*Int. J. Open Problems Complex Analysis, Vol. 13, No. 1, March 2021*
*ISSN 2074-2827;* *Copyright ©ICSRS Publication, 2021*
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# On Subclass of Analytic Functions Defined by Composition Operators

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Received 12 January 2021; Accepted 1 March 2021

**Abstract**

***In this work, we introduce a new subclass of analytic functions of composition operators and establish some properties namely, sufficient inclusion conditions, integral representations, univalency condition, coefficient inequalities and Fekete-Szegö problems.***

**Keywords:** *Analytic and univalent functions, integral and its inverse operator, Salagean derivative and its anti-derivative.*

**2020 Mathematical Subject Classification: 30C45.**

# 1 Introduction

Let $A$ denote the class of analytic functions in the unit disk

$$U = \{z \in C : |z| < 1\}$$

that have the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1}$$

Let $p$ denote the class of the functions

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{2}$$

analytic in $U$, satisfying $Rep(z) > 0$. Further, let $P(\beta)$ denote the subclass of $P$ with $Rep(z) > \beta$ for some real number $0 \leq \beta < 1$.

It is well-known that $f \in A$ is a starlike function of order $\beta$ (See [7]) denoted as $S^*(\beta)$ if

$$Re\frac{zf'(z)}{f(z)} > \beta.$$

Also, the class of bounded turning of order $\beta$ (see [15]) denoted as $R(\beta)$, if

$$Ref'(z) > \beta.$$

Using the Salagean differential operator introduced in [2], denoted by $D^n$ on $f(z)$, we have

$$Df(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k. \tag{3}$$

On the other hand, the integral operator [6] of Salagean type is

$$\mathcal{L}_{\sigma,\gamma}f(z) = \frac{(\lambda+\gamma)^{-\sigma}t^{\gamma-1}}{z^{\gamma}\Gamma - \sigma}\int_0^z (log\frac{z}{t})^{-\sigma-1}f(t)dt$$

on $f$. So, we have

$$\mathcal{L}_{\sigma,\gamma}f(z) = z + \sum_{k=2}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_k z^k. \tag{4}$$

We denote

$$\mathcal{L}_{\sigma,\gamma}(D^n f(z)) = D^n(\mathcal{L}_{\sigma,\gamma}f(z)) = L^n_{\sigma,\gamma}f(z). \tag{5}$$

Then

$$L^n_{\sigma,\gamma}f(z) = z + \sum_{k=2}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} k^n a_k z^k. \tag{6}$$

$n \in N \cup \{0\}, \sigma > 0, \gamma > -1$.

Note that $L^n_{1,0} = D^{n+1}f(z)$, $L^0_{1,0} = D^1 f(z) = zf'(z)$. then $L^0_{1,0} = zf'(z)$. From the series expansions of the operator $\mathcal{L}_{\sigma,\gamma}$ on $f(z)$, we have the recursive relation

$$z(\mathcal{L}_{\sigma,\gamma}f(z)' = (\gamma+1)\mathcal{L}_{\sigma,\gamma}f(z) - \gamma\mathcal{L}_{\sigma+1,\gamma}f(z). \tag{7}$$

Applying $D^n$ on (7), we have

$$L^{n+1}_{\sigma,\gamma}f(z) = (\gamma+1)L^n_{\sigma,\gamma}f(z) - \gamma L^n_{\sigma+1,\gamma}f(z). \tag{8}$$

Using the Salagean anti-derivative (see [2]) define as

$$I_n = I(I_{n-1}f(z)) = \int_0^z \frac{I_{n-1}f(t)}{t}dt$$

on $f$, we obtain

$$I_n = I(I_{n-1}f(z)) = z + \sum_{k=2}^{\infty}\frac{a_k}{k^n}z^k \tag{9}$$

and

$$\mathcal{J}_{\sigma,\gamma}f(z) = \frac{(\lambda+\gamma)^{\sigma}t^{\gamma-1}}{z^{\gamma}\Gamma\sigma}\int_0^z (log\frac{z}{t})^{\sigma-1}f(t)dt.$$

on $f$ (see [6]).

Therefore

$$\mathcal{J}_{\sigma,\gamma}f(z) = z + \sum_{k=2}^{\infty}\left(\frac{\gamma+1}{\gamma+k}\right)^{\sigma} a_k z^k. \tag{10}$$

We denote

$$I_n(\mathcal{J}_{\sigma,\gamma}f(z)) = \mathcal{J}_{\sigma,\gamma}(I_n f(z)) = J^n_{\sigma,\gamma}f(z). \tag{11}$$

Then

$$J^n_{\sigma,\gamma}f(z) = z + \sum_{k=2}^{\infty}\left(\frac{\gamma+1}{\gamma+k}\right)^{\sigma} \frac{a_k}{k^n} z^k. \tag{12}$$

It can be seen that

$$L^n_{\sigma,\gamma}(J^n_{\sigma,\gamma}f(z)) = J^n_{\sigma,\lambda}(L^n_{\sigma,\gamma}f(z)) = f(z). \tag{13}$$

The construction of new operator using composition and some other methods for subclasses of analytic and meromorphic functions in theory of geometric function has been considered by many researchers (see [8],[9],[10],[11],[12],[13],[14]).

Using the operator $L^n_{\sigma,\gamma}$, we introduce a new class defined as follows.
Definition 1. An analytic function $f \in A$ is said to belong to the class $B^n_{\sigma,\gamma}(\beta)$ if it satisfies the geometric condition

$$Re\frac{L^n_{\sigma,\gamma}f(z)}{z} > \beta, 0 \le \beta < 1. \tag{14}$$

**Remark 1**: If $n = 0$, $\sigma = 1$ and $\gamma = 0$, we have

$$Ref^{'}(z) > \beta.$$

The purpose of this paper is to study the subclass of analytic functions define by the composition of two operator denoted as $B^n_{\sigma,\gamma}(\beta)$ and investigate some properties namely, sufficient inclusion conditions, integral representations, univalency condition, coefficient inequalities and Fekete-Szegö problems.

The paper is organized as follows: in Section 2, relevant lemmas are stated, the main results are stated and proved in Section 3. Finally, Section 4 proposes suggestions for more results in this direction.

# 2 Preliminary Lemmas

**Lemma 1** [4]. Let $p(z)$ be analytic in $U$ with $p(0) = 1$. Suppose that

$$Re\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\beta - 1}{2\beta}.$$

Then

$$Rep(z) > 2^{1-\frac{1}{\beta}}, \frac{1}{2} \leq \beta < 1, z \in U. \tag{15}$$

and the constant $2^{1-\frac{1}{\beta}}$ is the best possible.
**Lemma 2** [1]. Let $p \in P$, then

$$|p_k| \leq 2, k = 1, 2, 3, \tag{16}$$

**Lemma 3**[5]. Let $p \in P$. Then for any real or complex number $\mu$, we have sharp inequalities

$$\left|p_2 - \mu\frac{p_1^2}{2}\right| \leq 2\max\{1, |1 - \mu|\}. \tag{17}$$

**Lemma 4**[3]. Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ and $\Phi(u, v)$ a complex valued function satisfying
(i) $\Phi(u, v)$ is continuous in a domain $\Omega$ of $C^2$.
(ii) $(1, 0) \in \Omega$ and $Re\Phi(1, 0) > 0$.
(iii) $Re\Phi(\beta + (1 - \beta)u_2 i, v_1) \leq \beta$ when $(\beta + (1 - \beta)u_2 i, v_1) \in \Omega$ and

$$2v_1 \leq -(1 - \beta)(1 + u_2^2)$$

for $0 \leq \beta < 1$. If $p \in P$ such that $(p(z), zp'(z)) \in \Omega$ and $Re(p(z), zp'(z)) > \beta$ for $z \in U$. Then $Rep(z) > \beta$ in $U$.

# 3 Main Results

**Theorem 1.** $B^{n+1}_{\sigma,\gamma}(\beta) \subset B^n_{\sigma,\gamma}(\beta)$.
**Proof.** Let

$$\frac{L^n_{\sigma,\gamma} f(z)}{z} = p(z) \tag{18}$$

$$L^n_{\sigma,\gamma} f(z) = zp(z)) \tag{19}$$

$$(L^n_{\sigma,\gamma} f(z))' = p(z) + zp'(z) \tag{20}$$

$$z(L^n_{\sigma,\gamma} f(z))' = zp(z) + z^2p'(z) \tag{21}$$

$$L^{n+1}_{\sigma,\gamma} f(z) = z^2p'(z) + zp(z) \tag{22}$$

which becomes

$$L^{n+1}_{\sigma,\gamma} f(z) = z\left(zp'(z) + p(z)\right) \tag{23}$$

so that if $f \in B^{n+1}_{\sigma,\gamma}(\lambda)$ then

$$Re\frac{L^{n+1}_{\sigma,\gamma} f(z)^{\lambda}}{z} = Re\left(zp'(z) + p(z)\right) > \beta. \tag{24}$$

Now define $\Phi(u,v) = u + v$. Noting that $Rep(z) > \beta$, then $\Phi$ satisfies all the conditions of Lemma 4, it follows that

$$Re\frac{L^{n+1}_{\sigma,\gamma} f(z)}{z} = Rep(z) > \beta. \tag{25}$$

meaning that $f \in B^{n}_{\sigma,\gamma}(\lambda)$.

**Theorem 2.** Let $f \in B^{n}_{\sigma,\gamma}(\beta)$. Then $f$ has the integral representation

$$f(z) = J^{n}_{\sigma,\lambda}\{zp(z)\}. \tag{26}$$

**Proof.** Since $f \in B^{n}_{\sigma,\gamma}(\beta)$, there exists $p(z) \in P(\beta)$ such that

$$\frac{L^{n}_{\sigma,\gamma} f(z)}{z} = p(z) \tag{27}$$

which becomes

$$L^{n}_{\sigma,\gamma} f(z) = zp(z). \tag{28}$$

By applying the inverse operator$J^{n}_{\sigma,\gamma} f(z)$, we have

$$f(z) = J^{n}_{\sigma,\lambda}\{zp(z)\}. \tag{29}$$

**Theorem 3.** If $f \in A$ satisfies

$$Re\left(\frac{L^{n+1}_{\sigma,\gamma} f(z)}{L^{n}_{\sigma,\gamma} f(z)}\right) > \frac{3\beta - 1}{2\beta} \tag{30}$$

then

$$Re\frac{L^{n}_{\sigma,\gamma} f(z)}{z} > 2^{1-1/\beta},$$

where $1/2 \le \beta < 1, z \in U$.

**Proof.** Let

$$p(z) = \frac{L^{n}_{\sigma,\gamma} f(z)}{z}.$$

Then

$$p'(z) = \frac{z(L^{n}_{\sigma,\gamma} f(z))' - L^{n}_{\sigma,\gamma} f(z)}{z^2}$$

$$\frac{zp^{'}(z)}{p(z)} = \frac{L^{n+1}_{\sigma,\gamma}f(z)}{L^{n}_{\sigma,\gamma}f(z)} - 1.$$

By the condition of the theorem, we have

$$Re\left(1 + \frac{zp^{'}(z)}{p(z)}\right) = Re\left(1 + \frac{L^{n+1}_{\sigma,\gamma}f(z)}{L^{n}_{\sigma,\gamma}f(z)} - 1\right) > \frac{3\beta - 1}{2\beta}$$

which is equivalent to

$$Re\left(1 + \frac{zp^{'}(z)}{p(z)}\right) = Re\left(\frac{L^{n+1}_{\sigma,\gamma}f(z)}{L^{n}_{\sigma,\gamma}f(z)}\right) > \frac{3\beta - 1}{2\beta}.$$

By Lemma 1, $Rep(z) > 2^{1-\frac{1}{\beta}}$, $1/2 \leq \beta < 1$ and the result follows.

**Corollary 4.** If $f \in A$, satisfies the condition (30), then $f(z) \in B^{n}_{\sigma,\gamma}(2^{1-1/\beta})$.

By letting $n = 0$ and $\beta = 1/2$, we have
**Corollary 5.** Suppose

$$Re\left(\frac{zf^{''}(z)}{f(z)} + 1\right) > \frac{1}{2}.$$

Then $Ref^{'}(z) > 1/2$.
**Theorem 6.** Let $f \in B^{n}_{\sigma,\gamma}(\lambda)$, then

$$|a_2| \leq 2(1-\beta)$$

$$|a_3| \leq \frac{2(1-\beta)(\gamma+1)^{\sigma}}{2^{n}(\gamma+2)^{\sigma}}. \tag{31}$$

The bounds are best possible. Equalities are obtained also by

$$f(z) = \left\{J^{n}_{\sigma,\gamma}z\left(\tfrac{1+(1-2\beta)z}{1-z}\right)\right\}$$

$$f(z) = z + 2(1-\beta)z^2 + \left(\frac{\gamma+1}{\gamma+2}\right)^{\sigma}\frac{2(1-\beta)}{2^{n}}z^3 + \left(\frac{\gamma+1}{\gamma+3}\right)^{\sigma}\frac{2(1-\beta)}{3^{n}}z^4 + \cdots. \tag{32}$$

**Proof.** Let $f \in B^{n}_{\sigma,\lambda}(\beta))$, then there exists $p \in P(\beta)$ such that

$$L^{n}_{\sigma,\lambda}f(z) = z(\beta + (1-\beta)p(z) \tag{33}$$

then

$$f(z) = J^{n}_{\sigma,\lambda}\{z(\beta + (1-\beta)p(z)\} \tag{34}$$

$$f(z) = J^n_{\sigma,\lambda}\{z + (1-\beta)c_1z^2 + (1-\beta)c_2z^3 + (1-\beta)c_3c^4 + \cdots\} \tag{35}$$

$$f(z) = z\left\{1 + \left(\frac{\gamma+1}{\gamma+1}\right)^\sigma (1-\beta)c_1 + \left(\frac{\gamma+1}{\gamma+2}\right)^\sigma \frac{1-\beta}{2^n}c_2z^2 + \left(\frac{\gamma+1}{\gamma+3}\right)^\sigma \frac{1-\beta}{3^n}c_3z^3 + \cdots\right\} \tag{36}$$

$$f(z) = z + \left(\frac{\gamma+1}{\gamma+1}\right)^\sigma (1-\beta)c_1z^2 + \left(\frac{\gamma+1}{\gamma+2}\right)^\sigma \frac{1-\beta}{2^n}c_2z^3 + \left(\frac{\gamma+1}{\gamma+3}\right)^\sigma \frac{1-\beta}{3^n}c_3z^4 + \cdots \tag{37}$$

Since

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots \tag{38}$$

and by comparing with respect to the power of $z$, we have that

$$a_2 = \left(\frac{\gamma+1}{\gamma+1}\right)^\sigma (1-\beta)c_1 \tag{39}$$

and

$$a_3 = \left(\frac{\gamma+1}{\gamma+2}\right)^\sigma \frac{1-\beta}{2^n}c_2. \tag{40}$$

By Lemma 2, we obtain the bound of $a_2$ and $a_3$.

**Theorem 7.** Let $f \in B^n_{\sigma,\lambda}(\beta)$, then for any real or complex number $\lambda$

$$|a_3 - \lambda a_2^2| = \frac{2(1-\beta)(\gamma+1)^\sigma}{2^n(\gamma+2)\sigma}\max\{1, |1-\alpha|\} \tag{41}$$

where

$$\alpha = \frac{2(1-\beta)(\gamma+1)^\sigma}{2^n(\gamma+2)^\sigma}.$$

**Proof.** From Theorem 6, we have

$$a_2 = (1-\beta)c_1. \tag{42}$$

and

$$a_3 = \left(\frac{\gamma+1}{\gamma+2}\right)^\sigma \frac{1-\beta}{2^n}c_2. \tag{43}$$

then

$$a_3 - \lambda a_2^2 = \left(\frac{\gamma+1}{\gamma+2}\right)^\sigma \frac{1-\beta}{2^n}c_2 - \lambda((1-\beta)c_1)^2. \tag{44}$$

equation becomes

$$a_3 - \lambda a_2^2 = \frac{2(1-\beta)(\gamma+1)^\sigma}{2^n(\gamma+2)^\sigma}\left[c_2 - \alpha\frac{c_1^2}{2}\right]. \tag{45}$$

By Lemma 2,

$$\left[c_2 - \alpha\frac{c_1^2}{2}\right] \leq 2\max\{1, |1-\alpha|\}$$

and the bound is obtained as desired.

# 4 Open Problem

The author suggests studying more classes defined by the composition of two operators as

$$L^n_{\sigma,\gamma}f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} k^n a_k z^k.$$

**ACKNOWLEDGEMENTS.** The author would like to thank the referee for making useful corrections which led to the improvement of the paper.

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