Int. J. Open Problems Complex Analysis, Vol. 13, No. 2, July 2021
ISSN 2074-2827; Copyright ©ICSRS Publication, 2021
www.i-csrs.org

On a Class of Uniformly Analytic Functions with q-Analogue

A. O. Mostafa and Z. M. Saleh

Department of Mathematics,Faculty of Science, Mansoura University
Mansoura 35516, Egypt
e-mail: adelaeg254@yahoo.com
e-mail: zeinabnsar2@gmail.com

Received 15 October 2020; Accepted 29 March 2021

Abstract

In this paper using a $q-$analogue operator, we define class of uniformly analytic functions and find coefficient bounds, radius of convexity, closure theorems and other properties for functions in this class.

Keywords: *Analytic function, univalent functions, coefficient estimates, radius of convexity, closure theorems, $q-$analogue operator.*

2020 Mathematical Subject Classification: 30C45.

1 Introduction

Let $\mathcal{S}$ denote the class of functions of the form

$$\mathcal{F}(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic univalent in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = f'(0)-1 = 0$. It is known that the calculus without the notion of limits is called $q-$calculus which has influenced many scientific fields due to its important applications.The generalization of derivative in $q-$calculus that is $q-$derivative was defined and studied by Jackson [12]. He defined the $q-$derivative operator ∇_q for $\mathcal{F} \in \mathcal{T},\ 0 < q < 1$, by (see also[3, 4, 5, 6] [10],[16], [20, 21]);

$$\nabla_q \mathcal{F}(z) = \begin{cases} \frac{\mathcal{F}(z)-\mathcal{F}(qz)}{(1-q)z} & ,z\neq 0 \\ \mathcal{F}'(0) & ,z=0 \end{cases},$$

that is

$$\nabla_q \mathcal{F}(z) = 1 - \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{2}$$

where

$$[j]_q = \frac{1-q^j}{1-q}, \; [0]_q = 0. \tag{3}$$

As $q \to 1^-$, $[j]_q = j$ and $\nabla_q \mathcal{F}(z) = \mathcal{F}'(z)$.

Using the $q-$derivative operator ∇_q, Mostafa and Saleh [14] defined the $\mathcal{H}^m_{\lambda,\mu,q}$ operator for $\lambda \geq \mu \geq 0$, $0 < q < 1$, by

$$\mathcal{H}^0_{\lambda,\mu,q}\mathcal{F}(z) = \mathcal{F}(z),$$

$$\mathcal{H}^1_{\lambda,\mu,q}\mathcal{F}(z) = \mathcal{H}_{\lambda,\mu,q}\mathcal{F}(z) = (1-\lambda+\mu)\mathcal{F}(z) + (\lambda-\mu)z\nabla_q\mathcal{F}(z) + \lambda\mu z^2\nabla_q^2\mathcal{F}(z),$$

$$\mathcal{H}^2_{\lambda,\mu,q}\mathcal{F}(z) = \mathcal{H}_{\lambda,\mu,q}(\mathcal{H}_{\lambda,\mu,q}\mathcal{F}(z)),$$

and

$$\begin{aligned} \mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z) &= \mathcal{H}_{\lambda,\mu,q}(\mathcal{H}^{m-1}_{\lambda,\mu,q}\mathcal{F}(z)) \\ &= z - \sum_{k=2}^{\infty} \chi^m_{q,k}(\lambda,\mu) a_k z^k, m \in \mathbb{N}, \end{aligned} \tag{4}$$

where

$$\chi^m_{q,k}(\lambda,\mu) = [1-\lambda+\mu+[k]_q(\lambda-\mu+\lambda\mu[k-1]_q)]^m. \tag{5}$$

Note that:
(i) $\lim_{q\to 1-} \mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z) = \mathcal{H}^m_{\lambda,\mu}\mathcal{F}(z)$ (see [1]);
(ii) $\mathcal{H}^m_{1,0,q}\mathcal{F}(z) = \mathcal{D}^m_q\mathcal{F}(z)$ (see [11], [22] and [8]);
(iii) $\mathcal{H}^m_{\lambda,0,q}\mathcal{F}(z) = \mathcal{D}^m_{\lambda,q}\mathcal{F}(z)$ (see Aouf et al. [9]);
(iv) $\lim_{q\to 1-} \mathcal{H}^m_{\lambda,0,q}\mathcal{F}(z) = \mathcal{D}^m_{\lambda}\mathcal{F}(z)$ (see Al-Oboudi [2]);
(v) $\lim_{q\to 1-} \mathcal{H}^m_{1,0,q}\mathcal{F}(z) = \mathcal{D}^m\mathcal{F}(z)$ (Sălăgean ([15])).

Definition 1.1. *For* $0 \leq \zeta \leq 1$, $\lambda \geq \mu \geq 0$, $0 < q < 1$, $m \in \mathbb{N}_0$, $0 \leq \beta < 1$, $\kappa \geq 0$ *and* $\mathcal{F} \in \mathcal{S}$, *such that* $\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z) \neq 0$ *for* $z \in \mathcal{D}/\{0\}$, *we say that* $\mathcal{F} \in \mathbb{G}^m_q(\lambda,\mu,\beta,\kappa,\zeta)$ *if*

$$Re\left\{\frac{z\nabla_q\mathcal{G}(z)}{\mathcal{G}(z)} - \beta\right\} \geq \kappa\left|\frac{z\nabla_q\mathcal{G}(z)}{\mathcal{G}(z)} - 1\right|, \tag{6}$$

where,

$$\begin{aligned} \mathcal{G}(z) &= (1-\zeta)\mathcal{H}_{q,\lambda,\mu}\mathcal{F}(z) + \zeta z \nabla_q \mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z) \\ &= z - \sum_{k=2}^{\infty} \chi^m_{q,k}(\lambda,\mu)\left[1+\zeta([k]_q-1)\right] a_k z^k, \end{aligned} \tag{7}$$

Note that: For different values of $q, \lambda, \mu, \beta, \kappa, \zeta$, we have:

(i) $\lim_{q\to 1-} \mathbb{G}^m_q(\lambda,\mu,\beta,\kappa,\zeta) = \mathbb{G}^m(\lambda,\mu,\beta,\kappa,\zeta) = \left\{\mathcal{F}(z) : Re\left\{\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)} - \beta\right\} \geq \kappa \left|\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)} - 1\right|\right\}$;

(ii) $\mathbb{G}^m_q(0,0,\beta,\kappa,0) = \mathbb{G}^m_q(\beta,\kappa) = \left\{\mathcal{F}(z) : Re\left\{\frac{z\nabla_q\mathcal{F}(z))}{\mathcal{F}(z)} - \beta\right\} \geq \kappa \left|\frac{z\nabla_q\mathcal{F}(z))}{\mathcal{F}(z)} - 1\right|\right\}$;

(iii) $\mathbb{G}^m_q(0,0,\beta,\kappa,1) = C^{*m}_q(\beta,\kappa) = \left\{\mathcal{F}(z) : Re\left\{\frac{\nabla_q(z\nabla_q\mathcal{F}(z))}{\nabla_q\mathcal{F}(z)} - \beta\right\} \geq \kappa \left|\frac{\nabla_q(z\nabla_q\mathcal{F}(z))}{\nabla_q\mathcal{F}(z)} - 1\right|\right\}$;

(iv) $\lim_{q\to 1-} \mathbb{G}^m_q(\lambda,0,\beta,\kappa,0) = S^m(\lambda,\beta,\kappa)$, (see Aouf and mostafa [7]);

(v) $\mathbb{G}^0_q(1,0,\beta,0,0) = S^*_q(\beta) = \left\{\mathcal{F}(z) : Re\left\{\frac{z\nabla_q\mathcal{F}(z))}{\mathcal{F}(z)}\right\} \geq \beta\right\}$, (Seoudy and Aouf [17]);

(vi) $\mathbb{G}^m_q(0,0,\beta,0,1) = K_q(\beta) = \left\{\mathcal{F}(z) : Re\left\{\frac{\nabla_q(z\nabla_q\mathcal{F}(z))}{\nabla_q\mathcal{F}(z)} - \beta\right\} \geq 0\right\}$, (Seoudy and Aouf [17]).

In the rest of the paper, we find coefficient bounds, radius of convexity, closure theorems and other properties for functions in the class $\mathbb{G}^{*m}_q(\lambda,\mu,\beta,\kappa,\zeta)$.

2 Main Results

Unless indicated, we assume that $0 \leq \zeta \leq 1$, $\lambda \geq \mu \geq 0$, $0 < q < 1$, $m \in \mathbb{N}_0, 0 \leq \beta < 1$, $\kappa \geq 0$, and $z \in \mathcal{D}$.

Theorem 2.1. *Let $\mathcal{F} \in \mathcal{S}$ given by (1). Then $\mathcal{F} \in \mathbb{G}^m_q(\lambda,\mu,\beta,\kappa,\zeta)$ if*

$$\sum_{k=2}^{\infty} \left[[k]_q(1+\kappa) - (\kappa+\beta)\right]\left[1+\zeta([k]_q-1)\right] \chi^m_{q,k}(\lambda,\mu)\,|a_k| \leq 1-\beta. \tag{8}$$

Proof. Let (8) holds. Then it suffices to show that

$$\kappa\left|\frac{z\nabla_q\mathcal{G}(z)}{\mathcal{G}(z)} - 1\right| - Re\left\{\frac{z\nabla_q\mathcal{G}(z)}{\mathcal{G}(z)} - 1\right\} \leq 1-\beta.$$

We have

$$\begin{aligned}
& \kappa\left|\frac{z\nabla_q\mathcal{G}(z)}{\mathcal{G}(z)}-1\right|-Re\left\{\frac{z\nabla_q\mathcal{G}(z)}{\mathcal{G}(z)}-1\right\} \\
\leq\ & (1+\kappa)\left|\frac{z\nabla_q\mathcal{G}(z)}{\mathcal{G}(z)}-1\right| \\
\leq\ & \frac{(1+\kappa)\sum_{k=2}^{\infty}([k]_q-1)\,[1+\zeta([k]_q-1)]\,\chi_{q,k}^m(\lambda,\mu)\,|a_k|\,|z|^{k-1}}{1-\sum_{k=2}^{\infty}[1+\zeta([k]_q-1)]\,\chi_{q,k}^m(\lambda,\mu)\,|a_k|\,|z|^{k-1}} \\
\leq\ & \frac{(1+\kappa)\sum_{k=2}^{\infty}([k]_q-1)\,[1+\zeta([k]_q-1)]\,\chi_{q,k}^m(\lambda,\mu)\,|a_k|}{1-\sum_{k=2}^{\infty}[1+\zeta([k]_q-1)]\chi_{q,k}^m(\lambda,\mu)\,|a_k|}.
\end{aligned}$$

From (8) the last expression is bounded above by $(1-\beta)$. Hence $\mathcal{F}(z)$ satisfies the condition (6).

Corollary 2.1. *Let $\mathcal{F}\in\mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$. Then*

$$a_k\leq\frac{1-\beta}{[[k]_q(1+\kappa)-(\kappa+\beta)]\,[1+\zeta([k]_q-1)]\,\chi_{q,k}^m(\lambda,\mu)}\ (k\geq 2). \tag{9}$$

The result is sharp for

$$\mathcal{F}(z)=z+\frac{1-\beta}{[[k]_q(1+\kappa)-(\kappa+\beta)]\,[1+\zeta([k]_q-1)]\,\chi_{q,k}^m(\lambda,\mu)}z^k\ (k\geq 2). \tag{10}$$

Let

$$\mathcal{F}_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k,\ (a_{k,j}\geq 0,\ j=1,2,...,n), \tag{11}$$

Theorem 2.2. *Let $\mathcal{F}_j(z)\in\mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$, $j=1,2,...,n$. Then*

$$g(z)=\sum_{j=1}^{n}d_j\mathcal{F}_j(z)\quad(d_j\geq 0), \tag{12}$$

is also in the class $\mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$, where

$$\sum_{j=1}^{n}d_j{=}1. \tag{13}$$

Proof. According to (12), we can write

$$g(z)=z+\sum_{k=2}^{\infty}\left(\sum_{j=1}^{n}d_ja_{k,j}\right)z^k. \tag{14}$$

Further, since $\mathcal{F}_j(z) \in \mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$, we get

$$\sum_{k=2}^{\infty} \left[[k]_q(1+\kappa) - (\kappa+\beta)\right] \left[1+\zeta([k]_q-1)\right] \chi_{q,k}^m(\lambda,\mu) a_{k,j} \leq 1-\beta. \tag{15}$$

Hence

$$\begin{aligned} & \sum_{k=2}^{\infty} \left[[k]_q(1+\kappa) - (\kappa+\beta)\right] \left[1+\zeta([k]_q-1)\right] \chi_{q,k}^m(\lambda,\mu) \left(\sum_{j=1}^{n} d_j a_{k,j}\right) \\ = & \sum_{j=1}^{n} d_j \left[\sum_{k=2}^{\infty} \left[[k]_q(1+\kappa) - (\kappa+\beta)\right] \left[1+\zeta([k]_q-1)\right] \chi_{q,k}^m(\lambda,\mu) a_{k,j}\right] \\ \leq & \left(\sum_{j=1}^{n} d_j\right)(1-\beta) = (1-\beta), \end{aligned} \tag{16}$$

which implies that $g(z) \in \mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$. Thus we have the theorem.

Theorem 2.3. *The class $\mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$ is closed under convex linear compination.*

Proof. Let $\mathcal{F}_j(z) \in \mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$, $j = 1,2$ and

$$g(z) = \zeta\mathcal{F}_1(z) + (1-\zeta)\mathcal{F}_2(z) \quad (0 \leq \zeta \leq 1). \tag{17}$$

Then by, taking $n = 2$, $d_1 = \zeta$ and $d_2 = 1-\zeta$ in Theorem 2, we have $g(z) \in \mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$.

Theorem 2.4. *Let $\mathcal{F}_1(z) = z$ and*

$$\mathcal{F}_k(z) = z + \frac{(1-\beta)}{\left[[k]_q(1+\kappa) - (\kappa+\beta)\right] \left[1+\zeta([k]_q-1)\right] \chi_{q,k}^m(\lambda,\mu)} z^k \quad (k \geq 2). \tag{18}$$

Then $\mathcal{F}(z) \in \mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$ if and only if it can be expressed in the form

$$\mathcal{F}(z) = \sum_{k=1}^{\infty} \eta_k \mathcal{F}_k(z), \tag{19}$$

where $\eta_k \geq 0$ $(k \geq 1)$ and

$$\sum_{k=1}^{\infty} \eta_k = 1. \tag{20}$$

Proof. Suppose that

$$\mathcal{F}(z)=\sum_{k=1}^{\infty}\eta_k\mathcal{F}_k(z)=z+\sum_{k=2}^{\infty}\frac{(1-\beta)}{[[k]_q(1+\kappa)-(\kappa+\beta)]\,[1+\zeta([k]_q-1)]\,\chi_{q,k}^{m}(\lambda,\mu)}\eta_k z^k.$$

Then it follows that

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{[[k]_q(1+\kappa)-(\kappa+\beta)]\,[1+\zeta([k]_q-1)]\,\chi_{q,k}^{m}(\lambda,\mu)}{(1-\beta)}.\\&\frac{(1-\beta)}{[[k]_q(1+\kappa)-(\kappa+\beta)]\,[1+\zeta([k]_q-1)]\,\chi_{q,k}^{m}(\lambda,\mu)}\eta_k\\=&\sum_{k=2}^{\infty}\eta_k=1-\eta_1\leq 1.\end{aligned}\tag{21}$$

So by Theorem 1, $\mathcal{F}(z)\in\mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$.

Conversely, assume that $\mathcal{F}(z)\in\mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$. Then

$$a_k\leq\frac{(1-\beta)}{[[k]_q(1+\kappa)-(\kappa+\beta)]\,[1+\zeta([k]_q-1)]\,\chi_{q,k}^{m}(\lambda,\mu)}\ (k\geq 2).\tag{22}$$

Setting

$$\eta_k=\frac{[[k]_q(1+\kappa)-(\kappa+\beta)]\,[1+\zeta([k]_q-1)]\,\chi_{q,k}^{m}(\lambda,\mu)}{(1-\beta)}a_k\ (k\geq 2),\tag{23}$$

and

$$\eta_1=1-\sum_{k=2}^{\infty}\eta_k,$$

we see that $\mathcal{F}(z)$ can be expressed in the form (19). This completes the proof.

Corollary 2.2. *The extreme points of* $\mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$ *are* $\mathcal{F}_k(z)$ $(k\geq 2)$ *given by Theorem 4.*

Theorem 2.5. *Let* $\mathcal{F}(z)\in\mathbb{G}_q^m(\lambda,\mu,\beta,\kappa,\zeta)$. *Then for* $0\leq\rho<1$, $k\geq 2$, $\mathcal{F}(z)$ *is*

(i) close -to- convex of order ρ in $|z|<r_1$, where

$$r_1=r_1(m,\lambda,\mu,\beta,\kappa,\zeta,\rho)=\inf_k\left[\frac{(1-\rho)\,[[k]_q(1+\kappa)-(\kappa+\beta)]\,[1+\zeta([k]_q-1)]\,\chi_{q,k}^{m}(\lambda,\mu)}{[k]_q(1-\beta)}\right]^{\frac{1}{k-1}}.\tag{24}$$

(ii) starlike of order ρ in $|z| < r_2$, where

$$r_2 = r_2(m, \lambda, \mu, \beta, \kappa, \zeta, \rho) = \inf_k \left[\frac{(1-\rho)\left[[k]_q(1+\kappa) - (\kappa+\beta)\right]\left[1+\zeta([k]_q - 1)\right]\chi^m_{q,k}(\lambda,\mu)}{([k]_q - \rho)(1-\beta)} \right]^{\frac{1}{k-1}}. \tag{25}$$

(iii) convex of order ρ in $|z| < r_3$, where

$$r_3 = r_2(m, \lambda, \mu, \beta, \kappa, \zeta, \rho) = \inf_k \left[\frac{(1-\rho)\left[[k]_q(1+\kappa) - (\kappa+\beta)\right]\left[1+\zeta([k]_q - 1)\right]\chi^m_{q,k}(\lambda,\mu)}{[k]_q([k]_q + 1 - \rho)(1-\beta)} \right]^{\frac{1}{k-1}}. \tag{26}$$

The results are sharp, for $\mathcal{F}(z)$ given by (10).
Proof. To prove (i) we must show that

$$\left|\mathcal{F}'(z) - 1\right| \leq 1 - \rho, \quad (|z| < r_1).$$

From (2), we have

$$\left|\mathcal{F}'(z) - 1\right| \leq \sum_{k=2}^{\infty} [k]_q a_k \left|z\right|^{k-1},$$

thus

$$\left|\mathcal{F}'(z) - 1\right| \leq 1 - \rho,$$

if

$$\sum_{k=2}^{\infty} \frac{[k]_q}{1-\rho} a_k \left|z\right|^{k-1} \leq 1. \tag{27}$$

But, by Theorem 1, (27) will be true if

$$\frac{[k]_q}{1-\rho}\left|z\right|^{k-1} \leq \frac{\left[[k]_q(1+\kappa) - (\kappa+\beta)\right]\left[1+\zeta([k]_q - 1)\right]\chi^m_{q,k}(\lambda,\mu)}{1-\beta},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\left[[k]_q(1+\kappa) - (\kappa+\beta)\right]\left[1+\zeta([k]_q - 1)\right]\chi^m_{q,k}(\lambda,\mu)}{[k]_q(1-\beta)} \right]^{\frac{1}{k-1}} \quad (k \geq 2),$$

which gives (24).

To prove (ii) and (iii) it is suffices to show

$$\left|\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1\right| \leq 1 - \rho, \quad (|z| < r_2), \tag{28}$$

$$\left|\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right| \leq 1 - \rho, \quad (|z| < r_3), \tag{29}$$

respectively, by using arguments as in proving (i), we have the results.

For $\mathcal{F}(z) \in \mathcal{S}$, given by 1, the sequence of partial sums is given by

$$\mathcal{F}_n(z) = z + \sum_{k=2}^{n} a_k z^k \quad (n \in \mathbb{N}\backslash\{1\}). \tag{30}$$

Now we will follow the work of [19] and also the works cited in [13, 18] on partial sums of analytic functions, to obtain the results. Let

$$\Phi_{q,k}^{m}(\lambda,\mu,\beta,\kappa) = [[k]_q(1+\kappa) - (\kappa+\beta)]\,[1+\zeta([k]_q-1)]\,\chi_{q,k}^{m}(\lambda,\mu). \tag{31}$$

Theorem 2.6. *If $\mathcal{F}(z) \in \mathcal{S}$, satisfies the condition (8), then*

$$Re\left(\frac{\mathcal{F}(z)}{\mathcal{F}_n(z)}\right) \geq \frac{\Phi_{q,n+1}^{m} - 1 + \beta}{\Phi_{q,n+1}^{m}}, \tag{32}$$

where

$$\Phi_{q,k}^{m} \geq \begin{cases} 1-\beta, & if\ k = 2,3,...,n \\ \Phi_{q,n+1}^{m}, & if\ k = n+1, n+2, ... \end{cases}. \tag{33}$$

The result (32) is sharp for

$$\mathcal{F}(z) = z + \frac{1-\beta}{\Phi_{q,n+1}^{m}} z^{n+1}. \tag{34}$$

Proof. Let

$$\begin{aligned} \frac{1+\omega(z)}{1-\omega(z)} &= \frac{\Phi_{q,n+1}^{m}}{1-\beta}\left[\frac{\mathcal{F}(z)}{\mathcal{F}_n(z)} - \frac{\Phi_{q,n+1}^{m} - 1 + \beta}{\Phi_{q,n+1}^{m}}\right] \\ &= \frac{1 + \sum_{k=2}^{n} a_k z^{k-1} + \left(\frac{\Phi_{q,n+1}^{m}}{1-\beta}\right)\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}}. \end{aligned} \tag{35}$$

It suffices to show that $|\omega(z)| \leq 1$. Now from (35) we have

$$\omega(z) = \frac{\left(\frac{\Phi_{q,n+1}^{m}}{1-\beta}\right)\sum_{k=n+1}^{\infty} a_k z^{k-1}}{2 + 2\sum_{k=2}^{n} a_k z^{k-1} + \left(\frac{\Phi_{q,n+1}^{m}}{1-\beta}\right)\sum_{k=n+1}^{\infty} a_k z^{k-1}}.$$

Hence we obtain

$$|\omega(z)| \leq \frac{\left(\frac{\Phi_{q,n+1}^{m}}{1-\beta}\right)\sum_{k=n+1}^{\infty} a_k}{2 - 2\sum_{k=2}^{n} a_k - \left(\frac{\Phi_{q,n+1}^{m}}{1-\beta}\right)\sum_{k=n+1}^{\infty} a_k}.$$

Now $|\omega(z)| \leq 1$ if and only if

$$2\left(\frac{\Phi^m_{q,n+1}}{1-\beta}\right)\sum_{k=n+1}^{\infty} a_k \leq 2-2\sum_{k=2}^{n} a_k,$$

or, equivalently

$$\sum_{k=2}^{n} a_k+\sum_{k=n+1}^{\infty}\left(\frac{\Phi^m_{q,n+1}}{1-\beta}\right)a_k \leq 1.$$

From (8), it is sufficient to show that

$$\sum_{k=2}^{n} a_k+\sum_{k=n+1}^{\infty}\left(\frac{\Phi^m_{q,n+1}}{1-\beta}\right)a_k \leq \sum_{k=2}^{\infty}\left(\frac{\Phi^m_{q,k}}{1-\beta}\right)a_k,$$

which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{\Phi^m_{q,k}-1+\beta}{1-\beta}\right)a_k+\sum_{k=n+1}^{\infty}\left(\frac{\Phi^m_{q,k}-\Phi^m_{q,n+1}}{1-\beta}\right)a_k \geq 0. \tag{36}$$

For $z = re^{i\pi/n}$ we have

$$\frac{\mathcal{F}(z)}{\mathcal{F}_n(z)} = 1+\frac{1-\beta}{\Phi^m_{q,n+1}}z^k \to 1-\frac{1-\beta}{\Phi^m_{q,n+1}}z^k = \frac{\Phi^m_{q,n+1}-1+\beta}{\Phi^m_{q,n+1}} \text{ where } r \to 1^-,$$

which shows that $\mathcal{F}(z)$ given by (34) gives the sharpness.

Theorem 2.7. *If $\mathcal{F}(z) \in \mathcal{S}$, satisfies the condition (8), then*

$$Re\left(\frac{\mathcal{F}_n(z)}{\mathcal{F}(z)}\right) \geq \frac{\Phi^m_{q,n+1}}{\Phi^m_{q,n+1}+1-\beta}, \tag{37}$$

where $\Phi^m_{q,n+1}$ is defined by (31) and satisfies (33) and $\mathcal{F}(z)$ given by (34) gives the sharpness.

Proof. The proof follows by defining

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{\Phi^m_{q,n+1}+1-\beta}{1-\beta}\left[\frac{\mathcal{F}_n(z)}{\mathcal{F}(z)}-\frac{\Phi^m_{q,n+1}}{\Phi^m_{q,n+1}+1-\beta}\right]. \tag{38}$$

The reminder part is as in Theorem 6. So, we omit it.

Theorem 2.8. *If $\mathcal{F}(z) \in \mathcal{S}$, satisfies the condition (8), then*

$$Re\left(\frac{\mathcal{F}'(z)}{\mathcal{F}'_n(z)}\right) \geq \frac{\Phi^m_{q,n+1}-(n+1)(1-\beta)}{\Phi^m_{q,n+1}}, \tag{39}$$

and

$$Re\left(\frac{\mathcal{F}_n'(z)}{\mathcal{F}'(z)}\right) \geq \frac{\Phi_{q,n+1}^m}{\Phi_{q,n+1}^m+(n+1)(1-\beta)}, \tag{40}$$

where $\Phi_{q,n+1}^m \geq (n+1)(1-\beta)$ *and*

$$\Phi_{q,k}^m \geq \begin{cases} [k]_q(1-\beta), & if\ k=2,3,...,n \\ [k]_q(\frac{\Phi_{q,n+1}^m}{n+1}), & if\ k=n+1,n+2,... \end{cases}, \tag{41}$$

$\mathcal{F}(z)$ *given by (34) gives the sharpness.*

Proof. We write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{\Phi_{q,n+1}^m}{(n+1)(1-\beta)}\left[\frac{\mathcal{F}'(z)}{\mathcal{F}_n'(z)} - \left(\frac{\Phi_{q,n+1}^m-(n+1)(1-\beta)}{\Phi_{q,n+1}^m}\right)\right],$$

where

$$\omega(z) = \frac{\left(\frac{\Phi_{q,n+1}^m}{(n+1)(1-\beta)}\right)\sum_{k=n+1}^{\infty}[k]_q a_k z^{k-1}}{2+2\sum_{k=2}^{n}[k]_q a_k z^{k-1}+\left(\frac{\Phi_{q,n+1}^m}{(n+1)(1-\beta)}\right)\sum_{k=n+1}^{\infty}[k]_q a_k z^{k-1}}.$$

Now $|\omega(z)| \leq 1$ if and only if

$$\sum_{k=2}^{n}[k]_q a_k + \left(\frac{\Phi_{q,n+1}^m}{(n+1)(1-\beta)}\right)\sum_{k=n+1}^{\infty}[k]_q a_k \leq 1,$$

From (8), it is sufficient to show that

$$\sum_{k=2}^{n}[k]_q a_k + \left(\frac{\Phi_{q,n+1}^m}{(n+1)(1-\beta)}\right)\sum_{k=n+1}^{\infty}[k]_q a_k \leq \sum_{k=2}^{\infty}\left(\frac{\Phi_{q,k}^m}{1-\beta}\right)a_k,$$

which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{\Phi_{q,k}^m-[k]_q(1-\beta)}{1-\beta}\right)a_k + \sum_{k=n+1}^{\infty}\left(\frac{(n+1)\Phi_{q,k}^m-[k]_q\Phi_{q,n+1}^m}{(n+1)(1-\beta)}\right)a_k \geq 0.$$

To prove the result (40), define the function $\omega(z)$ by

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{(n+1)(1-\beta)+\Phi_{q,n+1}^m}{(n+1)(1-\beta)}\left[\frac{\mathcal{F}_n'(z)}{\mathcal{F}'(z)} - \frac{\Phi_{q,n+1}^m}{(n+1)(1-\beta)+\Phi_{q,n+1}^m}\right],$$

and by similar arguments in first part we get desired result.

Remark 2.1. *Putting* $\kappa = 0$ *and letting* $q \to 1^-$ *in Theorems 7, 8 and 9, respectively, we obtain partial sum results for the class* $\mathbb{G}^m(\lambda,\mu,\beta,\zeta)$.

Remark 2.2. *For different values of* $\kappa, \zeta, \lambda, \mu, q$ *and* β *in our results, we have results for the special classes defined in the introduction.*

3 Conclusions

Using the operator $\mathcal{H}^m_{\lambda,\mu,q}$ $(\lambda \geq \mu \geq 0,\ 0 < q < 1)$ defined by the authors in [14], we defined the class $\mathbb{G}^m(\lambda, \mu, \beta, \zeta)$ of uniformly analytic functions and find coefficient bounds, radius of convexity, closure theorems and other properties for functions in this class.

4 Open Problem

The authors suggest studying other properties such that neighborhood and Hadamard for the class $\mathbb{G}^m_q(\lambda, \mu, \beta, \zeta)$ when $f(z)$ has the form

$$\mathcal{F}(z) = z - \sum_{k=2}^{\infty} a_k z^k.$$

Acknowledgment. The authors thanks the referees of the paper for their valuable comments.

References

[1] M. Al-Kaseasbeh and M. Darus, Uniformly geometric functions involving constructed operators, J. Compl. Anal., 2017 (2017), ID 5916805, 1-7.

[2] F. M. Al-Oboudi, On univalent functions defined by a generlalized Sălăgean operator, Int. J. Math. Math. Sci., 27 (2004), 1429–1436.

[3] M. H. Annby and Z. S. Mansour, $q-$Fractional Calculus Equations. Lecture Notes in Mathematics., Vol. 2056, Springer, Berlin 2012.

[4] M. K. Aouf, H. E. Darwish and G. S. Sălăgean, On a generalization of starlike functions with negative coefficients, Math., Tome 43 66 (2001), no. 1, 3–10.

[5] M. K. Aouf and A. O. Mostafa, Subordination results for analytic functions associated with fractional $q-$calculus operators with complex order, Afr. Mat., 31 (2020), 1387–1396.

[6] M. K. Aouf and A. O. Mostafa, Some subordinating results for classes of functions defined by Sălăgean type $q-$derivative operator, Filomat., 34 (2020), no. 7, 2283–2292.

[7] M. K. Aouf and A. O. Mostafa, Some properties of a subclass of uniformly convex functions with negative coefficients, Demonstratio Mathematica., 41(2008), no. 2, 353-370.

[8] M. K. Aouf, A. O. Mostafa and F. Y. AL-Quhali, Properties for class of $\beta-$ uniformly univalent functions defined by Sălăgean type $q-$difference operator, Int. J. Open Probl. Complex Anal., 11 (2019), no. 2, 1–16.

[9] M. K. Aouf, A. O. Mostafa and R. E. Elmorsy, Certain subclasses of analytic functions with varying arguments associated with $q-$difference operator, Afr. Math., 32 (2021), 621-630.

[10] A. Aral, V. Gupta and R. P. Agarwal, Applications of $q-$Calculus in Operator Theory, Springer, New York, 2013.

[11] M. Govindaraj and S. Sivasubramanian, On a class of analytic function related to conic domains involving $q-$calculus, Anal. Math., 43 (2017), no. 3, 475–487.

[12] F. H. Jackson, On $q-$functions and a certain difference operator, Trans. R. Soc. Edinb., 46 (1908), 253–281.

[13] A. O. Mostafa, On partial sums Some of certain analytic functions, Demonstratio Mathematica., 41(2008), no. 4, 779-789.

[14] A. O. Mostafa and Z. M. Saleh, A class of starlike functions of complex order defiend by difference operator, J. Fractional Calculus Appl., 13(2022), (To Appear).

[15] G. Sălăgean, Subclasses of univalent functions, Lecture note in Math., Springer-Verlag., 1013(1983), 362-372.

[16] T. M. Seoudy and M. K. Aouf, Coefficient estimates of new classes of $q-$starlike and $q-$convex functions of complex order, J. Math. Inequal., 10 (2016), no. 1, 135–145.

[17] T. M. Seoudy and M. K. Aouf, Convolution properties for certain classes of analytic functions defined by $q-$derivative operator, Abstract Appl. Anal., 2014(2014), 1-7.

[18] T. Sheil-Small. A note on partial sums of convex schlicht functions, Bull. London Math. Soc., 2(1970), 165-168.

[19] H. Silverman, Partial sums of starlike and convex functions, J. Math. Appl., 209(1997), 221-227.

[20] H. M. Srivastava, Operators of basic (or $q-$) calculus and fractional $q-$calculus and their applications in geometric function theory of complex analysis, Iran. J. Sci. Technol. Trans. Sci., 44(2020), 327–344.

[21] H. M. Srivastava, A. O. Mostafa, M. K. Aouf and H. M. Zayed, Basic and fractional $q-$calculus and associated Fekete–Szego problem for $p-$valently $q-$starlike functions and $p-$valently $q-$convex functions of complex order, Miskolc Math. Notes., 20 (2019), no. 1, 489–509.

[22] K. Vijaya, M. Kasthuri and G. Murugusundaramoorthy, Coefficient bounds for subclasses of bi-univalent functions defined by the Sălăgean derivative operator, Boletin de la Asociaciton, Matematica Venezolana., 21(2014), no. 2, 1-9.