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On Subclass of Analytic Bi-Close-to-Convex Functions

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Abstract

In this paper, we investigate and introduce a new subclass of the function class Σ of analytic bi-close-to-convex functions defined on U. For this new subclass, we determine the first two initial coefficient estimates $|a_1|$ and $|a_2|$. Furthermore, several new or known consequences of our result are mentioned.

Keywords: *Analytic functions, univalent and bi-univalent functions, close-to-convex functions, starlike functions, coefficient bounds, subordination.*

2020 Mathematical Subject Classification: 30C45.

1 Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $U := \{z \in C : |z| < 1\}$. Also let $S := \{f \in A : f \text{ is univalent in } U\}$. A function $f \in A$ is said to be starlike of order α,$(0 \leq \alpha < 1)$ if $\Re\left(zf'(z)/f(z)\right) > 0,\ (z \in U)$ and denoted by $S^*(\alpha)$. Also. a function $f \in A$ is said to be close-to-convex k if there is a function

$g \in S^*$ satisfies the inequality $\Re\left(zf'(z)/g(z)\right) > 0,\ (z \in U)$.

It is well known that

$$S^*(\alpha) \subset S^*(0) = S^* \subset S \quad \text{and} \quad S^* \subset k \subset S.$$

According to the Koebe one-quarter theorem, that every function $f \in A$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in U),$$

and

$$f^{-1}(f(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \frac{1}{4}),$$

where

$$F(w) = f^{-1}(w) = w + a_2 w^2 + (2a_2^2 - 3a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \quad (2)$$

For the subclass of close-to-convex analytic functions related to the starlike functions, Gao and Zhou [10] introduced the subclass K_s as follows:

Definition 1.1 *A function $f(z) \in S$ is said to be in the class K_s if there exists a function $g = z + b_2 z^2 + \cdots \in S^*(1/2)$ such that*

$$\Re\left(\frac{z^2 f'(z)}{-g(z)g(-z)}\right) > 0, \quad (z \in U).$$

For a brief history of the subclasses of close-to-convex functions, see (also [11], [12], [15]). Motivated the Gao and Zhou works, Wang and Chen [16] defined the following subclass of close-to-convex functions by using the principle of subordination.

Definition 1.2 *For $0 \leq \lambda \leq 1$ and a function ϕ with positive real part, a function $f \in \Sigma$ given by (1) is said to be in the class $H_\Sigma(\lambda, \psi)$ if it satisfies the following subordination condition:*

$$\frac{z^2 f'(z) + \lambda z^3 f''(z)}{-g(z)g(-z)} \prec \psi(z), z \in U, \quad (3)$$

where $g \in S^*(1/2)$.

Remark 1.3 *1. For $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, then, $H_\Sigma(\lambda, \psi) := K_s(\lambda, A, B)$ has been studied by Wang and Chen [15].*

2. For $\lambda = 0$ and $\phi(z) = \frac{1+\beta z}{1-\alpha\beta z}$ $(0 \leq \alpha \leq 1,\ 0 < \beta \leq 1)$, then, $H_\Sigma(\lambda, \psi) := K_s(\alpha, \beta)$ has been studied by Wang et al. [16].

3. *For* $\lambda = 0$ *and* $\phi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 < \beta \le 1)$*, then,* $H_\Sigma(\lambda, \psi) := K_s(\beta)$ *has been studied by Kowalczyk and Le-Bomba [12].*

Recently, Seker and Sumer Eker [13] introduced new subclasses of the bi-univalent function class Σ as following:

Definition 1.4 *A function* $f(z)$ *given by (1) is said to be in the class* $K_{\Sigma_s}(\alpha)$ *if there exist the functions*

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in S^*(1/2),\ G(w) = w + \sum_{n=2}^{\infty} d_n w^n \in S^*(1/2)$$

and satisfying the following conditions

$$f \in \Sigma \quad and \quad \left|\frac{-z^2 f'(z)}{-g(z)g(-z)}\right| \le \frac{\alpha\pi}{2},\ (0 < \alpha \le 1\ z \in U) \tag{4}$$

and

$$\left|\frac{-w^2 F'(z)}{-G(w)G(-w)}\right| \le \frac{\alpha\pi}{2},\ (0 < \alpha \le 1\ z \in U) \tag{5}$$

where where $F(w)$ *is given by (2).*

Definition 1.5 *A function* $f(z)$ *given by (1) is said to be in the class* $K_{\Sigma_s}(\beta)$ *if there exist the functions*

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in S^*(1/2),\ G(w) = w + \sum_{n=2}^{\infty} d_n w^n \in S^*(1/2)$$

and satisfying the following conditions

$$f \in \Sigma \quad and\ \Re\left\{\frac{-z^2 f'(z)}{-g(z)g(-z)}\right\} > \beta,\ (0 \le \beta < 1\ z \in U) \tag{6}$$

and

$$\Re\left\{\frac{-w^2 F'(z)}{-G(w)G(-w)}\right\} > \beta,\ (0 \le \beta < 1\ z \in U) \tag{7}$$

where $F(w)$ *is given by (2).*

Now, we define a new subclass of $f(z) \in A$ by means of the following definition.

Definition 1.6 *For* $0 \le \lambda \le 1$ *and a function* ϕ *with positive real part, a function* $f \in \Sigma$ *given by (1) is said to be in the class* $H_{\Sigma_s}(\lambda, \psi)$ *if there exist the functions*

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in S^*(1/2),\ G(w) = w + \sum_{n=2}^{\infty} d_n w^n \in S^*(1/2)$$

and satisfying the following conditions

$$\frac{z^2 f'(z) + \lambda z^3 f''(z)}{-g(z)g(-z)} \prec \psi(z), z \in U \tag{8}$$

and

$$\frac{w^2 g'(w) + \lambda w^3 g''(w)}{-g(w)g(-w)} \prec \psi(w), w \in U, \tag{9}$$

where $g = f^{-1}$ *is defined by (2).*

Remark 1.7 *For* $\lambda = 0$*, the function* $f \in \Sigma$ *is in* $H_\Sigma(\psi)$ *if and only if*

$$\frac{z^2 f'(z)}{-g(z)g(-z)} \prec \psi(z), z \in U \tag{10}$$

and

$$\frac{w^2 g'(w)}{-g(w)g(-w)} \prec \psi(w), w \in U, \tag{11}$$

where $g = f^{-1}$*.*

Recently, some several authors (see, for example, [6]-[14]) studied and obtained coefficient bounds for various subclasses of bi-univalent functions.

In the present paper, we introduce a new subclass of bi-close-to-convex functions and find an estimate on the coefficients $|a_2|$ and $|a_3|$. Also,we mention several new or known results . For proving the main basic results, we will need the following lemma.

Lemma 1.8 *If* $g(z) = z + \sum_{n=2}^{\infty} d_n z^n \in S^*(1/2)$*. Then*

$$\phi(z) = \frac{-g(z)g(z)}{z} = z + \sum_{n=2}^{\infty} D_{2n-1} z^{2n-1} \in S^* \in S, \tag{12}$$

where the coefficients of the odd-starlike function ϕ *satisfy the condition*

$$|D_{2n-1}| = |2d_{2n-1} - 2d_2 d_{2n-2} + \cdots + 2(-1)^n d_{n-1} d_{n+1} + (-1)^{n+1} d_n^2|$$

$$\leq 1 \qquad (n \geq 2). \tag{13}$$

2 Coefficient Bounds for the Function Class $H_\Sigma(\lambda, \phi)$

Theorem 2.1 *If f given by (1) is in the class $H_\Sigma(\lambda, \phi)$, then*

$$|a_2| \leq \min\left\{\frac{B_1}{2(1+\lambda)}, \frac{B_1\sqrt{B_1+1}}{\sqrt{3(1+2\lambda)B_1^2+4(B_1-B_2)(1+\lambda)}}\right\} \tag{14}$$

and

$$|a_3| \leq \min\left\{\frac{B_1^2}{4(1+2\lambda)^2}+\frac{B_1+1}{3(1+2\lambda)}, \frac{1+B_1-B_2}{3(1+2\lambda)}\right\}. \tag{15}$$

Proof. We can write the inequalities (8) and (9), as following equivalent relations

$$p(z) = \frac{z^2 f'(z) + \lambda z^3 f''(z)}{-g(z)g(-z)}$$

$$= \frac{zf'(z) + \lambda z^2 f''(z)}{\frac{-g(z)g(-z)}{z}} = \frac{zf'(z) + \lambda z^2 f''(z)}{\Psi(z)} \tag{16}$$

$$\prec \phi(z) \quad (z \in U) \tag{17}$$

and

$$q(w) = \frac{w^2 F'(w) + \lambda w^3 F''(w)}{-G(z)G(-z)}$$

$$= \frac{wF'(w) + \lambda w^2 F''(w)}{\frac{-G(w)G(-w)}{w}} = \frac{wF'(w) + \lambda w^2 F''(w)}{\Omega(w)} \tag{18}$$

$$\prec \phi(w) \quad (w \in U), \tag{19}$$

$$\tag{20}$$

respectively, where

$$\Psi(z) = \frac{-g(z)g(-z)}{z} \quad \text{and} \quad \Omega(w) = \frac{-G(w)G(-w)}{w}.$$

Now, we note that the conditions (8) and (9) are equivalent to

$$\frac{z^2 f'(z) + \lambda z^3 f''(z)}{-g(z)g(-z)} = \phi(u(z)) \tag{21}$$

and

$$\frac{w^2g'(w) + \lambda w^3g''(w)}{-g(w)g(-w)} = \phi(v(z)), \tag{22}$$

where, the functions u and v are analytic in the unit disk U with $u(0) = v(0) = 0$, and $|u(z)| < 1, |v(z)| < 1$ for all $z \in U$.
Assume that

$$p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + p_1z + p_2z^2 + \cdots \tag{23}$$

and

$$q(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + q_1z + q_2z^2 + \cdots . \tag{24}$$

It is clear that $\Re(p(z)) > 0$ and $\Re(q(z)) > 0$. From (23) and (24), we can derive that

$$u(z) = \frac{1}{2}p_1z + \frac{1}{2}(p_2 - \frac{1}{2}p_1^2)z^2 + \cdots \tag{25}$$

and

$$v(z) = \frac{1}{2}q_1z + \frac{1}{2}(q_2 - \frac{1}{2}q_2^1)z^2 + \cdots . \tag{26}$$

Combining (21), (22) , (25) and (26), we get

$$\frac{z^2f'(z) + \lambda z^3f''(z)}{-g(z)g(-z)} = 1 + \frac{1}{2}B_1p_1z + \left(\frac{1}{4}B_2p_1^2 + \frac{1}{2}B_1(p_2 - \frac{1}{2}p_1^2)\right)z^2 + \cdots \tag{27}$$

and

$$\frac{w^2g'(w) + \lambda w^3g''(w)}{-g(w)g(-w)} = 1 + \frac{1}{2}B_1q_1w + \left(\frac{1}{4}B_2q_1^2 + \frac{1}{2}B_1(q_2 - \frac{1}{2}q_1^2)\right)w^2 + \cdots . \tag{28}$$

Now, from (16) and (18), we get

$$2(1 + \lambda)a_2 = \frac{1}{2}B_1b_1, \tag{29}$$

$$3(1 + 2\lambda)a_3 - A_3 = \frac{1}{2}B_1(p_2 - \frac{1}{2}p_1^2) + \frac{1}{2}B_2p_1^2, \tag{30}$$

$$-2(1 + \lambda)a_2 = \frac{1}{2}B_1q_1, \tag{31}$$

$$3(1 + 2\lambda)(2a_2^2 - a_3) - C_3 = \frac{1}{2}B_1(q_2 - \frac{1}{2}q_1^2) + \frac{1}{2}B_2q_1^2. \tag{32}$$

From (29) and (31), we get

$$p_1 = -q_1, \tag{33}$$

and

$$8(1 + \lambda)^2a_2^2 = \frac{1}{4}B_1^2(p_1^2 + q_1^2) \tag{34}$$

By adding (30) to (32), and (34), we find that

$$6(1+2\lambda)a_2^2 = A_3 + C_3 + \frac{1}{2}B_1(p_2+q_2) - \frac{1}{4}(B_1-B_2)(p_1^2+q_1^2). \tag{35}$$

Applying Lemma 1.8 for the equations $\phi(z)$ and $\omega(w)$ with (12) hold, we obtain

$$|a_2| \leq \frac{B_1\sqrt{B_1+1}}{\sqrt{3(1+2\lambda)B_1^2 + 4(B_1-B_2)(1+\lambda)}}. \tag{36}$$

Next, in order to find the bound on $|a_3|$, by subtracting (30) from (32), we get

$$6(1+2\lambda)a_3 - 6(1+2\lambda)a_2^2 + C_3 - A_3 = \frac{1}{2}B_1(p_2-q_2),$$

or

$$a_3 = a_2^2 + \frac{B_1(p_2-q_2) - 2C_3 + 2A_3}{12(1+2\lambda)}. \tag{37}$$

Applying Lemma 1.8 for the equations $\phi(z)$ and $\omega(w)$ with (12) hold, we obtain

$$|a_3| \leq \frac{B_1^2}{4(1+2\lambda)^2} + \frac{B_1+1}{3(1+2\lambda)}. \tag{38}$$

On the other hand, by substituting the value of a_2^2 from (34) into (37), we have

$$a_3 = \frac{4A_3 - 4C_3 + 2B_1(p_2+q_2) - (B_1-B_2)(p_1^2+q_1^2)}{24(1+2\lambda)} + \frac{B_1(p_2-q_2) - 2C_3 + 2A_3}{12(1+2\lambda)}.$$

By using Lemma 1.1 for the equations $\phi(z)$ and $\omega(w)$ with (12) hold, we obtain

$$|a_3| \leq \frac{1+B_1-B_2}{3(1+2\lambda)}. \tag{39}$$

Letting $\lambda = 0$ in Theorem 2.1, we have the following Corollary.

Corollary 2.2 *If f given by (1) is in the class $H_\Sigma(\phi)$, then*

$$|a_2| \leq \min\left\{\frac{B_1}{2}, \frac{B_1\sqrt{B_1+1}}{\sqrt{3B_1^2 + 4(B_1-B_2)}}\right\} \tag{40}$$

and

$$|a_3| \leq \min\left\{\frac{B_1^2}{4} + \frac{B_1+1}{3}, \frac{1+B_1-B_2}{3}\right\}. \tag{41}$$

3 Conclusion

In this paper, we defined two new subclasses of bi-univalent functions in the open unit disk U. For the functions belonging to these subclasses, the estimates of second and third Taylor–Maclaurin coefficients are obtained. Also, some interesting corollaries and remarks are discussed.

4 Open Problems

For both subclasses, it is interesting to obtain the upper bounds of the Fekete-Szego inequality, and the bounds of the Hankel determinants. Also, to obtain the estimates of second and third Taylor–Maclaurin coefficients m-fold symmetric bi-univalent functions.

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