{2\lambda}\big|\Phi^D_h(f)(a,t)\big|^2 d\mu_k(a,t) \tag{3.42}$$
which establishes the Pitt inequality for the Dunkl wavelet transform in arbitrary space dimensions. □

Remark 3.1. *For $\lambda = 0$, equality holds in (3.39), which is in consonance with Plancherel's formula (2.32).*

Theorem 3.2. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. For any function $f \in S(\mathbb{R}^d)$, the following inequality holds:*
$$\int_{\mathbb{R}^{d+1}_+} \log||t||\,|\Phi^D_h(f)(a,t)|^2 d\mu_k(a,t) + C_h\int_{\mathbb{R}^d} \log||\xi||\,|\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big]C_h||f||^2_{L^2_k(\mathbb{R}^d)}. \tag{3.43}$$

Proof. For every $0 \leq \lambda < \frac{2\gamma+d}{2}$, we define

$$P(\lambda) = C_h \int_{\mathbb{R}^d} ||\xi||^{-2\lambda} \big|\mathcal{F}_D(f)(\xi)\big|^2 d\gamma_k(\xi) - C_k(\lambda) \int_{\mathbb{R}_+^{d+1}} ||t||^{2\lambda} \big|\Phi_h^D(f)(a,t)\big|^2 d\mu_k(a,t). \tag{3.44}$$

On differentiating (3.44) with respect to λ, we obtain

$$\begin{aligned} P'(\lambda) &= -2C_h \int_{\mathbb{R}^d} ||\xi||^{-2\lambda} \log ||\xi|| \big|\mathcal{F}_D(f)(\xi)\big|^2 d\gamma_k(\xi) \\ &-2C_k(\lambda) \int_{\mathbb{R}_+^{d+1}} ||t||^{2\lambda} \log ||t||\, |\Phi_h^D(f)(a,t)|^2 d\mu_k(a,t) - C_k'(\lambda) \int_{\mathbb{R}_+^{d+1}} ||t||^{2\lambda} \big|\Phi_h^D(f)(a,t)\big|^2 d\mu_k(a,t), \end{aligned} \tag{3.45}$$

where

$$C_k'(\lambda) \;=\; -C_k(\lambda)\Big(2\log 2 + \frac{\Gamma'(\frac{2\gamma+d-2\lambda}{4})}{\Gamma(\frac{2\gamma+d-2\lambda}{4})} + \frac{\Gamma'(\frac{2\gamma+d+2\lambda}{4})}{\Gamma(\frac{2\gamma+d+2\lambda}{4})}\Big). \tag{3.46}$$

For $\lambda = 0$, equation (3.46) yields

$$C_k'(0) = -2\Big[\log 2 + \frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})}\Big]. \tag{3.47}$$

By virtue of Dunkl Pitt's inequality (3.39), it follows that $P(\lambda) \leq 0$, for all $\lambda \in [0, \frac{2\gamma+d}{2})$ and

$$P(0) \;=\; C_h \int_{\mathbb{R}^d} \big|\mathcal{F}_D(f)(\xi)\big|^2 d\gamma_k(\xi) - C_k(0) \int_{\mathbb{R}_+^{d+1}} \big|\Phi_h^D(f)(a,t)\big|^2 d\mu_k(a,t) \tag{3.48}$$

$$\;=\; C_h ||f||^2_{L_k^2(\mathbb{R}^d)} - C_h ||f||^2_{L_k^2(\mathbb{R}^d)} = 0. \tag{3.49}$$

Therefore,

$$\begin{aligned} &-2C_h \int_{\mathbb{R}^d} \log ||\xi|| \big|\mathcal{F}_D(f)(\xi)\big|^2 d\gamma_k(\xi) - 2C_k(0) \int_{\mathbb{R}_+^{d+1}} \log ||t||\, |\Phi_h^D(f)(a,t)|^2 d\mu_k(a,t) \\ &-C_k'(0) \int_{\mathbb{R}_+^{d+1}} \big|\Phi_h^D(f)(a,t)\big|^2 d\mu_k(a,t) \leq 0. \end{aligned} \tag{3.50}$$

Applying Plancherel's formula (2.32) and the obtained estimate (3.47) of $C_k'(0)$, we get

$$\begin{aligned} &-2C_h \int_{\mathbb{R}^d} \log ||\xi|| \big|\mathcal{F}_D(f)(\xi)\big|^2 d\gamma_k(\xi) - 2 \int_{\mathbb{R}_+^{d+1}} \log ||t||\, \big|\Phi_h^D(f)(a,t)\big|^2 d\mu_k(a,t) \\ &\qquad +2\Big[\log 2 + \frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})}\Big] C_h ||f||^2_{L_k^2(\mathbb{R}^d)} \leq 0 \end{aligned}$$

or equivalently,

$$\begin{aligned} &\int_{\mathbb{R}_+^{d+1}} \log ||t||\, \big|\Phi_h^D(f)(a,t)\big|^2 d\mu_k(a,t) + C_h \int_{\mathbb{R}^d} \log ||\xi||\, |\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \\ &\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] C_h ||f||^2_{L_k^2(\mathbb{R}^d)}. \end{aligned} \tag{3.51}$$

Inequality (3.51) is the desired Beckner's uncertainty principle for the Dunkl wavelet transform in arbitrary space dimensions. □

4 Beckner's type inequalities for the Dunkl wavelet transforms

The Dunkl Beckner's inequality [25] is given by

$$\int_{\mathbb{R}^d} \log||t||\,|f(t)|^2 d\gamma_k(t)+\int_{\mathbb{R}^d} \log||\xi||\,|\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})}+\log 2\Big]\int_{\mathbb{R}^d} |f(t)|^2 d\gamma_k(t) \tag{4.52}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$. This inequality is related to the Heisenberg's uncertainty principle and for that reason it is often referred as the logarithmic uncertainty principle. Considerable attention has been paid to this inequality for its various generalizations, improvements, analogues, and their applications [30].

We now present an alternate proof of Theorem 3.2. The strategy of the proof is different of given in the previous section and is obtained directly from the Dunkl Beckner's inequality (4.52).

Proof. of Theorem 3.2. We replace f in (4.52) with $\Phi_h^D(f)(a,.)$, so that

$$\begin{aligned}&\int_{\mathbb{R}^d} \log||t||\,|\Phi_h^D(f)(a,t)|^2 d\gamma_k(t) + \int_{\mathbb{R}^d} \log||\xi||\big|\mathcal{F}_D[\Phi_h^D(f)(a,.)](\xi)\big|^2 d\gamma_k(\xi) \geq \\ &\Big(\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})}+\log 2\Big)\int_{\mathbb{R}^d} \big|\Phi_h^D(f)(a,t)\big|^2 d\gamma_k(t),\ \text{for all } a\in(0,\infty).\end{aligned} \tag{4.53}$$

Integrating (4.53) with respect to the measure $\frac{da}{a^{2\gamma+d+1}}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}_+^{d+1}} \log||t||\,|\Phi_h^D(f)(a,t)|^2 d\mu_k(a,t) + \int_{\mathbb{R}_+^{d+1}} \log||\xi||\big|\mathcal{F}_D[\Phi_h^D(f)(a,.)](\xi)\big|^2 d\mu_k(a,\xi) \geq \\ &\Big(\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})}+\log 2\Big)\int_{\mathbb{R}_+^{d+1}} \big|\Phi_h^D(f)(a,t)\big|^2 d\mu_k(a,t).\end{aligned} \tag{4.54}$$

Using Plancherel's formula (2.32), we get

$$\begin{aligned}&\int_{\mathbb{R}_+^{d+1}} \log||t||\,|\Phi_h^D(f)(a,t)|^2 d\mu_k(a,t) + \int_{\mathbb{R}_+^{d+1}} \log||\xi||\,|\mathcal{F}_D[\Phi_h^D(f)(a,.)](\xi)|^2 d\mu_k(a,\xi) \geq \\ &\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})}+\log 2\Big] C_h||f||^2_{L^2_k(\mathbb{R}^d)}.\end{aligned} \tag{4.55}$$

We shall now simplify the second integral of (4.55). By using Lemma 3.1 we infer that

$$\int_{\mathbb{R}_+^{d+1}} \log||\xi||\,|\mathcal{F}_D[\Phi_h^D(f)(a,.)](\xi)|^2 d\mu_k(a,\xi) = C_h\int_{\mathbb{R}^d} \log||\xi||\,|\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi). \tag{4.56}$$

Plugging the estimate (4.56) in (4.55) gives the desired inequality for the Dunkl wavelet transforms as

$$\begin{aligned}&\int_{\mathbb{R}_+^{d+1}} \log||t||\,|\Phi_h^D(f)(a,t)|^2 d\mu_k(a,t) + C_h\int_{\mathbb{R}^d} \log||\xi||\,|\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \\ &\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})}+\log 2\Big] C_h||f||^2_{L^2_k(\mathbb{R}^d)}.\end{aligned}$$

This completes the second proof of Theorem 3.2. □

Corollary 4.1. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$, such that $C_h = 1$. For any function $f \in S(\mathbb{R}^d)$, the following inequality holds:*

$$\Big\{\int_{\mathbb{R}^{d+1}_+} ||t||^2 \left|\Phi^D_h(f)(a,t)\right|^2 d\mu_k(a,t)\Big\}^{1/2}\Big\{\int_{\mathbb{R}^d} ||\xi||^2 |\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi)\Big\}^{1/2}$$
$$\geq \exp\Big(\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] ||f||^2_{L^2_k(\mathbb{R}^d)}\Big).$$

Proof. Involoving Jensen's inequality in (3.43) and the fact that $C_h = 1$, we obtain an analogue of the classical Heisenberg's uncertainty inequality for the Dunkl wavelet transforms as

$$\begin{aligned}
&\log\Big\{\int_{\mathbb{R}^{d+1}_+} ||t||^2 \left|\Phi^D_h(f)(a,t)\right|^2 d\mu_k(a,t)\int_{\mathbb{R}^d} ||\xi||^2 |\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi)\Big\}^{1/2}\\
= \ &\log\Big\{\int_{\mathbb{R}^{d+1}_+} ||t||^2 \left|\Phi^D_h(f)(a,t)\right|^2 d\mu_k(a,t)\Big\}^{1/2} + \log\Big\{\int_{\mathbb{R}^d} ||\xi||^2 |\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi)\Big\}^{1/2}\\
\geq \ &\int_{\mathbb{R}^{d+1}_+} \log ||t|| \left|\Phi^D_h(f)(a,t)\right|^2 d\mu_k(a,t) + \int_{\mathbb{R}^d} \log ||\xi|| \, |\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi)\\
\geq \ &\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] C_h ||f||^2_{L^2_k(\mathbb{R}^d)},
\end{aligned}$$

which yields the result. $\square$

5 Concentration uncertainty principles for the Dunkl wavelet transforms

In this Section, we derive two concentration uncertainty principles for the Dunkl wavelet transforms as an analog of the Benedick-Amrein-Berthier and local uncertainty principles in the time-frequency analysis.

5.1 Benedick-Amrein-Berthier's uncertainty principle for the Dunkl wavelet transforms

Recently Ghobber and Jaming in [21] have proved the Benedicks-Amrein-Berthier uncertainty principle for the Dunkl transform which states that if E_1 and E_2 are two subsets of $\mathbb{R}^d$ with finite measure, then there exist a positive constant $C_k(E_1, E_2)$ such that for any $f \in L^2_k(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} |f(t)|^2 d\gamma_k(t) \leq C_k(E_1,E_2)\Big\{\int_{\mathbb{R}^d\setminus E_1} |f(t)|^2 d\gamma_k(t) + \int_{\mathbb{R}^d\setminus E_2} |\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi)\Big\}. \quad (5.57)$$

In this Subsection, our primary interest is to establish the Benedick-Amrein-Berthier uncertainty principle for the Dunkl wavelet transforms in arbitrary space dimensions by employing the inequality (5.57). In this direction, we have the following main theorem.

Theorem 5.1. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. For any arbitrary function $f \in L^2_k(\mathbb{R}^d)$, we have the following uncertainty inequality*

$$\int_0^\infty \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^D(f)(a,t)\right|^2 d\mu_k(a,t) + C_h \int_{\mathbb{R}^d\setminus E_2} |\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \frac{C_h||f||^2_{L^2_k(\mathbb{R}^d)}}{C_k(E_1,E_2)} \quad (5.58)$$

where $C_k(E_1,E_2)$ the constant given in relation (5.57), E_1 and E_2 are two subsets of $\mathbb{R}^d$ such that $\gamma_k(E_i) < \infty$, $i = 1,2$.

Proof. Since for all $a \in (0,\infty)$, $\Phi_h^D(f)(a,.) \in L^2_k(\mathbb{R}^d)$, whenever $f \in L^2_k(\mathbb{R}^d)$, so we can replace the function f appearing in (5.57) with $\Phi_h^D(f)(a,.)$ to get

$$\begin{aligned}&\int_{\mathbb{R}^d} \left|\Phi_h^D(f)(a,t)\right|^2 d\gamma_k(t) \leq \\ &C_k(E_1,E_2)\Big\{\int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^D(f)(a,t)\right|^2 d\gamma_k(t) + \int_{\mathbb{R}^d\setminus E_2} \left|\mathcal{F}_D\left[\Phi_h^D(f)(a,t)\right](\xi)\right|^2 d\gamma_k(\xi)\Big\}.\end{aligned} \quad (5.59)$$

By integrating (5.59) with respect the measure $\frac{da}{a^{2\gamma+d+1}}$, we obtain

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^d} \left|\Phi_h^D(f)(a,t)\right|^2 d\mu_k(a,t) \leq \\ &C_k(E_1,E_2)\Big\{\int_0^\infty \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^D(f)(a,t)\right|^2 d\mu_k(a,t) + \int_0^\infty \int_{\mathbb{R}^d\setminus E_2} \left|\mathcal{F}_D\Big[\Phi_h^D(f)\Big](a,.)\Big](\xi)\right|^2 d\mu_k(a,\xi)\Big\}.\end{aligned}$$

Using Lemma 3.1 together with Plancherel's formula (2.32), the above inequality becomes

$$\begin{aligned}&\frac{C_h||f||^2_{L^2_k(\mathbb{R}^d)}}{C_k(E_1,E_2)} \leq \\ &\int_0^\infty \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^D(f)(a,t)\right|^2 d\mu_k(a,t) + \int_{\mathbb{R}^d\setminus E_2} \int_0^\infty a^{2\gamma+d} |\mathcal{F}_D(f)(\xi)|^2 |\mathcal{F}_D(h)(-a\xi)|^2 d\mu_k(a,\xi)\end{aligned}$$

which further implies

$$\begin{aligned}&\frac{C_h||f||^2_{L^2_k(\mathbb{R}^d)}}{C_k(E_1,E_2)} \leq \\ &\int_0^\infty \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^D(f)(a,t)\right|^2 d\mu_k(a,t) + \int_{\mathbb{R}^d\setminus E_2} |\mathcal{F}_D(f)(\xi)|^2 \Big\{\int_0^\infty |\mathcal{F}_D(h)(-a\xi)|^2 \frac{da}{a}\Big\} d\gamma_k(\xi).\end{aligned}$$

Thus using the fact that h is Dunkl wavelet on $\mathbb{R}^d$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^D(f)(a,t)\right|^2 d\mu_k(a,t) + C_h \int_{\mathbb{R}^d\setminus E_2} |\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \frac{C_h||f||^2_{L^2_k(\mathbb{R}^d)}}{C_k(E_1,E_2)}$$

which is the desired Benedick-Amrein-Berthier's uncertainty principle for the Dunkl wavelet transforms in arbitrary space dimensions. $\square$

5.2 Local-type Uncertainty Principle for the Dunkl wavelet Transforms

We begin this subsection by recalling the local uncertainty principle for the Dunkl transform.

Proposition 5.1. *([21]). Let E be a subset of $\mathbb{R}^d$ with finite measure $0<\gamma_k(E)<\infty$. For $0<s<\frac{2\gamma+d}{2}$, there exist a positive constant $C(k,s)$ such that for any $f\in L^2_k(\mathbb{R}^d)$*

$$\int_E \big|\mathcal{F}_D(f)(\xi)\big|^2 d\gamma_k(\xi) \leq C(k,s)\big(\gamma_k(E)\big)^{\frac{2s}{2\gamma+d}} || \, ||x||^s f||^2_{L^2_k(\mathbb{R}^d)}. \tag{5.60}$$

The main objective of this Subsection is to establish the local uncertainty principles for the Dunkl wavelet transforms in arbitrary space dimensions by employing the previous inequalities.

Theorem 5.2. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. Let $E\subset\mathbb{R}^d$ such that $0<\gamma_k(E)<\infty$. Let $0<s<\frac{2\gamma+d}{2}$. For any $f\in L^2_k(\mathbb{R}^d)$, we have*

$$\int_E |\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \leq \frac{C(k,s)(\gamma_k(E))^{\frac{2s}{2\gamma+d}}}{C_h}\int_{\mathbb{R}^{d+1}_+} ||x||^{2s}\big|\Phi^D_h(f)(a,x)\big|^2 d\mu_k(a,x). \tag{5.61}$$

where $C(k,s)$, the constant given in Proposition 5.1.

Proof. Since $\Phi^D_h(f)(a,.)\in L^2_k(\mathbb{R}^d)$, whenever $f\in L^2_k(\mathbb{R}^d)$, so we can replace the function f appearing in (5.60) with $\Phi^D_h(f)(a,.)$, to get for all $a\in(0,\infty)$,

$$\int_E \big|\mathcal{F}_D\big[\Phi^D_h(f)(a,x)\big](\xi)\big|^2 d\gamma_k(\xi) \leq C(k,s)\big(\gamma_k(E)\big)^{\frac{2s}{2\gamma+d}} || \, ||x||^s \Phi^D_h(f)(a,.)||^2_{L^2_k(\mathbb{R}^d)}. \tag{5.62}$$

For explicit expression of (5.62), we shall integrate this inequality with respect to the measure $\frac{da}{a^{2\gamma+d+1}}$ to get

$$\int_0^\infty \int_E \big|\mathcal{F}_D\big[\Phi^D_h(f)(a,t)\big](\xi)\big|^2 d\mu_k(a,\xi) \leq C(k,s)\big(\gamma_k(E)\big)^{\frac{2s}{2\gamma+d}} \int_{\mathbb{R}^{d+1}_+} ||x||^{2s}\big|\Phi^D_h(f)(a,x)|^2 d\mu_k(a,x)$$

which together with Lemma 3.1 and Fubini's theorem gives

$$\begin{aligned}&\int_E |\mathcal{F}_D(f)(\xi)|^2\Big(\int_0^\infty |\mathcal{F}_D(h)(-a\xi)|^2\frac{da}{a}\Big)d\gamma_k(\xi) \leq \\ &C(k,s)\big(\gamma_k(E)\big)^{\frac{2s}{2\gamma+d}} \int_{\mathbb{R}^{d+1}_+} ||x||^{2s}\big|\Phi^D_h(f)(a,x)|^2 d\mu_k(a,x).\end{aligned} \tag{5.63}$$

Using the hypothesis on h, inequality (5.63) reduces to

$$C_h\int_E \big|\mathcal{F}_D(f)(\xi)\big|^2 d\gamma_k(\xi) \leq C(k,s)\big(\gamma_k(E)\big)^{\frac{2s}{2\gamma+d}} \int_{\mathbb{R}^{d+1}_+} ||x||^{2s}\big|\Phi^D_h(f)(a,x)|^2 d\mu_k(a,x).$$

Or equivalently,

$$\int_E \big|\mathcal{F}_D(f)(\xi)\big|^2 d\gamma_k(\xi) \leq \frac{C(k,s)\big(\gamma_k(E)\big)^{\frac{2s}{2\gamma+d}}}{C_h} \int_{\mathbb{R}^{d+1}_+} ||x||^{2s}\big|\Phi^D_h(f)(a,x)|^2 d\mu_k(a,x).$$

This completes the proof of (5.61). $\square$

6 Dunkl logarithmic Sobolev inequalities and applications

This Section is devoted to establish new Dunkl logarithmic Sobolev inequalities. Next we use these inequalities to obtain Dunkl logarithm Sobolev type uncertainty inequalities for the Dunkl wavelet transform. To facilitate our intention, we start with the following definitions:

Definition 6.1. *(i) The Dunkl transform of a distribution u in $\mathcal{S}'(\mathbb{R}^d)$ is defined by*

$$\langle \mathcal{F}_D(u), \phi \rangle = \langle u, \mathcal{F}_D^{-1}(\phi) \rangle, \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^d). \tag{6.64}$$

(ii) Let u be in $\mathcal{S}'(\mathbb{R}^d)$. We recall that

$$\mathcal{F}_D(T_j u) = i\xi_j \mathcal{F}_D(u), \quad j = 1, ..., d. \tag{6.65}$$

Definition 6.2. *[37] Let $s \in \mathbb{R}$. The Dunkl Sobolev space $H_k^s(\mathbb{R}^d)$ of order s on $\mathbb{R}^d$ is defined by*

$$H_k^s(\mathbb{R}^d) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^d) : (1 + ||\xi||^2)^{\frac{s}{2}} \mathcal{F}_D(f) \in L_k^2(\mathbb{R}^d) \Big\}. \tag{6.66}$$

Remark 6.1. *Using Parseval's formula (2.14) and relation (6.65) we can see that*

$$H_k^1(\mathbb{R}^d) = \Big\{ f \in L_k^2(\mathbb{R}^d) : \nabla_k f \in L_k^2(\mathbb{R}^d) \Big\}, \tag{6.67}$$

where ∇_k denotes the Dunkl nabla operator given by $\nabla_k = \Big(T_1, ..., T_d \Big)$. Fore more details on the Dunkl Sobolev spaces we refer the reader to [36].

Definition 6.3. *For $1 \leq p < \infty$ and $b > 0$, the weighted Lebesgue space on $\mathbb{R}^d$ is defined by*

$$L_{k,b}^p(\mathbb{R}^d) = \Big\{ f \in L_k^p(\mathbb{R}^d) : \langle t \rangle^b f \in L_k^p(\mathbb{R}^d) \Big\}, \tag{6.68}$$

where $\langle t \rangle$ is the weight function given by $\langle t \rangle = (1 + ||t||^2)^{1/2}$, $t \in \mathbb{R}^d$.

Theorem 6.1. *Let $1 < b < \infty$. For any $f \in L_{k,b}^1(\mathbb{R}^d) \backslash \{0\}$ we have*

$$-\int_{\mathbb{R}^d} |f(x)| \log \frac{|f(x)|}{\|f\|_{L_k^1(\mathbb{R}^d)}} d\gamma_k(x) \leq (d + 2\gamma) \int_{\mathbb{R}^d} |f(x)| \log(C(d,k,b)(1 + ||x||^b) d\gamma_k(x), \tag{6.69}$$

where

$$C(d,k,b) = \Big(\frac{d_k \Gamma(\frac{d+2\gamma}{b}) \Gamma(\frac{d+2\gamma}{b'})}{b\Gamma(2\gamma + d)} \Big)^{\frac{1}{d+2\gamma}}$$

is the best possible and $1/b + 1/b' = 1$. Moreover, it is attained up to conformal automorphism by

$$f(x) = (1 + ||x||^b)^{-(d+2\gamma)}.$$

Proof. It suffices to prove that this inequality (6.69) holds for f belongs to $L^1_{k,b}(\mathbb{R}^d)$ with $\|f\|_{L^1_k(\mathbb{R}^d)} = 1$. We first show that the right-hand side is well-defined. In fact, for $f \in L^1_{k,b}(\mathbb{R}^d)$, set a measure $d\lambda_k$ by

$$d\lambda_k(x) = |f(x)| d\gamma_k(x).$$

We note that $\int_{\mathbb{R}^d} d\lambda_k(x) = 1$. By Jensen's inequality,

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)| \log(1 + ||x||^b) d\gamma_k(x) &= \int_{\mathbb{R}^d} \log(1 + ||x||^b) d\lambda_k(x) \\ &\leq \log \int_{\mathbb{R}^d} (1 + ||x||^b) d\lambda_k(x) \\ &= \log \int_{\mathbb{R}^d} (1 + ||x||^b) |f(x)| d\gamma_k(x) \\ &\leq \log \int_{\mathbb{R}^d} \langle x \rangle^b |f(x)| d\gamma_k(x) < \infty.\end{aligned}$$

Let ϕ be given by

$$\phi(x) = \mathcal{C}(d, k, b)(1 + ||x||^b)^{-(d+2\gamma)},$$

where

$$\mathcal{C}(d, k, b) = \frac{b\Gamma(2\gamma + d)}{d_k \Gamma(\frac{d+2\gamma}{b})\Gamma(\frac{d+2\gamma}{b'})}$$

so that $\|\phi\|_{L^1_k(\mathbb{R}^d)} = 1$. Then, considering the relative entropy of f and ϕ, by Jensen's inequality, then we have

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)| \log \frac{\phi(x)}{|f(x)|} d\gamma_k(x) &= \int_{\mathbb{R}^d} \log \frac{\phi(x)}{|f(x)|} d\lambda_k(x) \\ &\leq \log \int_{\mathbb{R}^d} \frac{\phi(x)}{|f(x)|} d\lambda_k(x) \\ &\leq \log \int_{\mathbb{R}^d} \phi(x) d\gamma_k(x) = 0.\end{aligned}$$

Thus, we have

$$-\int_{\mathbb{R}^d} |f(x)| \log |f(x)| d\gamma_k(x) \leq -\int_{\mathbb{R}^d} |f(x)| \log |\phi(x)| d\gamma_k(x).$$

Hence we obtain the desired result

$$-\int_{\mathbb{R}^d} |f(x)| \log |f(x)| d\gamma_k(x) \leq (d + 2\gamma) \int_{\mathbb{R}^d} |f(x)| \log(1 + ||x||^b) d\gamma_k(x) - \log \mathcal{C}(d, k, b).$$

Moreover, since the equality can be valid in this inequality depends only when the equality holds in Jensen's inequality, the equality holds if and only if

$$f(x) = \mathcal{C}(d, k, b)(1 + ||x||^b)^{-(d+2\gamma)}.$$

□

Motivated by Beckner's method and by simple argument based on a sharp form of Dunkl Pitt's inequality we obtain the following logarithmic estimate of uncertainty.

Theorem 6.2. *For any arbitrary $f \in S(\mathbb{R}^d)$ there exist a constant $B_{d,k}$ independent of f such that we have*

$$\int_{\mathbb{R}^d} \log|f(x)|\,|f(x)|^2 d\gamma_k(x) \leq \frac{d+2\gamma}{2}\int_{\mathbb{R}^d} \log(||\xi||)\,|\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) - B_{d,k}||f||^2_{L^2_k(\mathbb{R}^d)}. \tag{6.70}$$

Proof. The Dunkl Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^d$ (cf. [27]) state that:

$$|\int_{\mathbb{R}^d} \mathcal{I}^k_\lambda f(x)\, g(x) d\gamma_k(x)| \leq A(p,\lambda)||f||_{L^p_k(\mathbb{R}^d)}||g||_{L^p_k(\mathbb{R}^d)}, \tag{6.71}$$

where $\mathcal{I}^k_\lambda$ designate the Dunkl Riesz potentials given by

$$\mathcal{I}^k_\lambda f(x) = \frac{1}{d_k}\int_{\mathbb{R}^d} \frac{\tau_y f(x)}{||y||^{2\gamma+d-\lambda}} d\gamma_k(y),$$

$\lambda = (2\gamma+d)(\frac{2}{p}-1)$ and $A(p,\lambda) > 0$ depends only on p and λ. So using (6.71) we deduce the following sharp form of Dunkl Pitt's inequality

$$\Big|\int_{\mathbb{R}^d} ||\xi||^{(2\gamma+d)(1-\frac{2}{p})}|\mathcal{F}_D f(\xi)|^2 d\gamma_k(\xi)\Big| \leq A(p,\lambda)||f||^2_{L^p_k(\mathbb{R}^d)}. \tag{6.72}$$

Using (6.72) is an equality at the point $p=2$, we deduce that we can be differentiated with respect p to produce inequality (6.70). □

Remark 6.2. *Motivated by Beckner's method we note that if $|f|$ is radial decreasing and C denoting a generic constant*

$$|f(x)| \leq \frac{C}{||x||^{\gamma+\frac{d}{2}}} \quad \text{or} \quad \frac{d+2\gamma}{2}\log||x|| \leq -\log|f(x)| + \log C.$$

Then by (4.52) we infer

$$\int_{\mathbb{R}^d} \log|f(x)|\,|f(x)|^2 d\gamma_k(x) \leq$$

$$\frac{d+2\gamma}{2}\int_{\mathbb{R}^d} \log(||\xi||)\,|\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) + \Big[\log(\frac{C}{2}) - \frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})}\Big]||f||^2_{L^2_k(\mathbb{R}^d)}.$$

Now we state that logarithmic Beckner's inequality (6.70) and the main result (6.69) is a dual relation in the following sense.

Theorem 6.3. *For any $f \in H^1_k(\mathbb{R}^d) \cap L^1_{k,1}(\mathbb{R}^d)$,*

$$\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}||f||^2_{L^2_k(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} |f(x)|^2\log(C(k,d)\langle x\rangle^2)d\gamma_k(x) + \int_{\mathbb{R}^d} \log(K(k,d)||\xi||)\,|\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi), \tag{6.73}$$

where

$$C(k,d) = C(d,k,2) = \Big(\frac{d_k\Gamma^2(\frac{d+2\gamma}{2})}{2\Gamma(2\gamma+d)}\Big)^{\frac{1}{2\gamma+d}} \quad \text{and} \quad K(k,d) := \exp\Big(\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})} - \frac{2B_{d,k}}{2\gamma+d}\Big).$$

Proof. Clearly, the inequality (6.73) holds for $f \equiv 0$. Let f be in $H^1_k(\mathbb{R}^d) \cap L^1_{k,1}(\mathbb{R}^d) \setminus \{0\}$. The inequality (6.69) with $b = 2$, $f \to |f|^2$ is

$$-\int_{\mathbb{R}^d} |f(x)|^2 \log \frac{|f(x)|^2}{\|f\|^2_{L^2_k(\mathbb{R}^d)}} d\gamma_k(x) \leq (d + 2\gamma) \int_{\mathbb{R}^d} |f(x)|^2 \log(C(k,d)\langle x\rangle^2) d\gamma_k(x). \quad (6.74)$$

This inequality corresponds with the logarithmic Beckner's inequality (6.70) can be written as:

$$\int_{\mathbb{R}^d} \log \frac{|f(x)|^2}{\|f\|^2_{L^2_k(\mathbb{R}^d)}} |f(x)|^2 d\gamma_k(x) \leq \frac{d+2\gamma}{2} \int_{\mathbb{R}^d} \log(||\xi||^2) \, |\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) - 2B_{d,k} ||f||^2_{L^2_k(\mathbb{R}^d)}. \quad (6.75)$$

Combining the inequalities (6.74) and (6.75), we obtain

$$\tfrac{2B_{d,k}}{d+2\gamma} \|f\|^2_{L^2_k(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} |f(x)|^2 \log(C(k,d)\langle x\rangle^2) d\gamma_k(x) + \frac{1}{2} \int_{\mathbb{R}^d} \log(||\xi||^2) |\mathcal{F}_D(f)(x)|^2 d\gamma_k(\xi).$$

Finally by simple calculations we infer that

$$\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})} \|f\|^2_{L^2_k(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} |f(x)|^2 \log(C(k,d)\langle x\rangle^2) d\gamma_k(x) + \int_{\mathbb{R}^d} \log(K(k,d)||\xi||) |\mathcal{F}_D(f)(x)|^2 d\gamma_k(\xi).$$

□

Corollary 6.1. *For any* $f \in H^1_k(\mathbb{R}^d) \cap L^1_{k,1}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |f(x)|^2 d\gamma_k(x) \leq 2C(k,d)K(k,d) e^{-\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}} \Big(\int_{\mathbb{R}^d} ||x||^2 |f(x)|^2 d\gamma_k(x) \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^d} |\nabla_k f(x)|^2 d\gamma_k(x) \Big)^{\frac{1}{2}}. \quad (6.76)$$

Proof. By the inequality (6.73), Jensen's inequality and (6.65), we get

$$\begin{aligned} \frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})} &\leq \log \Big(C(k,d) \int_{\mathbb{R}^d} (1 + ||x||^2) \frac{|f(x)|^2}{||f||^2_{L^2_k(\mathbb{R}^d)}} d\gamma_k(x) \Big) \\ &+ \frac{1}{2} \log \Big(K^2(k,d) \int_{\mathbb{R}^d} ||\xi||^2 \frac{|\mathcal{F}_D(f)(\xi)|^2}{||f||^2_{L^2_k(\mathbb{R}^d)}} d\gamma_k(\xi) \Big) \\ &\leq \log \Big\{ \frac{C(k,d)K(k,d)}{\|f\|^3_{L^2_k(\mathbb{R}^d)}} \int_{\mathbb{R}^d} (1 + ||x||^2) |f(x)|^2 d\gamma_k(x) \Big(\int_{\mathbb{R}^d} |\nabla_k f(x)|^2 d\gamma_k(x) \Big)^{\frac{1}{2}} \Big\}, \end{aligned}$$

that is,

$$\|f\|^3_{L^2_k(\mathbb{R}^d)} \leq C(k,d)K(k,d) e^{-\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}} \Big(||f||^2_{L^2_k(\mathbb{R}^d)} + \| \, ||x|| f \|^2_{L^2_k(\mathbb{R}^d)} \Big) \|\nabla_k f\|_{L^2_k(\mathbb{R}^d)}. \quad (6.77)$$

If we define f^λ by

$$f^\lambda(x) := \lambda^{\frac{d+2\gamma}{2}} f(\lambda x) \ \text{ for } \ x \in \mathbb{R}^d, \ \lambda > 0,$$

then we have

$$\|f^\lambda\|_{L^2_k(\mathbb{R}^d)} = \|f\|_{L^2_k(\mathbb{R}^d)}, \| \, ||x|| f^\lambda\|_{L^2_k(\mathbb{R}^d)} = \lambda^{-1} \| \, ||x|| f\|_{L^2_k(\mathbb{R}^d)}, \ \|\nabla_k f\|_{L^2_k(\mathbb{R}^d)} = \lambda \|\nabla_k f\|_{L^2_k(\mathbb{R}^d)}.$$

Applying (6.77) to f^λ, we get

$$\|f\|^3_{L^2_k(\mathbb{R}^d)} \leq C(k,d)K(k,d)e^{-\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}}\Big(\lambda\|f\|^2_{L^2_k(\mathbb{R}^d)} + \lambda^{-1}\| \, ||x||f\|^2_{L^2_k(\mathbb{R}^d)}\Big)\|\nabla_k f\|_{L^2_k(\mathbb{R}^d)}.$$

Optimizing the right-hand side with

$$\lambda = \frac{\| \, ||x||f\|_{L^2_k(\mathbb{R}^d)}}{\|f\|_{L^2_k(\mathbb{R}^d)}},$$

we obtain the desired inequality (6.76). □

In the following we prove another version of uncertainty inequalities for the Dunkl wavelet transform:

Theorem 6.4. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. For any arbitrary function $f \in H^1_k(\mathbb{R}^d) \cap L^1_{k,1}(\mathbb{R}^d)$ we have*

$$\begin{aligned} \frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})} C_h ||f||^2_{L^2_k(\mathbb{R}^d)} \;\; \leq & \int_{\mathbb{R}^{d+1}_+} \left|\Phi^D_h(f)(a,t)\right|^2 \log\Big(C(k,d)(1+||t||^2)\Big) d\mu_k(a,t) \\ + & \; C_h \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2 \log(K(k,d)||\xi||) d\gamma_k(\xi) \end{aligned} \tag{6.78}$$

whenever the L.H.S of (6.78) is defined.

Proof. As a consequence of inequality (6.73), we have for all $a \in (0,\infty)$

$$\begin{aligned} \frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})} \int_{\mathbb{R}^d} \left|\Phi^D_h(f)(a,t)\right|^2 d\gamma_k(t) \;\; \leq & \int_{\mathbb{R}^d} \left|\Phi^D_h(f)(a,t)\right|^2 \log\Big(C(k,d)(1+||t||^2)\Big) d\gamma_k(t) \\ + & \int_{\mathbb{R}^d} \left|\mathcal{F}_D[\Phi^D_h(f)(a,.)](\xi)\right|^2 \log(K(k,d)||\xi||) d\gamma_k(\xi), \end{aligned}$$

which upon integration yields with the measure $\frac{da}{a^{2\gamma+d+1}}$

$$\begin{aligned} \frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})} \int_{\mathbb{R}^{d+1}_+} \left|\Phi^D_h(f)(a,t)\right|^2 d\mu_k(a,t) \;\; \leq & \int_{\mathbb{R}^{d+1}_+} \left|\Phi^D_h(f)(a,t)\right|^2 \log\big(C(k,d)(1+||t||^2)\big) d\mu_k(a,t) \\ + & \int_{\mathbb{R}^{d+1}_+} \log(K(k,d)||\xi||) \left|\mathcal{F}_D[\Phi^D_h(f)(a,.)](\xi)\right|^2 d\mu_k(a,\xi). \end{aligned} \tag{6.79}$$

Using Lemma 3.1, for the second integral on the L.H.S of (6.79) and invoking (2.32), we get

$$\int_{\mathbb{R}^{d+1}_+} \left|\Phi^D_h(f)(a,t)\right|^2 \log\Big(C(k,d)(1+||t||^2)\Big) d\mu_k(a,t) + C_h \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2 \log(K(k,d)||\xi||) d\gamma_k(\xi)$$

$$\geq \frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})} C_h ||f||^2_{L^2_k(\mathbb{R}^d)}.$$

This completes the proof of Theorem 6.4. □

Based on the Dunkl logarithm Sobolev type uncertainty inequality (6.78), we shall derive another uncertainty principle for the Dunkl wavelet transform in arbitrary space dimensions.

Theorem 6.5. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$, such that $C_h = 1$. Then, for any arbitrary function $f \in H^1_k(\mathbb{R}^d) \cap L^1_{k,1}(\mathbb{R}^d) \setminus \{0\}$, we have*

$$\int_{\mathbb{R}^{d+1}_+} ||t||^2 \big|\Phi^D_h(f)(a,t)\big|^2 d\mu_k(a,t) \geq \frac{\exp\left(\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\right)}{C(k,d)K(k,d)\|\nabla_k f\|_{L^2_k(\mathbb{R}^d)}} \|f\|^3_{L^2_k(\mathbb{R}^d)} - ||f||^2_{L^2_k(\mathbb{R}^d)}. \tag{6.80}$$

Proof. Let f be in $H^1_k(\mathbb{R}^d) \cap L^1_{k,1}(\mathbb{R}^d) \setminus \{0\}$. For $C_h = 1$, we infer from (6.78) that

$$\begin{aligned} \frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}||f||^2_{L^2_k(\mathbb{R}^d)} \leq & \int_{\mathbb{R}^{d+1}_+} \big|\Phi^D_h(f)(a,t)\big|^2 \log(C(k,d)(1+||t||^2))d\mu_k(a,t) \\ & + \int_{\mathbb{R}^d} \log(K(k,d)||\xi||)\,|\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi). \end{aligned} \tag{6.81}$$

Using Jensen's inequality in (6.81), we can deduce that

$$\begin{aligned} \frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})} \leq & \log C(k,d)\Big(\int_{\mathbb{R}^{d+1}_+} \frac{\big|\Phi^D_h(f)(a,t)\big|^2}{||f||^2_{L^2_k(\mathbb{R}^d)}}(1+||t||^2)d\mu_k(a,t)\Big) \\ & + \frac{1}{2}\int_{\mathbb{R}^d} \log(K^2(k,d)||\xi||^2)\frac{|\mathcal{F}_D(f)(\xi)|^2}{||f||^2_{L^2_k(\mathbb{R}^d)}} d\gamma_k(\xi). \end{aligned} \tag{6.82}$$

To obtain a fruitful estimate of the second integral of (6.82), we set

$$d\varrho_k(\xi) = \frac{|\mathcal{F}_D(f)(\xi)|^2}{||f||^2_{L^2_k(\mathbb{R}^d)}} d\gamma_k(\xi), \text{ so that } \int_{\mathbb{R}^d} d\varrho_k(\xi) = 1. \tag{6.83}$$

Again by employing the Jensen's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \log(K^2(k,d)||\xi||^2)|\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi) &= \|f\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \log(K^2(k,d)||\xi||^2)|d\rho_k(\xi) \\ &\leq ||f||^2_{L^2_k(\mathbb{R}^d)} \log\Big\{K^2(k,d)\int_{\mathbb{R}^d} ||\xi||^2|d\rho_k(\xi)\Big\} \\ &\leq ||f||^2_{L^2_k(\mathbb{R}^d)} \log\Big\{\frac{K^2(k,d)}{\|f\|^2_{L^2_k(\mathbb{R}^d)}}\int_{\mathbb{R}^d} ||\xi||^2||\mathcal{F}_D(f)(\xi)|^2 d\gamma_k(\xi)\Big\} \\ &\leq ||f||^2_{L^2_k(\mathbb{R}^d)} \log\Big\{\frac{K^2(k,d)}{||f||^2_{L^2_k(\mathbb{R}^d)}}\int_{\mathbb{R}^d} |\nabla_k f(t)|^2 d\gamma_k(t)\Big\}. \end{aligned} \tag{6.84}$$

Using the expression (6.84) in (6.82), we infer

$$\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})} \leq \log\Big(\frac{C(k,d)K(k,d)}{||f||^3_{L^2_k(\mathbb{R}^d)}}\Big\{\int_{\mathbb{R}^{d+1}_+} \big|\Phi^D_h(f)(a,t)\big|^2(1+||t||^2)d\mu_k(a,t)\Big\}\|\nabla_k f\|_{L^2_k(\mathbb{R}^d)}\Big). \tag{6.85}$$

Expression (6.85) can be rewritten in a lucid manner as

$$\Big\{\int_{\mathbb{R}^{d+1}_+}\big|\Phi_h^D(f)(a,t)\big|^2\Big(1+||t||^2\Big)d\mu_k(a,t)\Big\}\Big\{\int_{\mathbb{R}^d}|\nabla_k f(t)|^2 d\gamma_k(t)\Big\}^{1/2}\geq \frac{\exp\left(\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\right)}{C(k,d)K(k,d)}||f||^3_{L^2_k(\mathbb{R}^d)}. \tag{6.86}$$

Applying Plancherel's formula (2.32) with $C_h = 1$, we get

$$\Big\{\int_{\mathbb{R}^{d+1}_+}||t||^2\big|\Phi_h^D(f)(a,t)\big|^2 d\mu_k(a,t)\Big\}\Big\{\int_{\mathbb{R}^d}|\nabla_k f(t)|^2 d\gamma_k(t)\Big\}^{1/2}\geq \frac{\exp\left(\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\right)}{C(k,d)K(k,d)}||f||^3_{L^2_k(\mathbb{R}^d)}$$
$$-||f||^2_{L^2_k(\mathbb{R}^d)}\|\nabla_k f\|_{L^2_k(\mathbb{R}^d)},$$

which upon simplification gives the desired inequality

$$\int_{\mathbb{R}^{d+1}_+}||t||^2\big|\Phi_h^D(f)(a,t)\big|^2 d\mu_k(a,t)\geq \frac{\exp\left(\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\right)}{C(k,d)K(k,d)\|\nabla_k f\|_{L^2_k(\mathbb{R}^d)}}||f||^3_{L^2_k(\mathbb{R}^d)}-||f||^2_{L^2_k(\mathbb{R}^d)}.$$

This completes the proof of the theorem. □

Remark 6.3. *We note that we have studied these types of uncertainty principles and others for some integral transforms as the Dunkl Gabor transform, the* (k, a)*-generalized wavelet transform, the k-Hankel Gabor transform and others integral transforms. These studies have given some papers. We cite as examples [41, 42, 43]. We mention also that Shah et al. in [1, 52], have been studying the same uncertainty principles studied in this paper for the continuous Shearlet transform and non-isotropic angular Stockwell transform.*

7 Open Problem

In the present paper, we have successfully studied new uncertainty principles associated with the Dunkl wavelet transforms. The obtained results have a novelty and contribution to the literature. It is our hope that this work motivate the researchers to find explicitly expression of the constant $A(p, \lambda)$ given in the Dunkl Hardy-Littlewood-Sobolev inequality (6.71).

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