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Coefficient Estimates of Te-Univalent Functions Associated with the Dziok-Srivastava Operator

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Abstract

In this paper, the concept of bi-univalency is extended to the Te-univalency associated with an operator. Furthermore, we introduce a new subclass of analytic and Te-univalent functions in the open unit disk associated with the Dziok-Srivastava operator. We find estimates for the first two Taylor-Maclaurin coefficients $|a_2|$ ***and*** $|a_3|$ ***for functions in this subclass, and we obtain an estimation for the Fekete-Szegő problem for this function class. Our results generalize and improve some previously published results.***

Keywords: *Univalent functions, Bi-univalent functions, Te-univalent functions, Hadamard product, linear operators, coefficient bounds.*

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1 Introduction

Let A denote the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also let S denote the class of all functions in A which are univalent in U. According to the Koebe one-quarter theorem Duren [6], it ensures that the images of U

under every univalent functions $f \in S$ contains a disk of radius $\frac{1}{4}$. Thus, every univalent function f on U has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$h\,(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \quad (2)$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of all bi-univalent functions in U given by (1). Some examples of functions in the class Σ are $\frac{z}{1-z}$, $-\log(1-z)$, and $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$.

For the function $f \in A$, let the operator $T : A \to A$ defined as:

$$Tf(z) = z + \sum_{n=2}^{\infty} t_n a_n z^n. \quad (3)$$

In this paper, the concept of bi-univalency is extended to the class of functions given by (3) defined on U. For this purpose, let T_S denote the class of all functions given by (3), which are univalent in U. It is well known that every function $Tf \in T_S$ has an inverse $(Tf)^{-1}$, defined by

$$(Tf)^{-1}((Tf)(z)) = z \quad (z \in U)$$

and

$$Tf((Tf)^{-1}(w)) = w \quad \left(|w| < r_0(Tf); r_0(Tf) \geq \frac{1}{4}\right),$$

where

$$(Tf)^{-1}(w) = w - t_2 a_2 w^2 + (2t_2^2 a_2^2 - t_3 a_3)w^3 - (5t_2^3 a_2^3 - 5t_2 t_3 a_2 a_3 + t_4 a_4)w^4 + \cdots. \quad (4)$$

A function f given by (1) is said to be Te-univalent in U associated with T, if both Tf and $(Tf)^{-1}$ are univalent in U. Let T_Σ denote the class of all functions given by (1) which are Te-univalent in U associated with T.

Remark 1.1

(i) For $Tf = f$ *we have* $T_\Sigma = \Sigma$;

(ii) If $t_n \neq 1$ *for some* n, *we have*

$$Tf(Th(w)) = w + 2\left[t_3 - t_2^2\right] a_2^2 w^3 + ... \neq w,$$

where h *given by* (2).

For function f given by (1) and ϕ given by

$$\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) of f and ϕ is defined by

$$(f * \phi)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (\phi * f)(z).$$

For complex parameters $\alpha_1, ..., \alpha_q$, $\beta_1, ..., \beta_s$, and non-positive b's, the generalized hypergeometric function ${}_qF_s$ is defined by the following infinite series:

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n .... (\alpha_q)_n}{(\beta_1)_n ... (\beta_s)_n} \frac{z^n}{n!},$$

where $q \leq s+1, q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in U$, and $(\theta)_n$ is the Pochhammer symbol (or shift factorial) defined, in terms of the Gamma function Γ, by

$$(\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1 & for \quad n = 0 \\ \theta(\theta+1)...(\theta+n-1) & for \quad n \in \mathbb{N} \end{cases}$$

Corresponding a function $h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ defined by

$$h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z\ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) \quad (z \in U),$$

Dziok and Srivastava [7] considered a linear operator

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : A \to A$$

defined by the following Hadamard product:

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z), \tag{5}$$

where $q \leq s+1; q, s \in \mathbb{N}_0; z \in U$.

If $f \in A$ is given by (1), then we have

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = z + \sum_{n=2}^{\infty} \Gamma_n[\alpha_1; \beta_1] a_n z^n \quad (z \in U), \tag{6}$$

where

$$\Gamma_n[\alpha_1;\beta_1] = \frac{(\alpha_1)_{n-1} .... (\alpha_q)_{n-1}}{(\beta_1)_{n-1} ... (\beta_s)_{n-1}} \frac{1}{(n-1)!}. \tag{7}$$

To make the notation simple, we write

$$H_{q,s}[\alpha_1;\beta_1] f = H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f.$$

The linear operator $H_{q,s}[\alpha_1;\beta_1]$ is a generalization of many other linear operators considered earlier.

It easily follows from (6) that

$$z\left(H_{q,s}[\alpha_1;\beta_1] f(z) \right)' = \alpha_1 H_{q,s}[\alpha_1+1;\beta_1] f(z) - (\alpha_1 - 1) H_{q,s}[\alpha_1;\beta_1] f(z). \tag{8}$$

Let $T_S^{q,s}[\alpha_1;\beta_1]$ denote the class of all functions given by (6), which are univalent in U. It is well known that every function $H_{q,s}[a_1;\beta_1] f \in T_S^{q,s}[\alpha_1;\beta_1]$ has an inverse $(H_{q,s}[\alpha_1;\beta_1] f)^{-1}$, defined by

$$g(H_{q,s}[\alpha_1;\beta_1] f(z)) = z \quad (z \in U)$$

and

$$H_{q,s}[\alpha_1;\beta_1] f(g(w)) = w \quad \left(|w| < r_0(H_{q,s}[\alpha_1;\beta_1] f); r_0(H_{q,s}[\alpha_1;\beta_1] f) \geq \frac{1}{4} \right),$$

where

$$\begin{aligned} g(w) &= (H_{q,s}[\alpha_1;\beta_1] f)^{-1}(w) \\ &= w - \Gamma_2[\alpha_1;\beta_1] a_2 w^2 + \left[2(\Gamma_2[\alpha_1;\beta_1])^2 a_2^2 - \Gamma_3[\alpha_1;\beta_1] a_3 \right] w^3 \\ &\quad - \left[5(\Gamma_2[\alpha_1;\beta_1])^3 a_2^3 - 5\Gamma_2[\alpha_1;\beta_1]\Gamma_3[\alpha_1;\beta_1] a_2 a_3 + \Gamma_4[\alpha_1;\beta_1] a_4 \right] w^4 + \cdots \end{aligned} \tag{9}$$

and $\Gamma_n[\alpha_1;\beta_1]$ is given by (7).

A function f given by (1) is said to be Te-univalent in U associated with the operator $H_{q,s}[a_1;\beta_1]$, if both $H_{q,s}[\alpha_1;\beta_1] f$ and $(H_{q,s}[\alpha_1;\beta_1] f)^{-1}$ are univalent in U. Let $T_\Sigma^{q,s}[\alpha_1;\beta_1]$ denote the class of all functions given by (1), which are Te-univalent in U associated with $H_{q,s}[\alpha_1;\beta_1]$.

Remark 1.2

(i) *For* $q = 2, s = 1, \alpha_1 = \beta_1 = c,$ *and* $\alpha_2 = 1,$ *we have* $T_\Sigma^{2,1}[c, 1; c] = \Sigma.$

(ii) *If* $\Gamma_n[\alpha_1;\beta_1] \neq 1$ *for some* n, *we have:*

$$H_{q,s}[\alpha_1;\beta_1] f(H_{q,s}[\alpha_1;\beta_1] h(w)) = w + 2\left[\Gamma_3[\alpha_1;\beta_1] - (\Gamma_2[\alpha_1;\beta_1])^2 \right] a_2^2 w^3 + ... \neq w,$$

where h, *is given by* (2).

For two functions f and g, which are analytic in U, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function s, which (by definition) is analytic in U with $s(0) = 0$ and $|s(z)| < 1$ for all $z \in U$, such that $f(z) = g(s(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g., [5] , and [12]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let φ is an analytic function with positive real part in the unit disk U, which satisfies the following conditions:

$$\varphi(0) = 1 \text{ and } \varphi'(0) > 0$$

and is so constrained that $\varphi(U)$ is symmetric with respect to the real axis. Ma and Minda [11] unified various subclasses of starlike and convex functions consist of functions $f \in A$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \varphi(z)$ and $1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)$ respectively. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex (see [1]). Many interesting examples of the functions of the class Σ, together with various other properties and characteristics associated with bi-univalent functions can be found in the earlier work studied by Lewin [10], Brannan and Clunie [3], Netanyahu [13] and others. Brannan and Taha [4] (see also [23]) introduced certain subclasses of bi-univalent functions similar to the familiar subclasses of univalent functions consisting of starlike, convex and strongly starlike functions. They investigated the bound on the initial coefficients of the classes bi-starlike and bi-convex functions. Recently, many researchers (see [17, 22, 24, 9, 1, 19]) introduced and investigated some new subclasses of Σ and obtained bounds for the initial coefficients of the function given by (1). For a brief history and interesting examples in the class Σ (see [21]).

Earlier in 1933, Fekete and Szego [8] made use of Lowner's parametric method in order to prove that, if $f \in S$ and is given by (1),

$$\left|a_3 - \mu a_2^2\right| \leq 1 + 2\exp\left(-\frac{2\lambda}{1-\lambda}\right) \quad (0 \leq \lambda \leq 1).$$

For some history of Feketo-Szegő problem for class of starlike, convex and close-to-convex functions, refer to work produced by by Srivastava et al. [20]. Besides that, some authors [14, 16, 25] have studied the Feketo-Szegő inequalities for certain subclasses of bi-univalent functions.

The object of the present paper is to introduce a new subclass of analytic and Te-univalent functions in the open unit disk associated with the operator $H_{q,s}[\alpha_1;\beta_1]$ based on the Ma-Minda concept, and the bound for second and third coefficients of functions in this class are obtained. Also the Fekete-Szegő inequality is determined for this function class. Our results generalize and improve several well-known results in [1, 9, 14, 15, 16, 21, 25] and these are pointed out.

In order to derive our main results, we have to recall here the following lemmas.

Lemma 1.3 *[18] If* $p \in \mathcal{P}$ *then* $|c_k| \leq 2$ *for each* k*, where* $\mathcal{P}$ *is the family of all functions* p *analytic in* U *for which* $Rep(z) > 0$*,* $p(z) = 1+c_1z+c_2z^2+c_3z^3+\cdots$ *for* $z \in U$*.*

Lemma 1.4 *[26] Let* $k, l \in \mathbb{R}$ *and* $z_1, z_2 \in \mathbb{C}$*. If* $|z_1| < R$ *and* $|z_2| < R$ *then*

$$|(k+l)\,z_1 + (k-l)\,z_2| \leq \begin{cases} 2\,|k|\,R & for \quad |k| \geq |l| \\ 2\,|l|\,R & for \quad |k| \leq |l| \end{cases}$$

2 Coefficient bounds of the function class $T_{\Sigma}^{q,s}\left[\alpha_1;\beta_1,\varphi,\lambda\right]$

We begin this section by assuming that φ is an analytic function with positive real part in U, which satisfies the following conditions:

$$\varphi\,(0) = 1 \text{ and } \varphi'\,(0) > 0$$

and is so constrained that $\varphi\,(U)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form:

$$\varphi\,(z) = 1 + B_1z + B_2z^2 + B_3z^3 + ... \qquad (z \in U) \tag{10}$$

where $B_1, B_2, B_3, ...$ are real and $B_1 > 0$.

Unless otherwise mentioned, we assume throughout this paper that, the function φ satisfies the above conditions, $q \leq s+1\,(q, s \in \mathbb{N}_0)$, $\alpha_1 > 0$, and $z \in U$.

Definition 2.1 *A function* f *given by* (1) *is said to be in the class* $T_{\Sigma}^{q,s}\left[\alpha_1;\beta_1,\varphi,\lambda\right]$*, if the following subordination conditions hold true:*

$$(1-\lambda)\,\frac{H_{q,s}\left[\alpha_1;\beta_1\right]f\,(z)}{z} + \lambda\,\left(\,H_{q,s}\left[\alpha_1;\beta_1\right]\;f\,(z)\right)' \prec \varphi\,(z)\,, \tag{11}$$

and

$$(1-\lambda)\,\frac{g(w)}{w} + \lambda g'(w) \prec \varphi\,(z)\,, \tag{12}$$

where the functions $H_{q,s}\left[\alpha_1;\beta_1\right]f$ *and* g *are given by* (6) *and* (9)*, respectively.*

Note that for appropriate values of parameters involved in the class $T_{\Sigma}^{q,s}[\alpha_1;\beta_1,\varphi,\lambda]$, there are several previously defined classes. These special cases are given as follows:

(i) $T_{\Sigma}^{2,1}[a,b;c,\varphi,\lambda]$ improves the class $\mathcal{B}_{\Sigma}^{a,b,c}(\varphi,\lambda)$ which was introduced and studied by Omar et al. [16];

(ii) $T_{\Sigma}^{2,1}[a,b;c,\varphi,1]$ improves the class $\mathcal{H}_{\Sigma}^{a,b,c}(\varphi)$ which was introduced and studied by Omar et al. [16];

(iii) $T_{\Sigma}^{2,1}[c,1;c,\varphi,\lambda]=\mathcal{B}_{\Sigma}(\varphi,\lambda)$, where the class $\mathcal{B}_{\Sigma}(\varphi,\lambda)$ was introduced and studied by Omar et al. [14];

(iv) $T_{\Sigma}^{2,1}[c,1;c,\varphi,1]=\mathcal{H}_{\Sigma}(\varphi)$, where the class $\mathcal{H}_{\Sigma}(\varphi)$ was introduced and studied by Ali et al. [1];

(v) $T_{\Sigma}^{2,1}\left[c,1;c,\left(\frac{1+z}{1-z}\right)^{\zeta},\lambda\right]=\mathcal{B}_{\Sigma}(\zeta,\lambda)$ $(0<\zeta\leq 1)$, where the class $\mathcal{B}_{\Sigma}(\zeta,\lambda)$ was introduced and studied by Frasin and Aouf [9];

(vi) $T_{\Sigma}^{2,1}\left[c,1;c,\frac{1+(1-2\eta)z}{1-z},\lambda\right]=\mathcal{B}_{\Sigma}(\eta,\lambda)$ $(0\leq\eta<1)$, where the class $\mathcal{B}_{\Sigma}(\eta,\lambda)$ was introduced and studied by Frasin and Aouf [9];

(vii) $T_{\Sigma}^{2,1}\left[c,1;c,\left(\frac{1+z}{1-z}\right)^{\zeta},1\right]=\mathcal{H}_{\Sigma}^{\zeta}$ $(0<\zeta\leq 1)$, where the class $\mathcal{H}_{\Sigma}^{\zeta}$ was introduced and studied by Srivastava et al. [21];

(viii) $T_{\Sigma}^{2,1}\left[c,1;c,\frac{1+(1-2\eta)z}{1-z},1\right]=\mathcal{H}_{\Sigma}(\eta)$ $(0\leq\eta<1)$, where the class $\mathcal{H}_{\Sigma}(\eta)$ was introduced and studied by Srivastava et al. [21].

Theorem 2.2 *If f given by* (1) *be in the class $T_{\Sigma}^{q,s}[\alpha_1;\beta_1,\varphi,\lambda]$, then*

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\Gamma_2[\alpha_1;\beta_1]\sqrt{\left|(2\lambda+1)B_1^2+(B_1-B_2)(1+\lambda)^2\right|}}\tag{13}$$

and

$$|a_3|\leq\frac{B_1}{(2\lambda+1)\Gamma_3[\alpha_1;\beta_1]}+\frac{B_1^2}{(\lambda+1)^2\Gamma_3[\alpha_1;\beta_1]}\tag{14}$$

where $\Gamma_n[\alpha_1;\beta_1]$ is given by (7).

Proof. If $f\in T_{\Sigma}^{q,s}[\alpha_1;\beta_1,\varphi,\lambda]$, from (11), (12), and the definition of subordination it follows that there exist two functions where u and v in U are analytic functions with $u(0)=v(0)=0$ and $|u(z)|<1$, $|v(w)|<1$ for all $z,w\in U$, such that

$$(1-\lambda)\frac{H_{q,s}[\alpha_1;\beta_1]f(z)}{z}+\lambda\left(H_{q,s}[\alpha_1;\beta_1]f(z)\right)'=\varphi(u(z)),\tag{15}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda g^{'}(w)=\varphi\left(v\left(w\right)\right). \tag{16}$$

We define the functions p and q in $\mathcal{P}$ given by

$$p\left(z\right)=\frac{1+u\left(z\right)}{1-u\left(z\right)}=1+p_1z+p_2z^2+p_3z^3+..., \tag{17}$$

and

$$q\left(z\right)=\frac{1+v\left(z\right)}{1-v\left(z\right)}=1+q_1z+q_2z^2+q_3z^3+.... \tag{18}$$

It follows from (17) and (18) that

$$u\left(z\right)=\frac{p\left(z\right)-1}{p\left(z\right)+1}=\frac{p_1}{2}z+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)z^2+..., \tag{19}$$

and

$$v\left(z\right)=\frac{q\left(z\right)-1}{q\left(z\right)+1}=\frac{q_1}{2}z+\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)z^2+.... \tag{20}$$

Using (19) and (20) with (10) lead us to

$$\varphi\left(u\left(z\right)\right)=\varphi\left(\frac{p\left(z\right)-1}{p\left(z\right)+1}\right)=1+\frac{B_1p_1}{2}z+\left[\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)B_1+\frac{1}{4}p_1^2B_2\right]z^2+...,$$

and

$$\varphi\left(v\left(z\right)\right)=\varphi\left(\frac{q\left(z\right)-1}{q\left(z\right)+1}\right)=1+\frac{B_1q_1}{2}z+\left[\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)B_1+\frac{1}{4}q_1^2B_2\right]z^2+....$$

On the other hand,

$$\begin{gathered}(1-\lambda)\tfrac{H_{q,s}[\alpha_1;\beta_1]f(z)}{z}+\lambda\left(\;H_{q,s}\left[\alpha_1;\beta_1\right]\;f\left(z\right)\right)^{'}=\\ 1+\left(\lambda+1\right)\Gamma_2\left[\alpha_1;\beta_1\right]a_2z+\left(2\lambda+1\right)\Gamma_3\left[\alpha_1;\beta_1\right]a_3z^2+...,\end{gathered}$$

and

$$\begin{gathered}(1-\lambda)\tfrac{g(w)}{w}+\lambda g^{'}(w)=\\ 1-\left(\lambda+1\right)\Gamma_2\left[\alpha_1,\beta,\gamma\right]a_2w+\left(2\lambda+1\right)\left(2\left(\Gamma_2\left[\alpha_1;\beta_1\right]\right)^2a_2^2-\Gamma_3\left[\alpha_1;\beta_1\right]a_3\right)w^2+....\end{gathered}$$

Now, equating the coefficients in (15) and (16), we get

$$\left(\lambda+1\right)\Gamma_2\left[\alpha_1;\beta_1\right]a_2=\frac{B_1p_1}{2}, \tag{21}$$

$$\left(2\lambda+1\right)\Gamma_3\left[\alpha_1;\beta_1\right]a_3=\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)B_1+\frac{1}{4}p_1^2B_2, \tag{22}$$

$$-(\lambda+1)\Gamma_2[\alpha_1;\beta_1]a_2=\frac{B_1q_1}{2} \tag{23}$$

and

$$(2\lambda+1)\left(2(\Gamma_2[\alpha_1;\beta_1])^2a_2^2-\Gamma_3[\alpha_1;\beta_1]a_3\right)=\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)B_1+\frac{1}{4}q_1^2B_2. \tag{24}$$

From (21) and (23), we get

$$p_1=-q_1 \tag{25}$$

and

$$2(\lambda+1)^2(\Gamma_2[\alpha_1;\beta_1])^2a_2^2=\frac{B_1^2}{4}\left(p_1^2+q_1^2\right). \tag{26}$$

Now from (22), (24) and (26), we obtain

$$\begin{aligned}2(2\lambda+1)(\Gamma_2[\alpha_1;\beta_1])^2a_2^2 &= \frac{B_1}{2}(p_2+q_2)+\frac{B_2-B_1}{4}\left(p_1^2+q_1^2\right)\\ &= \frac{B_1}{2}(p_2+q_2)+\frac{2(B_2-B_1)(\lambda+1)^2(\Gamma_2[\alpha_1;\beta_1])^2a_2^2}{B_1^2}.\end{aligned}$$

Therefore, we have

$$a_2^2=\frac{B_1^3(p_2+q_2)}{4(\Gamma_2[\alpha_1;\beta_1])^2\left[(2\lambda+1)B_1^2+(B_1-B_2)(1+\lambda)^2\right]} \tag{27}$$

Applying Lemma 1.3 for the coefficients p_2 and q_2, we immediately have

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\Gamma_2[\alpha_1;\beta_1]\sqrt{\left|(2\lambda+1)B_1^2+(B_1-B_2)(1+\lambda)^2\right|}}.$$

This gives the bound on $|a_2|$ as asserted in (13).

Next, in order to find the bound on $|a_3|$, by subtracting (24) from (22) and using (25), we get

$$\begin{aligned}&2(2\lambda+1)\Gamma_3[\alpha_1;\beta_1]a_3-2(2\lambda+1)(\Gamma_2[\alpha_1;\beta_1])^2a_2^2\\ =\ &\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)B_1+\frac{1}{4}p_1^2B_2-\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)B_1-\frac{1}{4}q_1^2B_2\\ =\ &\frac{1}{2}B_1(p_2-q_2).\end{aligned} \tag{28}$$

It follows from (26) and (28) that

$$2(2\lambda+1)\Gamma_3[\alpha_1;\beta_1]a_3=\frac{B_1^2(2\lambda+1)(p_1^2+q_1^2)}{4(\lambda+1)^2}+\frac{1}{2}B_1(p_2-q_2)$$

and then,

$$a_3 = \frac{B_1^2\left(p_1^2+q_1^2\right)}{8\left(\lambda+1\right)^2\Gamma_3\left[\alpha_1;\beta_1\right]} + \frac{B_1\left(p_2-q_2\right)}{4\left(2\lambda+1\right)\Gamma_3\left[\alpha_1;\beta_1\right]}$$

Applying Lemma 1.3 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{B_1^2}{\left(\lambda+1\right)^2\Gamma_3\left[\alpha_1;\beta_1\right]} + \frac{B_1}{\left(2\lambda+1\right)\Gamma_3\left[\alpha_1;\beta_1\right]}.$$

This completes the proof of Theorem 2.2.

Taking $q = 2, s = 1, \alpha_1 = a, \alpha_2 = b$, and $\beta_1 = c$ in Theorem 2.2, we obtain the following corollary which improves the result of Omar et al. [[15],Theorem 2].

Corollary 2.3 *If f given by* (1) *be in the class $T_{\Sigma}^{2,1}\left[a,b;c,\varphi,\lambda\right]$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\Gamma_2\left[a,b;c\right]\sqrt{\left|\left(2\lambda+1\right)B_1^2+\left(B_1-B_2\right)\left(1+\lambda\right)^2\right|}}$$

and

$$|a_3| \leq \frac{B_1}{\left(2\lambda+1\right)\Gamma_3\left[a,b;c\right]} + \frac{B_1^2}{\left(\lambda+1\right)^2\Gamma_3\left[a,b;c\right]}$$

where

$$\Gamma_n\left[a,b;c\right] = \frac{\left(a\right)_{n-1}\left(b\right)_{n-1}}{\left(c\right)_{n-1}}\frac{1}{\left(n-1\right)!}.$$

Taking $\lambda = 1$ in Corollary 2.3, we obtain the following corollary which improves the result of Omar et al. [[15], Theorem 1].

Corollary 2.4 *If f given by* (1) *be in the class $T_{\Sigma}^{2,1}\left[a,b;c,\varphi,1\right]$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\Gamma_2\left[a,b;c\right]\sqrt{\left|3B_1^2+4\left(B_1-B_2\right)\right|}}$$

and

$$|a_3| \leq \frac{B_1}{\Gamma_3\left[a,b;c\right]}\left(\frac{1}{3}+\frac{B_1}{4}\right)$$

where

$$\Gamma_n\left[a,b;c\right] = \frac{\left(a\right)_{n-1}\left(b\right)_{n-1}}{\left(c\right)_{n-1}}\frac{1}{\left(n-1\right)!}.$$

Remark 2.5

(i) Taking $q = 2, s = 1, \alpha_1 = \beta_1 = c$, and $\alpha_2 = 1$ in Theorem 2.2, we obtain the result obtained by Omar et al.[[14], Theorem 2.2];

(ii) *Taking* $q=2, s=1, \alpha_1=\beta_1=c,$ *and* $\alpha_2=\lambda=1$ *in Theorem 2.2, we obtain the result obtained by Ali et al.[[1], Theorem 2.1];*

(iii) *Taking* $q=2, s=1, \alpha_1=\beta_1=c, \alpha_2=1,$ *and* $\varphi=\left(\frac{1+z}{1-z}\right)^{\zeta}$ $(0<\zeta\leq 1)$ *in Theorem 2.2, we obtain the result obtained by Frasin and Aouf [[9], Theorem 2.2];*

(iv) *Taking* $q=2, s=1, \alpha_1=\beta_1=c, \alpha_2=1,$ *and* $\varphi=\frac{1+(1-2\eta)z}{1-z}$ $(0\leq\eta<1)$ *in Theorem 2.2, we obtain the result obtained by Frasin and Aouf [[9], Theorem 3.2];*

(v) *Taking* $q=2, s=1, \alpha_1=\beta_1=c, \alpha_2=\lambda=1,$ *and* $\varphi=\left(\frac{1+z}{1-z}\right)^{\zeta}$ $(0<\zeta\leq 1)$ *in Theorem 2.2, we obtain the result obtained by Srivastava et al.[[21], Theorem 1];*

(vi) *Taking* $q=2, s=1, \alpha_1=\beta_1=c, \alpha_2=\lambda=1,$ *and* $\varphi=\frac{1+(1-2\eta)z}{1-z}$ $(0\leq\eta<1)$ *in Theorem 2.2, we obtain the result obtained by Srivastava et al.[[21], Theorem 2] .*

3 Fekete-Szegő Problem for Function Class $T_{\Sigma}^{q,s}[\alpha_1;\beta_1,\varphi,\lambda]$

Theorem 3.1 *If* f *given by* (1) *be in the class* $T_{\Sigma}^{q,s}[\alpha_1;\beta_1,\varphi,\lambda]$, *then*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{B_1}{(2\lambda+1)\Gamma_3[\alpha_1;\beta_1]} & for\quad \left|1-\mu\frac{\Gamma_3[\alpha_1;\beta_1]}{(\Gamma_2[\alpha_1;\beta_1])^2}\right|\leq\left|1+\frac{(B_1-B_2)(1+\lambda)^2}{(2\lambda+1)B_1^2}\right| \\ \frac{B_1^3\left|1-\mu\frac{\Gamma_3[\alpha_1;\beta_1]}{(\Gamma_2[\alpha_1;\beta_1])^2}\right|}{\Gamma_3[\alpha_1;\beta_1]\left|(2\lambda+1)B_1^2+(B_1-B_2)(1+\lambda)^2\right|} & for\quad \left|1-\mu\frac{\Gamma_3[\alpha_1;\beta_1]}{(\Gamma_2[\alpha_1;\beta_1])^2}\right|\geq\left|1+\frac{(B_1-B_2)(1+\lambda)^2}{(2\lambda+1)B_1^2}\right|\end{cases}$$

where $\mu\in\mathbb{C}$, *and* $\Gamma_k[\alpha_1;\beta_1], k=1,2,$ *are given by* (7).

Proof. If $f\in T_{\Sigma}^{q,s}[\alpha_1;\beta_1,\varphi,\lambda]$ like in the proof of Theorem 2.2, from (28) we have,

$$a_3-\frac{(\Gamma_2[\alpha_1;\beta_1])^2}{\Gamma_3[\alpha_1;\beta_1]}a_2^2=\frac{B_1(p_2-q_2)}{4(2\lambda+1)\Gamma_3[\alpha_1;\beta_1]} \tag{29}$$

. Multiplying (27) by $\left(\frac{(\Gamma_2[\alpha_1;\beta_1])^2}{\Gamma_3[\alpha_1;\beta_1]}-\mu\right)$ we get

$$\left(\frac{(\Gamma_2[\alpha_1;\beta_1])^2}{\Gamma_3[\alpha_1;\beta_1]}-\mu\right)a_2^2=\frac{B_1^3\left(1-\mu\frac{\Gamma_3[\alpha_1;\beta_1]}{(\Gamma_2[\alpha_1;\beta_1])^2}\right)(p_2+q_2)}{4\Gamma_3[\alpha_1;\beta_1]\left[(2\lambda+1)B_1^2+(B_1-B_2)(1+\lambda)^2\right]} \tag{30}$$

Adding (29) and (30), it follows that

$$a_3-\mu a_2^2=\frac{B_1}{4\Gamma_3\left[\alpha_1;\beta_1\right]}\left[\left(L\left(\mu\right)+\frac{1}{(2\lambda+1)}\right)p_2+\left(L\left(\mu\right)-\frac{1}{(2\lambda+1)}\right)q_2\right] \tag{31}$$

and it follows

$$\left|a_3-\mu a_2^2\right|=\frac{B_1}{4\Gamma_3\left[\alpha_1;\beta_1\right]}\left|\left(L\left(\mu\right)+\frac{1}{(2\lambda+1)}\right)p_2+\left(L\left(\mu\right)-\frac{1}{(2\lambda+1)}\right)q_2\right| \tag{32}$$

where

$$L\left(\mu\right)=\frac{B_1^2\left(1-\mu\frac{\Gamma_3[\alpha_1;\beta_1]}{(\Gamma_2[\alpha_1;\beta_1])^2}\right)}{\left[(2\lambda+1)\,B_1^2+(B_1-B_2)\,(1+\lambda)^2\right]}$$

Since $B_1, B_2, B_3, ...$ are real and $B_1>0$, $|p_2|\leq 2$ and $|q_2|\leq 2$ and applying Lemma 1.4, we conclude that

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{B_1}{(2\lambda+1)\Gamma_3[\alpha_1;\beta_1]} & for \quad \left|L\left(\mu\right)\right|\leq\frac{1}{(2\lambda+1)}\\ \\ \frac{B_1}{\Gamma_3[\alpha_1;\beta_1]}\left|L\left(\mu\right)\right| & for \quad \left|L\left(\mu\right)\right|\geq\frac{1}{(2\lambda+1)}\end{cases}$$

This completes the proof of Theorem 3.1.

Taking $\mu=0$ in Theorem 3.1, we obtain the following corollary which improves the corresponding result in Theorem 2.2.

Corollary 3.2 *If f given by* (1) *be in the class $T_\Sigma^{q,s}\left[\alpha_1;\beta_1,\varphi,\lambda\right]$, then*

$$|a_3|\leq\begin{cases}\frac{B_1}{(2\lambda+1)\Gamma_3[\alpha_1;\beta_1]} & for \quad \frac{B_1-B_2}{B_1^2}\in\left(-\infty,\frac{-2(2\lambda+1)}{(1+\lambda)^2}\right]\cup[0,\infty)\\ \\ \frac{B_1^3}{\Gamma_3[\alpha_1;\beta_1]\left|(2\lambda+1)B_1^2+(B_1-B_2)(1+\lambda)^2\right|} & for \quad \frac{B_1-B_2}{B_1^2}\in\left[\frac{-2(2\lambda+1)}{(1+\lambda)^2},\frac{-(2\lambda+1)}{(1+\lambda)^2}\right)\cup\left(\frac{-(2\lambda+1)}{(1+\lambda)^2},0\right]\end{cases}$$

where $\Gamma_k\left[\alpha_1;\beta_1\right], k=1,2,$ are given by(7).

Taking $q=2, s=1, \alpha_1=a, \alpha_2=b$, and $\beta_1=c$ in Theorem 3.1, we obtain the following corollary which improves the result of Omar et al.[[16], Theorem 2].

Corollary 3.3 *If f given by* (1) *be in the class $T_\Sigma^{2,1}\left[a,b;c,\varphi,\lambda\right]$, then*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{B_1}{(2\lambda+1)\Gamma_3[a,b;c]} & for \quad \left|1-\mu\frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\right|\leq\left|1+\frac{(B_1-B_2)(1+\lambda)^2}{(2\lambda+1)B_1^2}\right|\\ \\ \frac{B_1^3\left|1-\mu\frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\right|}{\Gamma_3[a,b;c]\left|(2\lambda+1)B_1^2+(B_1-B_2)(1+\lambda)^2\right|} & for \quad \left|1-\mu\frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\right|\geq\left|1+\frac{(B_1-B_2)(1+\lambda)^2}{(2\lambda+1)B_1^2}\right|\end{cases}$$

where $\mu\in\mathbb{C}$, and $\Gamma_n\left[a,b;c\right]=\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}}\frac{1}{(n-1)!}$.

Taking $\lambda = 1$ in Corollary 3.3, we obtain the following corollary which improves the result of Omar et al. [[16], Theorem 1].

Corollary 3.4 *If f given by (1) be in the class $T_{\Sigma}^{2,1}\left[a,b;c,\varphi,1\right]$, then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{B_1}{3\Gamma_3[a,b;c]} & for \quad \left|1-\mu\frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\right| \leq \left|1+\frac{4(B_1-B_2)}{3B_1^2}\right| \\ \frac{B_1^3\left|1-\mu\frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\right|}{\Gamma_3[a,b;c]\left|3B_1^2+4(B_1-B_2)\right|} & for \quad \left|1-\mu\frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\right| \geq \left|1+\frac{4(B_1-B_2)}{3B_1^2}\right| \end{cases}$$

where $\mu \in \mathbb{C}$, and $\Gamma_n\left[a,b;c\right] = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}}\frac{1}{(n-1)!}$.

Remark 3.5

(i) Taking $q = 2, s = 1, \alpha_1 = \beta_1 = c$, and $\alpha_2 = 1$ in Theorem 3.1, we obtain the result obtained by Omar et al.[[14], Theorem 2.7];

(ii) Taking $q = 2, s = 1, \alpha_1 = \beta_1 = c$, and $\alpha_2 = \lambda = 1$ in Theorem 3.1, we obtain the result obtained by Zaprawa [[25], Theorem 1];

(iii) Taking $q = 2, s = 1, \alpha_1 = \beta_1 = c$, and $\alpha_2 = 1$ in Corollary 3.2, we obtain the result obtained by Omar et al.[[14], Corollary 2.9];

(iv) Taking $q = 2, s = 1, \alpha_1 = \beta_1 = c$, and $\alpha_2 = \lambda = 1$ in Corollary 3.2, we obtain the result obtained by Zaprawa [[25], Corollary 2].

4 Conclusions

In this paper, we defined a new subclasses of analytic and Te-univalent functions in the open unit disk associated with the operatorr $H_{q,s}\left[\alpha_1;\beta_1\right]$ based on the Ma-Minda concept. For the functions belonging to these subclasses, the estimates of second and third Taylor–Maclaurin coefficients are obtained. Also the Fekete-Szegő inequality is determined for this function class.

5 Open Problem

We mention that all the above estimations for the first two Taylor-Maclaurin coefficients and Fekete-Szegő problem for the function class $T_{\Sigma}^{q,s}\left[\alpha_1;\beta_1,\varphi,\lambda\right]$ are not sharp. To find the sharp upper bounds for the above function class, it still is an interesting open problem, as well as for $|a_n|$, $n \geq 4$.

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