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Further precision on the growth of solutions to linear differential equations with entire coefficients

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Abstract

In this work, we investigate the iterated type of growth of solutions to linear differential equations with entire coefficients to provide further precise on their growth. For that, we use Nevanlinna theory of meromorphic functions in the complex plane and Wiman-Valiron theory for entire functions.

Keywords: *Linear differential equations, growth of solutions, order and type of entire functions, Nevanlinna theory.*

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1 Introduction

Throughout this work, we use the fundamental results and the standard notations of the Nevanlinna value distribution theory (see [13, 16, 26]). In addition, for a non-constant entire function $f : \mathbb{C} \to \mathbb{C}$, we will use the notations $\sigma_{n,M}(f)$, $\sigma_{n,T}(f)$ to denote the n-iterated order and $\tau_{n,M}(f)$, $\tau_{n,T}(f)$

to denote the n-iterated type of f defined by

$$\sigma_{n,M}(f)=\limsup_{r\to+\infty}\frac{\log_{n+1}M(r,f)}{\log r},\ \sigma_{n,T}(f)=\limsup_{r\to+\infty}\frac{\log_{n}T(r,f)}{\log r},$$

$$\tau_{n,M}(f)=\limsup_{r\to+\infty}\frac{\log_{n}M(r,f)}{r^{\sigma_n}},\ \tau_{n,T}(f)=\limsup_{r\to+\infty}\frac{\log_{n-1}T(r,f)}{r^{\sigma_n}}$$

where $M(r,f)=\max_{|z|=r}|f(z)|$ and $T(r,f)$ is the Nevanlinna characteristic function, $\sigma_n=\sigma_{n,M}(f)=\sigma_{n,M}(f)$ and $\log_{n+1}x=\log\log_n x,\ n\in\mathbb{N}$, with $\log_1 x=\log x$ and $\log_0 x=x$. It is well known that $\sigma_{n,M}(f)=\sigma_{n,T}(f)$ for $n\geq 1$ and $\tau_{n,M}(f)=\tau_{n,T}(f)$ for $n\geq 2$, while the equality $\tau_{1,M}(f)=\tau_{1,T}(f)$ is not valid: for example, if $f(z)=e^z$ then $\tau_{1,M}(f)=1$ and $\tau_{1,T}(f)=\frac{1}{\pi}$. If there is no ambiguity we use the notations $\sigma_n(f),\ \tau_n(f)$ and for $n=1$ we write briefly $\sigma(f),\ \tau_M(f),\ \tau_T(f)$. By the well known inequality $T(r,f)\leq\log^+M(r,f)$, we get $\tau_T(f)\leq\tau_M(f)$. In the other side, in [4] Goldberg and Ostrovskii proved the following inequalities

$$\tau_M(f)\leq\pi\sigma\csc(\pi\sigma)\,\tau_T(f)\ \text{ if } 0<\sigma=\sigma(f)\leq 1/2;$$

$$\tau_M(f)\leq\pi\sigma\tau_T(f)\ \text{ if } 1/2<\sigma<\infty;$$

while $\tau_M(f)$ and $\tau_T(f)$ are equal to 0 and $+\infty$ simultaneously.

Nevanlinna theory of meromorphic functions and Wiman-Valiron theory for entire functions are a powerful tool in the field of complex differential equations. For an introduction to the theory of differential equations in the complex plane by using Nevanlinna theory and Wiman-Valiron theory; see, for example, [16, 17, 18]. Active research in this field was started by Wittich [25] and his students in the 1950s and 1960s. The order of growth of solutions to the differential equation

$$f^{(k)}+A_{k-1}(z)\,f^{(k-1)}+...+A_0(z)\,f=0, \tag{1}$$

is one of the aims in studying complex differential equations. It is well known that all solutions of (1) are entire functions when all the coefficients $A_0\not\equiv 0, A_1,...,A_{n-1}$ are entire. It can be observed in many results that the studying of the order of growth of non trivial solutions of (1) is based on the domination of one coefficient on the others and essentially when A_0 is the dominant coefficient, see, for example, [1, 5, 20]. In general, when the domination is weak, the study becomes more difficult and requires new methods. Many researchers have recently tried to investigate the case when the coefficients have the same order of growth, see for example [2, 3, 8, 9, 11, 12, 21]. On the other hand, the results obtained concerning the growth of the solutions are only on the order. In this paper, we will investigate the type of solutions to certain class of linear differential equations to provide further precise on their growth.

2 Main results

Theorem 2.1 *Let $A(z)$ be an entire function of finite order $0 < \sigma(A) = \sigma < \infty$ and of finite type $0 < \tau_M(A) = \tau < \infty$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f^{(k)} + A(z) f = 0, \ k \geq 1, \tag{2}$$

satisfies $\sigma_2(f) = \sigma$ and $\tau_2(f) = \frac{\tau}{k}$.

Corollary 2.2 *Let n be a positive integer; a, b be complex numbers such that $|b| < |a|$ and $A(z) \not\equiv 0$, $B(z)$ be entire functions with $\max\{\sigma(A), \sigma(B)\} < m \ (m \in \mathbb{N} \setminus \{0\})$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f^{(k)} + \left(B(z) e^{bz^m} + A(z) e^{az^m}\right) f = 0, \ k \geq 1,$$

satisfies $\sigma_2(f) = m$ and $\tau_2(f) = \frac{|a|}{k}$.

Example 2.3 *$f_1(z) = \exp\{e^z\}$ and $f_2(z) = \int \exp\{-2e^z\} dz$ form the fundamental system of solutions of the differential equation*

$$f'' - \left(e^z + e^{2z}\right) f = 0. \tag{3}$$

We have $\tau_M(-e^z - e^{2z}) = 2$, $\tau_2(f_1) = 1$, $\tau_2(f_2) = \tau_2(f_2') = 1$. Then, every solution $f \not\equiv 0$ of (3), satisfies $\sigma_2(f) = 1$ and $\tau_2(f) = 1$.

Theorem 2.4 *Let $P(z) = a_n z^n + ... + a_0$ be a polynomial of degree $n \ (a_n \neq 0)$. Then, every solution $f \not\equiv 0$ of the differential equation*

$$f^{(k)} + P(z) f = 0, \ k \geq 1, \tag{4}$$

satisfies $\limsup\limits_{r \to +\infty} \frac{\nu_f(r)}{r^\sigma} = |a_n|^{\frac{1}{k}}$ and $\tau_M(f) \geq \frac{\log 2}{2^\sigma} |a_n|^{\frac{1}{k}}$, where $\sigma = \sigma(f) = 1 + \frac{n}{k}$ and $\nu_f(r)$ is the central index of f.

Theorem 2.5 *Let $A(z)$, $B(z)$ be entire functions satisfying $0 < \sigma(A) = \sigma < \infty$, $0 < \tau_M(A) = \tau < \infty$, $\sigma(B) \leq \sigma(A)$ and $\tau_M(B) < \tau_M(A)$ if $\sigma(B) = \sigma(A)$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f'' + B(z) f' + A(z) f = 0, \tag{5}$$

satisfies $\sigma_2(f) = \sigma$ and $\tau_2(f) = \frac{\tau}{2}$.

Corollary 2.6 *Let $A(z) \not\equiv 0, B(z)$ be entire functions satisfying max$\{\sigma(A), \sigma(B)\} < m, (m \in \mathbb{N} \setminus \{0\})$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f'' + B(z) f' + A(z) \exp\{az^m\} f = 0$$

satisfies $\sigma_2(f) = m$ and $\tau_2(f) = \frac{|a|}{2}, (a \in \mathbb{C} \setminus \{0\})$.

Theorem 2.7 *Let $A_j(z)$ $(j = 0, 1, ..., k-1)$ be entire functions satisfying $0 < \sigma(A_0) = \sigma < \infty$, $0 < \tau(A_0) = \tau < \infty$, $\sigma(A_j) \leq \sigma(A_0)$ and $\tau(A_j) < \tau(A_0)$ if $\sigma(A_j) = \sigma(A_0)$ $(j = 1, ..., k-1)$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_0(z) f = 0, \ k \geq 3, \tag{6}$$

satisfies $\sigma_2(f) = \sigma$ and $\frac{\tau - \tau^*}{k} \leq \tau_2(f) \leq \tau$, where $\tau^* = \max\{\tau(A_j) : \sigma(A_j) = \sigma(A_0)\}$.

Theorem 2.8 *Let $A_j(z)$ $(j = 0, 1, ..., k-1)$ be entire functions satisfying $0 < \sigma_n(A_0) = \sigma < \infty$, $0 < \tau_n(A_0) = \tau < \infty$, $\sigma_n(A_j) \leq \sigma_n(A_0)$ and $\tau_n(A_j) < \tau_n(A_0)$ if $\sigma_n(A_j) = \sigma_n(A_0)$ $(j = 1, ..., k-1)$ $(n \geq 2)$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_0(z) f = 0, \ k \geq 1, \tag{7}$$

satisfies $\sigma_{n+1}(f) = \sigma$, $\tau_{n+1}(f) = \tau$.

3 Preliminary lemmas

Lemma 3.1 *Let $h(z)$ be an entire function of finite n-iterated order $0 < \sigma_n(h) = \sigma < \infty$ and of finite n-iterated type $0 < \tau_{n,M}(h) = \tau < \infty$, where $n \geq 1$ is an integer. Then, for any given $\varepsilon > 0$ there exists a set $F \subset [1, \infty)$ of infinite logarithmic measure such that for all z satisfying $|z| = r \in F$ we have*

$$\exp_n\{(\tau - \varepsilon) r^{\sigma}\} \leq M(r, h) \leq \exp_n\{(\tau + \varepsilon) r^{\sigma}\}; \tag{8}$$

where $\exp_n = \exp\exp ... \exp$, n times.

Proof. By the definition of $\tau_{n,M}(h) = \tau$, for any $\varepsilon > 0$ there exists r_0 such that for $r \geq r_0$ we have

$$M(r, h) \leq \exp_n\{(\tau + \varepsilon) r^{\sigma}\}. \tag{9}$$

Now by [21, 2], for any given $\varepsilon > 0$ there exists a set $F \subset [1, \infty)$ of infinite logarithmic measure such that for all z satisfying $|z| = r \in F$ we have

$$\exp_n\{(\tau - \varepsilon) r^{\sigma}\} \leq M(r, h). \tag{10}$$

By combining (9) and (10), we obtain (8).

Lemma 3.2 *If $f(z)$ is an entire function of finite n-iterated order $0 < \sigma_n(f) < \infty$ $(n \geq 1)$, then $\tau_{n,M}(f') = \tau_{n,M}(f)$.*

Proof. It is well known that $\sigma(f') = \sigma(f)$ (see [24, 19]); and then $\sigma_n(f') = \sigma_n(f)$ for every $n \geq 1$. The equalities

$$f(z) = \int_0^z f'(t)\,dt + f(0), \quad f'(t) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(t-z)^2}dt,$$

where Γ is the circle $|t-z| = 1$, yield the inequalities

$$\frac{1}{r}\left(M(r,f) - |f(0)|\right) \leq M(r,f') \leq M(r+1,f). \tag{11}$$

From the first inequality of (11), we get

$$\tau_{n,M}(f) = \limsup_{r\to+\infty}\frac{\log_n M(r,f)}{r^{\sigma_n}} \leq \limsup_{r\to+\infty}\frac{\log_n M(r,f')}{r^{\sigma_n}} = \tau_{n,M}(f'), \tag{12}$$

where $\sigma_n = \sigma_n(f) = \sigma_n(f')$. In the other side, from the second inequality of (11), we obtain

$$\limsup_{r\to+\infty}\frac{\log_n M(r,f')}{r^{\sigma_n}\left(1+\frac{1}{r}\right)^{\sigma_n}} \leq \limsup_{r\to+\infty}\frac{\log_n M(r+1,f)}{(r+1)^{\sigma_n}} = \tau_{n,M}(f);$$

which implies $\tau_{n,M}(f') \leq \tau_{n,M}(f)$. So, we conclude that $\tau_{n,M}(f') = \tau_{n,M}(f)$.

We signal here that Lemma 3.2 is provided in [22] by the same method but instead of $r+1$ they have taken βr with $\beta \to 1$ which leads to a mistake in the proof by taking, for example, $\beta = 1 + \exp\{-e^r\}$.

Lemma 3.3 *[22] Let f and g be entire functions satisfying $0 < \sigma_n(f) < \sigma_n(g) < \infty$ or $\sigma_n(f) = \sigma_n(g)$ with $0 < \tau_n(f) < \tau_n(g) < \infty$. Then*
i) $\sigma_n(f+g) = \sigma_n(f)$ and $\tau_n(f+g) = \tau_n(g)$ for $n \geq 1$.
ii) $\sigma_n(f.g) = \sigma_n(f)$ and $\tau_n(f.g) = \tau_n(g)$ for $n \geq 2$.

Lemma 3.4 *[14] Let $f(z)$ be an entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r,f)$. Then for all $|z|$ outside a set E of finite logarithmic measure, we have*

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^k (1 + o(1)), \ k \in \mathbb{N},$$

where $\nu_f(r)$ is the central index of $f(z)$.

Lemma 3.5 *[15, 20] Let $f(z)$ be a transcendental entire function with $0 < \sigma_{n+1}(f) = \sigma < \infty, \ 0 < \tau_{n+1}(f) = \tau < \infty, \ n \geq 1$. Then*

$$\limsup_{r\to+\infty}\frac{\log_n \nu_f(r)}{r^\sigma} = \tau,$$

where $\nu_f(r)$ is the central index of $f(z)$.

Lemma 3.6 *[6] Let g be a meromorphic function; let $\alpha > 0$ be given real constants and $k \in \mathbb{N}$; then there exists a set $E \subset (1, \infty)$ that has a finite logarithmic measure and a constant $A > 0$ that depends only on α and k such that for all $r = |z|$ satisfying $r \notin E$, we have*

$$\left| \frac{g^{(k)}(z)}{g(z)} \right| \leq A \left[T(\alpha r, g) \right]^{2k}.$$

Lemma 3.7 *[16] Let $P(z) = a_n z^n + ... + a_0$ be a polynomial of degree n. Then, for any given $\varepsilon > 0$ there exists $n_0 > 0$ such that for all $r = |z| > n_0$ the inequalities*

$$(1 - \varepsilon) |a_n| r^n \leq |P(z)| \leq (1 + \varepsilon) |a_n| r^n$$

hold.

4 Proof of Theorems

Proof of Theorem 2.1. By [21, Theorem 1], every solution $f \not\equiv 0$ of (2) satisfies $\sigma_2(f) = \sigma$. We have to prove $\tau_2(f) = \frac{\tau}{k}$. For $k = 1$, it is well known that every non trivial solution of (2) has the form $f(z) = c \exp\{F(z)\}$ where $c \neq 0$ and $F'(z) = -A(z)$. We have $\tau_M(F) = \tau_M(F') = \tau_M(A) = \tau$, and then $\tau_2(f) = \tau$. Now, for $k \geq 2$, from (2), we can write

$$\frac{f^{(k)}(z)}{f(z)} = -A(z) \tag{13}$$

By Lemma 3.1, for any given $\varepsilon > 0$ there exists a set $F \subset [1, \infty)$ of infinite logarithmic measure such that for all z satisfying $|z| = r \in F$ we have

$$\exp\{(\tau - \varepsilon) r^\sigma\} \leq M(r, A) \leq \exp\{(\tau + \varepsilon) r^\sigma\}. \tag{14}$$

By Lemma 3.4, we have

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z} \right)^k (1 + o(1)), \tag{15}$$

where $|z| = r$ is outside a set E of finite logarithmic measure and $f(z) = M(r, f)$. By (13)-(15), we get

$$\exp\{(\tau - \varepsilon) r^\sigma\} \leq \left(\frac{\nu_f(r)}{r} \right)^k (1 + o(1)) \leq \exp\{(\tau + \varepsilon) r^\sigma\}. \tag{16}$$

From (16) and since $\varepsilon > 0$ is arbitrary, we get

$$\limsup_{r \to +\infty} \frac{\log \nu_f(r)}{r^\sigma} = \frac{\tau}{k};$$

and by Lemma 3.5, we obtain

$$\tau_2(f) = \frac{\tau}{k}.$$

Proof of Theorem 2.4. By [7], every solution $f \not\equiv 0$ of (4) satisfies $\sigma(f) = 1 + \frac{n}{k}$, $(k \geq 2)$. Also for $k = 1$, it is clear that $\sigma(f) = 1 + n$. From (4), we can write

$$\frac{f^{(k)}(z)}{f(z)} = -P(z). \tag{17}$$

By Lemma 3.7, for any given $\varepsilon > 0$ there exists $n_0 > 0$ such that for all $r = |z| > n_0$, we have

$$(1-\varepsilon)|a_n| r^n \leq |P(z)| \leq (1+\varepsilon)|a_n| r^n. \tag{18}$$

By (15), (17) and (18), we get

$$(1-\varepsilon)|a_n| r^n \leq \left(\frac{\nu_f(r)}{r}\right)^k (1+o(1)) \leq (1+\varepsilon)|a_n| r^n. \tag{19}$$

From (19) and since $\varepsilon > 0$ is arbitrary, then

$$\limsup_{r\to+\infty} \frac{\nu_f(r)}{r^\sigma} = |a_n|^{\frac{1}{k}}. \tag{20}$$

where $\sigma = \sigma(f) = 1 + \frac{n}{k}$. Now we proceed to prove $\tau_M(f) \geq \frac{\log 2}{2^\sigma}|a_n|^{\frac{1}{k}}$. By [20, formula 3.8], we have

$$\nu_f(r)\log 2 \leq \log M(2r, f) + c, \ c > 0, \tag{21}$$

which implies

$$\frac{\log 2}{2^\sigma}\frac{\nu_f(r)}{r^\sigma} \leq \frac{\log M(2r, f)}{(2r)^\sigma} + \frac{c}{(2r)^\sigma}. \tag{22}$$

By (20) and (22), we obtain

$$\tau_M(f) \geq \frac{\log 2}{2^\sigma}|a_n|^{\frac{1}{k}}.$$

Proof of Theorem 2.5. Suppose that $f \not\equiv 0$ is a solution of (5). Set $f = g.h$, where g and h are entire functions. We have $f' = g'.h + g.h'$ and $f'' = g''.h + 2g'.h' + g.h''$. Substituting f, f', f'' in (5), we get

$$h.g'' + (2h' + B.h)g' + (h'' + Bh' + Ah)g = 0.$$

By taking $h(z)=\exp\left\{-\frac{B(z)}{2}\right\}$ as a solution of $2h'+B.h=0$, the equation (5) becomes

$$g''+\left(\frac{(B')^2}{4}-\frac{B''}{2}-\frac{B'B}{2}+A\right)g=0. \tag{23}$$

By Lemma 3.2 and Lemma 3.3, $\frac{(B')^2}{4}-\frac{B''}{2}-\frac{B'B}{2}+A$ is an entire function of order $\sigma=\sigma(A)$ and of type $\tau=\tau_M(A)$; and by Theorem 2.1, every solution $g\not\equiv 0$ of (23) satisfies $\sigma_2(g)=\sigma$ and $\tau_2(g)=\frac{\tau}{2}$. From the assumptions, Lemma 3.3 and by taking account that $\sigma_2(h)=\sigma(B)$ and $\tau_2(h)=\tau(B)$, we conclude that $\sigma_2(f)=\sigma_2(g.h)=\sigma_2(g)=\sigma$ and

$$\tau_2(f)=\tau_2(g.h)=\tau_2(g)=\frac{\tau}{2}.$$

Proof of Theorem 2.7. Suppose that $f\not\equiv 0$ is a solution of (6). Then by [21], we have $\sigma_2(f)=\sigma$. We start to prove $\tau_2(f)\leq\tau$. From (6), we can write

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq|A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+...+|A_1(z)|\left|\frac{f'(z)}{f(z)}\right|+|A_0(z)|. \tag{24}$$

By the assumptions, for all $j=0,1,...,k-1$ and for any given $\varepsilon>0$ there exists r_0 such that for all $|z|=r\geq r_0$, we have

$$|A_j(z)|\leq\exp\{(\tau+\varepsilon)r^\sigma\}. \tag{25}$$

By Lemma 3.4 and (24)-(25), for $|f(z)|=M(r,f)$ and for all $|z|=r$ outside a set E of finite logarithmic measure, we obtain

$$\left(\frac{\nu_f(r)}{r}\right)^k(1+o(1))\leq k\exp\{(\tau+\varepsilon)r^\sigma\}\left(\frac{\nu_f(r)}{r}\right)^{k-1}(1+o(1)),$$

which implies

$$\nu_f(r)(1+o(1))\leq kr\exp\{(\tau+\varepsilon)r^\sigma\}. \tag{26}$$

By Lemma 3.4 and (26), we obtain

$$\tau_2(f)\leq\tau. \tag{27}$$

In the other hand, From (7), we can write

$$|A_0(z)|\leq\left|\frac{f^{(k)}(z)}{f(z)}\right|+|A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+...+|A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{28}$$

By Lemma 3.1, for any given $\varepsilon>0$ there exists a set $F\subset[1,\infty)$ of infinite logarithmic measure such that for all z satisfying $|z|=r\in F$, we have

$$\exp_n\{(\tau-\varepsilon)r^\sigma\}\leq|A_0(z)|. \tag{29}$$

By the assumptions, for all $j = 1, ..., k-1$ and $\frac{\tau-\tau^*}{2} > \varepsilon > 0$, there exists r_1 such that for all $|z| = r \geq r_1$, we have

$$|A_j(z)| \leq \exp\{(\tau^* + \varepsilon) r^\sigma\}. \tag{30}$$

By Lemma 3.4, (28) and (30), for $r \in F \setminus E$, we have

$$\exp\{(\tau - \varepsilon) r^\sigma\} \leq c_1 \exp\{(\tau^* + \varepsilon) r^\sigma\} \left(\frac{\nu_f(r)}{r}\right)^k (1 + o(1)),$$

where $c_1 > 0$; which implies

$$\frac{\tau - \tau^*}{k} \leq \tau_2(f).$$

Proof of Theorem 2.8. Suppose that $f \not\equiv 0$ is a solution of (7). Then by [21, 2], we have $\sigma_{n+1}(f) = \sigma$. We have to prove $\tau_{n+1}(f) = \tau$, for $n \geq 2$. From (7), we can write

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |A_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + ... + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right| + |A_0(z)|. \tag{31}$$

By the assumptions, for all $j = 0, 1, ..., k-1$ and for any given $\varepsilon > 0$ there exists r_0 such that for all $|z| = r \geq r_0$, we have

$$|A_j(z)| \leq \exp_n\{(\tau + \varepsilon) r^\sigma\}. \tag{32}$$

By Lemma 3.4 and (31)-(32), for $|f(z)| = M(r, f)$ and for all $|z| = r$ outside a set E of finite logarithmic measure, we obtain

$$\left(\frac{\nu_f(r)}{r}\right)^k (1 + o(1)) \leq k \exp_n\{(\tau + \varepsilon) r^\sigma\} \left(\frac{\nu_f(r)}{r}\right)^{k-1} (1 + o(1)),$$

and so

$$\nu_f(r)(1 + o(1)) \leq kr \exp_n\{(\tau + \varepsilon) r^\sigma\}. \tag{33}$$

By Lemma 3.4 and (33), we get the inequality

$$\tau_{n+1}(f) \leq \tau. \tag{34}$$

In the other hand, From (7), we can write

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + ... + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right|. \tag{35}$$

By Lemma 3.1, for any given $\varepsilon > 0$ there exists a set $F \subset [1, \infty)$ of infinite logarithmic measure such that for all z satisfying $|z| = r \in F$, we have

$$\exp_n\{(\tau - \varepsilon) r^\sigma\} \leq |A_0(z)|. \tag{36}$$

By the assumptions, for all $j = 1, ..., k-1$ and for $\varepsilon > 0$ such that $\tau - 3\varepsilon > \max\{\tau_n(A_j) : \sigma_n(A_j) = \sigma_n(A_0)\}$, there exists r_1 such that for all $|z| = r \geq r_1$, we have

$$|A_j(z)| \leq \exp_n\{(\tau - 3\varepsilon) r^\sigma\}. \quad (37)$$

By Lemma 3.6, (35) and (37), for $r \in F \setminus E$, we have

$$\exp_n\{(\tau - \varepsilon) r^\sigma\} \leq c_2 \exp_n\{(\tau - 3\varepsilon) r^\sigma\} T(r, f)^{2k} (1 + o(1)), \quad (38)$$

where $c_2 > 0$ and $n \geq 2$; which implies

$$\exp_n\{(\tau - 2\varepsilon) r^\sigma\} \leq c_2 T(r, f)^{2k} (1 + o(1));$$

from which, we obtain the second inequality

$$\tau_{n+1}(f) \geq \tau. \quad (39)$$

From (34) and (39), we conclude that $\tau_{n+1}(f) = \tau$.

5 Open Problem

To our knowledge, this is the first work that investigates the type of growth of solutions to linear differential equations with entire coefficients and it remains some open questions:
1) Can we get the exact value of $\tau_M(f)$ in Theorem 2.4 and study equations more general than (4)?
2) Can we improve the result of Theorem 2.7 by precising $\tau_2(f)$? We expect that $\tau_2(f) = \frac{\tau}{k}$ as in Theorem 2.1 and Theorem 2.5.
3) The case of meromorphic coefficients remains to be studied.

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