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On Certain Analytic Functions Defined by Frasin Differential Operator

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Abstract

In this current paper, we establish a new class of analytic function $\mathfrak{BO}(\kappa, \mu, \eta, \varrho, \upsilon)$ defined by comprehensive differential operator of holomorphic functions involving binomial series. We investigate the adequate condition for a function $\varphi(z)$ to be in this class.

Keywords: *Frasin differential operator, starlike function, analytic function, convex function.*
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1 Introduction

Let $\mathfrak{A}$ denote the class of functions of the form

$$\varphi(z) = z + \sum_{\tau=2}^{\infty} a_\tau z^\tau = z + a_2 z^2 + \cdots \tag{1}$$

which are analytic in the open unit disk

$$\mathfrak{U} = \{z \in \mathcal{C} : |z| < 1\},$$

and $\mathcal{W}(\mathfrak{U})$ be the space of holomorphic functions in $\mathfrak{U}$. Let

$$\mathfrak{A}_q = \left\{\varphi \in \mathcal{W}(\mathfrak{U}) : \varphi(z) = z + e_{q+1} z^{q+1} + e_{q+2} z^{q+2} + \cdots \right\} \tag{2}$$

with $\mathfrak{A}_1 = \mathfrak{A}$, $z \in \mathfrak{U}$ and

$$\mathcal{W}[e,q] := \left\{\varphi \in \mathcal{W}(\mathcal{U}) : \varphi(z) = e + e_q z^q + e_{q+1} z^{q+1} + \cdots\right\} \tag{3}$$

for $e \in \mathcal{C}$, $z \in \mathfrak{U}$ and $q \in \mathcal{N}$.

Let $\mathcal{S} \subset \mathfrak{A}$ denote the class of univalent functions in $\mathfrak{U}$ and by $\mathcal{S}^*(\varrho) \subset \mathcal{S}$ denote the class of starlike functions of order ϱ, $0 \leq \varrho < 1$ which satisfy the condition

$$\Re\left(\frac{z\varphi'(z)}{\varphi(z)}\right) > \varrho. \tag{4}$$

Also, a function $\varphi(z)$ belonging to $\mathfrak{K}(\varrho) \subset \mathcal{S}$ is said to be convex of order ϱ, $0 \leq \varrho < 1$ in $\mathfrak{U}$, if and only if

$$\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > \varrho, \tag{5}$$

and denote by $\mathfrak{R}(\varrho) \subset \mathcal{S}$ the class of bounded turning functions which satisfy the condition

$$\Re(\varphi'(z)) > \varrho,$$

where $z \in \mathfrak{U}$.

If φ and g are analytic function in $\mathfrak{U}$, we say that φ is subordinate to g, written $\varphi \prec g$, if there is a function w analytic in $\mathfrak{U}$, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in \mathfrak{U}$ such that $\varphi(z) = g(w(z))$ for all $z \in \mathfrak{U}$. If g is univalent, then $\varphi \prec g$ if and only if $\varphi(0) = g(0)$ and $\varphi(\mathfrak{U}) \subseteq g(\mathfrak{U})$.
Frasin [8] (also see [2],[10]) introduced the differential operator $D^{\eta}_{\kappa,\mu}\varphi(z)$ defined as follows:

$$D^0\varphi(z) = \varphi(z)$$
$$D^1_{\kappa,\mu}\varphi(z) = (1-\mu)^{\kappa}\varphi(z) + (1-(1-\mu)^{\kappa})z\varphi'(z) = D_{\kappa,\mu}\varphi(z)$$
$$D^{\eta}_{\kappa,\mu}\varphi(z) = D_{\kappa,\mu}(D^{\eta-1}\varphi(z))$$

where $\eta \in \mathcal{N}$, then we have

$$D^{\eta}_{\kappa,\mu}\varphi(z) = z + \sum_{\tau=2}^{\infty}\left(1 + (\tau-1)\sum_{s=1}^{\kappa}\binom{\kappa}{s}(-1)^{s+1}\mu^s\right)^{\eta} a_\tau z^\tau \tag{6}$$

Using (6), we have

$$C^{\kappa}_{s}(\mu)z(D^{\eta}_{\kappa,\mu}\varphi(z))' = D^{\eta+1}_{\kappa,\mu}\varphi(z) - (1 - C^{\kappa}_{s}(\mu))D^{\eta}_{\kappa,\mu}\varphi(z)$$

where $\mu > 0$, $\kappa \in \mathcal{N}$, $\eta \in \mathcal{N}_0$ and $C^{\kappa}_{s}(\mu) := \sum_{s=1}^{\kappa}\binom{\kappa}{s}(-1)^{s+1}\mu^s$.

Remark 1.1 *We observe that*

1. *When $\kappa = 1$, we obtain the Al-Oboudi differential operator [1].*

2. *When $\kappa = \mu = 1$, we obtain the Salagean operator [9].*

Lemma 1.2 *[7] Let ρ be holomorphic in $\mathfrak{U}$ with $\rho(0) = 1$, if*

$$\Re\left(1 + \frac{z\rho'(z)}{\rho(z)}\right) > \frac{3\varrho - 1}{2\varrho},$$

then $\Re(\rho(z)) > \varrho$ in $\mathfrak{U}$, $z \in \mathfrak{U}$ and $\frac{1}{2} \leq \varrho < 1$.

2 Main Results

Definition 2.1 *A function $\varphi \in \mathfrak{A}$, $\mu > 0$, $\kappa \in \mathcal{N}$, $\eta \in \mathcal{N}_0$, $C_s^\kappa(\mu) := \sum_{s=1}^{\kappa} \binom{\kappa}{s}(-1)^{s+1}\mu^s$, $\upsilon \geq 0$ and $0 \leq \varrho < 1$ is in the class $\mathfrak{BO}(\kappa, \mu, \eta, \varrho, \upsilon)$ if*

$$\left|\frac{D_{\kappa,\mu}^{\eta+1}\varphi(z)}{z}\left(\frac{z}{D_{\kappa,\mu}^{\eta}\varphi(z)}\right)^{\upsilon} - 1\right| < 1 - \varrho. \tag{7}$$

where $z \in \mathfrak{U}$.

Remark 2.2 *The family $\mathfrak{BO}(\kappa, \mu, \eta, \varrho, \upsilon)$ have various new classes of analytic univalent functions as well as some very well-known ones. In place of "equivalence" we are going to take "contained in" as it was discussed in [3]. For example,*

- *Given $\eta = 0$, $\kappa = \mu = 1$ and $\upsilon = 1$, we have the class $\mathfrak{BO}(1, 1, 0, \varrho, 1)$ contained in $\mathcal{S}^*(\varrho)$.*

- *Given $\eta = 1$, $\kappa = \mu = 1$ and $\upsilon = 1$, we have the class $\mathfrak{BO}(1, 1, 1, \varrho, 1)$ contained in $\mathfrak{K}(\varrho)$.*

- *Given $\eta = 0$, $\kappa = \mu = 1$ and $\upsilon = 0$, we have the class $\mathfrak{BO}(1, 1, 0, \varrho, 0)$ contained in $\mathfrak{R}(\varrho)$.*

- *Given $\kappa = 1$ and $\upsilon = 1$, then*

$$\mathfrak{BO}(1, \mu, \eta, \varrho, \upsilon) = \mathfrak{BO}(\mu, \eta, \varrho, \upsilon)$$

  investigated and studied by Catas and Lupas [5].

- *Given $\kappa = \mu = 1$ and $\upsilon = 1$, then*

$$\mathfrak{BO}(1, 1, \eta, \varrho, \upsilon) = \mathfrak{BO}(\eta, \varrho, \upsilon)$$

  investigated and studied by Catas and Lupas [4].

- *Given* $\eta = 0$, $\kappa = \mu = 1$ *and* $\upsilon = 2$, *then*

$$\mathfrak{BO}(1, 1, 0, \varrho, 2) = \mathfrak{B}(\varrho)$$

investigated by Frasin and Darus [6].

- *Given* $\eta = 0$ *and* $\kappa = \mu = 1$ *then*

$$\mathfrak{BO}(1, 1, 0, \varrho, \upsilon) = \mathfrak{B}(\varrho, \upsilon)$$

investigated by Frasin and Jahangiri [7].

Theorem 2.3 *If for the function* $\varphi \in \mathfrak{A}$, $\mu > 0$, $\kappa \in \mathcal{N}$, $\eta \in \mathcal{N}_0$, $C_s^\kappa(\mu) := \sum_{s=1}^{\kappa} \binom{\kappa}{s}(-1)^{s+1}\mu^s$, $\upsilon \geq 0$ *and* $1/2 \leq \varrho < 1$, *if*

$$\frac{D_{\kappa,\mu}^{\eta+2}\varphi(z)}{C_s^\kappa(\mu) D_{\kappa,\mu}^{\eta+1}\varphi(z)} - \frac{\upsilon D_{\kappa,\mu}^{\eta+1}\varphi(z)}{C_s^\kappa(\mu) D_{\kappa,\mu}^{\eta}\varphi(z)} + \frac{\upsilon - 1}{C_s^\kappa(\mu)} + 1 \prec \psi z + 1, \quad z \in \mathfrak{U}, \tag{8}$$

where $\psi = \frac{3\varrho - 1}{2\varrho}$, *then* $\varphi(z) \in \mathfrak{BO}(\kappa, \mu, \eta, \varrho, \upsilon)$.

Proof. If we denote by

$$\rho(z) = \frac{D_{\kappa,\mu}^{\eta+1}\varphi(z)}{z} \left(\frac{z}{D_{\kappa,\mu}^{\eta}\varphi(z)}\right)^{\upsilon}$$

where $\rho(z) = 1 + \rho_1 z + \rho_2 z + \cdots$, $\rho(z) \in \mathcal{W}[1, 1]$, by simplification

$$\ln(\rho(z)) = \ln(D_{\kappa,\mu}^{\eta+1}\varphi(z)) - \ln(z) + \upsilon \ln(z) - \upsilon \ln(D_{\kappa,\mu}^{\eta}\varphi(z))$$

and by simple differentiation we get

$$\frac{\rho'(z)}{\rho(z)} = \frac{D_{\kappa,\mu}^{\eta+2}\varphi(z)}{zC_s^\kappa(\mu) D_{\kappa,\mu}^{\eta+1}\varphi(z)} - \frac{\upsilon D_{\kappa,\mu}^{\eta+1}\varphi(z)}{zC_s^\kappa(\mu) D_{\kappa,\mu}^{\eta}\varphi(z)} - \frac{1 - C_s^\kappa(\mu)}{zC_s^\kappa(\mu)} + \frac{\upsilon(1 - C_s^\kappa(\mu))}{zC_s^\kappa(\mu)} - \frac{1}{z} + \frac{\upsilon}{z}$$

Multiply through by z and simplifying we have,

$$\frac{z\rho'(z)}{\rho(z)} = \frac{D_{\kappa,\mu}^{\eta+2}\varphi(z)}{C_s^\kappa(\mu) D_{\kappa,\mu}^{\eta+1}\varphi(z)} - \frac{\upsilon D_{\kappa,\mu}^{\eta+1}\varphi(z)}{C_s^\kappa(\mu) D_{\kappa,\mu}^{\eta}\varphi(z)} + \frac{\upsilon - 1}{C_s^\kappa(\mu)}$$

In the interpretation of the Theorem 2.3, it implies that

$$\Re\left(1 + \frac{z\rho'(z)}{\rho(z)}\right) > \frac{3\varrho - 1}{2\varrho}$$

Hence, by Lemma 1.2, we get

$$\Re\left(\frac{D_{\kappa,\mu}^{\eta+1}\varphi(z)}{z} \left(\frac{z}{D_{\kappa,\mu}^{\eta}\varphi(z)}\right)^{\upsilon}\right) > \varrho$$

therefore,$\varphi(z) \in \mathfrak{BO}(\kappa, \mu, \eta, \varrho, \upsilon)$ by Definition 2.1.

We have the subsequent corollaries as a result of the above theorem.

Taking $\eta = 0$, $\varrho = \frac{1}{2}$, $\kappa = \mu = 1$ and $\upsilon = 1$, we have

Corollary 2.4 *Suppose $\varphi(z) \in \mathfrak{A}$ and*

$$\Re\left(\frac{z\varphi''(z)}{\varphi'(z)} - \frac{z\varphi'(z)}{\varphi(z)}\right) > -\frac{3}{2} \quad z \in \mathfrak{U}.$$

Then $\varphi(z)$ is starlike of order $\frac{1}{2}$.

Taking $\eta = 1$, $\varrho = \frac{1}{2}$, $\kappa = \mu = 1$ and $\upsilon = 1$, we have

Corollary 2.5 *Suppose $\varphi(z) \in \mathfrak{A}$ and*

$$\Re\left(\frac{2z\varphi''(z) + z^2\varphi'''(z)}{z\varphi''(z) + \varphi'(z)} - \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\frac{1}{2} \quad z \in \mathfrak{U}.$$

Then

$$\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > \frac{1}{2}.$$

That is, $\varphi(z)$ is convex of order $\frac{1}{2}$.

Taking $\eta = 1$, $\varrho = \frac{1}{2}$, $\kappa = \mu = 1$ and $\upsilon = 0$, we have

Corollary 2.6 *Suppose $\varphi(z) \in \mathfrak{A}$ and*

$$\Re\left(\frac{2z\varphi''(z) + z^3\varphi'''(z)}{z^2\varphi''(z) + z\varphi'(z)}\right) > -\frac{1}{2} \quad z \in \mathfrak{U}.$$

Then

$$\Re\left(z\varphi''(z) + \varphi'(z)\right) > \frac{1}{2}.$$

Taking $\eta = 0$, $\varrho = \frac{1}{2}$, $\kappa = \mu = 1$ and $\upsilon = 0$, we have

Corollary 2.7 *Suppose $\varphi(z) \in \mathfrak{A}$ and*

$$\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > \frac{1}{2} \quad z \in \mathfrak{U}.$$

Then

$$\Re\left(\varphi'(z)\right) > \frac{1}{2}.$$

Also, if the function $\varphi(z)$ is convex of order $\frac{1}{2}$ then $\varphi(z) \in \mathfrak{BO}(1, 1, 0, \frac{1}{2}, 0)$ which is contained in $\mathfrak{R}(1/2)$.

3 Open Problem

The open problem is to determine a generic class of univalent functions such has $\mathfrak{BO}(\kappa,\mu,\eta,\varrho,\upsilon)$, $\mu > 0$, $\kappa \in \mathbb{N}$, $\eta \in \mathcal{N}_0$, $C_s^\kappa(\mu) := \sum_{s=1}^{\kappa}\binom{\kappa}{s}(-1)^{s+1}\mu^s$, $\upsilon \geq 0$ and $1/2 \leq \varrho < 1$ is contained inside and possible to obtain. Compare the new results with the results given by [7].

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