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Upper Bounds for Class of Analytic Functions Defined by q-Difference Operator

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Abstract

In this paper using a q-difference operator, we define a class of univalent functions and obtained upper bounds for functions in it.

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1 Introduction

The class of univalent analytic functions of the form

$$F(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ z \in \mathcal{D} = \{z \in \mathbb{C} :\mid z \mid< 1\}, \tag{1}$$

is denoted by S.

For $F \in S, \ 0 < q < 1$, the $q-$difference operator Δ_q is given by [12] (see also [2, 3,4,5,6,7],[11],[21, 22,23]);

$$\Delta_q F(z) = \begin{cases} \frac{F(z)-F(qz)}{(1-q)z} & , z \neq 0 \\ F^{'}(0) & , z = 0 \end{cases},$$

that is

$$\Delta_q F(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{2}$$

where

$$[j]_q = \frac{1-q^j}{1-q}, \ [0]_q = 0. \tag{3}$$

As $q \to 1^-$, $[k]_q = k$ and $\Delta_q F(z) = F'(z)$.

For $F \in S$, the generalized Sălăgean operator is defined by Al-Oboudi [1] as

$$D_\delta^m F(z) = z + \sum_{k=2}^{\infty} [1 + \delta(k-1)]^m a_k z^k, \ \delta \geq 0, \ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{4}$$

Note that: $D_1^m F(z) = D^m F(z)$ (see [20]).

Pommerenke [19] (see also [16]) defined the Hankel determinent for $\eta \geq 1$, $\gamma \geq 0$ as

$$H_\eta(\gamma) = \begin{vmatrix} a_\gamma & a_{\gamma+1} & a_{\gamma+\eta-1} \\ a_{\gamma+1} & a_{\gamma+2} & a_{\gamma+\eta} \\ & & \\ a_{\gamma+\eta-1} & a_{\gamma+\eta} & a_{\gamma+2\eta-2} \end{vmatrix} \ (a_1 = 1), \tag{5}$$

where a_γ's are the coefficients of various power of z in $F(z)$ defined by (1).

This determinant has also been considered by several authors, for example $H_2(1) = a_3 - a_2^2$, is known as the Fekete-Szego functional (see Fekete-Szego [10] who generalized the estimate to $|a_3 - \mu a_2^2|$ where μ is real).

In the case $\eta = 2$ and $\gamma = 2$, the Hankel determinant $H_\eta(\gamma)$ is

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left| a_2 a_4 - a_3^2 \right|. \tag{6}$$

For more studies of $H_\eta(\gamma)$ see [9, 13, 17].

Using (3) and (4), we define the following class.

Definition 1 *Let $F \in S$, $\zeta, \delta \geq 0$, $m \in \mathbb{N}_0$. Then $F \in S_q^m(\delta, \zeta)$ if and only if*

$$\mathrm{Re}\{\frac{z\Delta_q(D_\delta^m F(z))}{D_\delta^m F(z)} + \zeta \frac{z^2 \Delta_q(\Delta_q D_\delta^m F(z))}{D_\delta^m F(z)}\} > 0. \tag{7}$$

Note that:

(i) $S_q^m(\delta, 0) = S_q^m(\delta) = \{F \in S : \mathrm{Re}[\frac{z\Delta_q(D_\delta^m F(z))}{D_\delta^m F(z)}] > 0\}$;

(ii) $S_q^0(\delta, 0) = S_q(0)$ (see Seoudy and Aouf [21]);

(iii) $S_q^m(1,\zeta) = S_q^m(\zeta) = \{F \in S : \mathrm{Re}[\frac{z(D^m F(z))'}{D^m F(z)} + \zeta \frac{z^2(D^m F(z))''}{D^m F(z)}] > 0\}$;

(iv) $lim_{q\to 1-} S_q^m(\delta,\zeta) = S^m(\delta,\zeta) = \{F \in S : \mathrm{Re}[\frac{z(D_\delta^m F(z))'}{D_\delta^m F(z)} + \zeta \frac{z^2(D_\delta^m F(z))''}{D_\delta^m F(z)}] > 0\}$.

So, see that (7) modifies the definition of Patil and Khairnar [18].

2 Main Results

Unless indicated, we assume that $0 < q < 1$, $m \in \mathbb{N}_0$, $z \in \mathcal{D}$, $F(z)$ given by (1) and $\delta, \zeta \geq 0$.

To prove our main results we shall need the following lemmas. Let P be the family of all functions p analytic in $\mathcal{D}$ for which $R\{p(z)\} > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + ... \tag{8}$$

Lemma 1 [8] Let $p \in P$, then $|c_k| \leq 2,\ k = 1, 2, ...$ and the inequality is sharp.

Lemma 2 [14] Let $p \in P$, then

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2) \\ 4c_3 &= c_1^3 + 2xc_1(4 - c_1^2) - x^2 c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \end{aligned} \tag{9}$$

for some x and y such that $|x| \leq 1$, $|y| \leq 1$.

Lemma 3 [15] If $p \in P$ is of the form (8) and ν is a complex number, then

$$\left|c_2 - \nu c_1^2\right| \leq 2\max\{1; |2\nu - 1|\}.$$

Theorem 1 Let $F(z) \in S_q^m(\delta,\zeta)$, then

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{4}{(1+2\delta)^{2m}([2]_q[3]_q\zeta + ([3]_q - 1))^2}. \tag{10}$$

Proof. Let $F(z) \in S_q^m(\delta,\zeta)$ then, there exist $p(z) \in P$ such that

$$z\Delta_q(D_\delta^m F(z)) + \zeta z^2 \Delta_q(\Delta_q D_\delta^m F(z)) = D_\delta^m F(z) p(z) \quad for\ some\ z \in \mathcal{D}\cdot \tag{11}$$

Therefore,

$$\begin{aligned} z\Delta_q(D_\delta^m F(z)) + \zeta z^2 \Delta_q(\Delta_q D_\delta^m F(z)) &= z + (1+\delta)^m a_2 [2]_q (1+\zeta) z^2 \\ &\quad + (1+2\delta)^m a_3 [3]_q (1 + [2]_q \zeta) z^3 \\ &\quad + (1+3\delta)^m a_4 [4]_q (1 + [3]_q \zeta) z^4 + ... \end{aligned} \tag{12}$$

and

$$
\begin{aligned}
D_{\delta}^{m}F(z)p(z) &= z+(c_1+(1+\delta)^m a_2)z^2 \\
&\quad +(c_2+c_1a_2(1+\delta)^m+(1+2\delta)^m a_3)z^3 \qquad (13)\\
&\quad +(c_3+c_2a_2(1+\delta)^m+c_1a_3(1+2\delta)^m+(1+3\delta)^m a_4)z^4+\ldots.
\end{aligned}
$$

Equating the coefficients of (12) and (13):

$$a_2=\frac{c_1}{(1+\delta)^m([2]_q\zeta+([2]_q-1))}, \tag{14}$$

$$
\begin{aligned}
a_3 &= \frac{c_2}{(1+2\delta)^m([2]_q[3]_q\zeta+([3]_q-1))} \\
&\quad +\frac{c_1^2}{(1+2\delta)^m([2]_q\zeta+([2]_q-1))([2]_q[3]_q\zeta+([3]_q-1))}, \qquad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
a_4 &= \frac{c_1^3}{(1+3\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+([2]_q-1))([2]_q[3]_q\zeta+([3]_q-1))} \\
&\quad +\frac{c_1c_2([2]_q\zeta(1+[3]_q)+[2]_q+[3]_q-2)}{(1+3\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+([2]_q-1))([2]_q[3]_q\zeta+([3]_q-1))} \\
&\quad +\frac{c_3}{(1+3\delta)^m([3]_q[4]_q\zeta+([4]_q-1))}. \qquad (16)
\end{aligned}
$$

From (14), (15) and (16), we have

$$
\left|a_2a_4-a_3^2\right| = \left|
\begin{array}{c}
\frac{c_1^4}{(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+([2]_q-1))^2([2]_q[3]_q\zeta+([3]_q-1))} \\
+\frac{c_1^2c_2([2]_q\zeta(1+[3]_q)+[2]_q+[3]_q-2)}{(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+([2]_q-1))^2([2]_q[3]_q\zeta+([3]_q-1))} \\
+\frac{c_1c_3}{(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+([2]_q-1))} \\
-\{[\frac{c_2}{(1+2\delta)^m([2]_q[3]_q\zeta+([3]_q-1))}+\frac{c_1^2}{(1+2\delta)^m([2]_q[3]_q\zeta+([3]_q-1))([2]_q\zeta+([2]_q-1))}]^2\}
\end{array}
\right|. \tag{17}
$$

By using Lemma 2,

$$
\left|a_2a_4-a_3^2\right| = \left|
\begin{array}{c}
\frac{c_1^4}{(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+([2]_q-1))^2([2]_q[3]_q\zeta+([3]_q-1))} \\
+\frac{c_1^2([2]_q\zeta(1+[3]_q)+[2]_q+[3]_q-2)[\frac{c_1^2+x(4-c_1^2)}{2}]}{(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+([2]_q-1))^2([2]_q[3]_q\zeta+([3]_q-1))} \\
+\frac{c_1[\frac{c_1^3+2xc_1(4-c_1^2)-x^2c_1(4-c_1^2)+2y(1-|x|^2)(4-c_1^2)}{4}]}{(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+([2]_q-1))}-\frac{c_1^4}{(1+2\delta)^{2m}([2]_q[3]_q\zeta+([3]_q-1))^2([2]_q\zeta+([2]_q-1))^2} \\
-\frac{[\frac{c_1^2+x(4-c_1^2)}{2}]^2}{(1+2\delta)^{2m}([2]_q[3]_q\zeta+([3]_q-1))^2}-\frac{2c_1^2[\frac{c_1^2+x(4-c_1^2)}{2}]}{(1+2\delta)^{2m}([2]_q[3]_q\zeta+([3]_q-1))^2([2]_q\zeta+([2]_q-1))}
\end{array}
\right|. \tag{18}
$$

Substituting for c_2 and c_3 from (9) and since $|c_1| \leq 2$ by Lemma 1, let $c_1 = c$ and assuming without restriction that $c \in [0, 2]$ we obtain, by triangle inequality,

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| \leq\ & \frac{c^4}{(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta + ([4]_q - 1))([2]_q\zeta + ([2]_q - 1))^2([2]_q[3]_q\zeta + ([3]_q - 1))} \\ & + \frac{c^4([2]_q\zeta(1+[3]_q) + [2]_q + [3]_q - 2)}{2(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta + ([4]_q - 1))([2]_q\zeta + ([2]_q - 1))^2([2]_q[3]_q\zeta + ([3]_q - 1))} \\ & + \frac{pc^2(4-c^2)([2]_q\zeta(1+[3]_q) + [2]_q + [3]_q - 2)}{2(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta + ([4]_q - 1))([2]_q\zeta + ([2]_q - 1))^2([2]_q[3]_q\zeta + ([3]_q - 1))} \\ & + \frac{c^4 + 2pc^2(4-c^2) - p^2c^2(4-c^2) + 2(4-c^2)(1-p^2)}{4(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta + ([4]_q - 1))([2]_q\zeta + ([2]_q - 1))} \\ & + \frac{c^4}{(1+2\delta)^{2m}([2]_q[3]_q\zeta + ([3]_q - 1))^2([2]_q\zeta + ([2]_q - 1))^2}\end{aligned}$$

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| \leq\ & \frac{c^2(c^2 + p(4-c^2)}{(1+2\delta)^{2m}([2]_q[3]_q\zeta + ([3]_q - 1))^2([2]_q\zeta + ([2]_q - 1))} \\ & + \frac{c^4 + 2pc^2(4-c^2) + p^2(4-c^2)^2}{4(1+2\delta)^{2m}([2]_q[3]_q\zeta + ([3]_q - 1))^2} \\ \leq\ & G(p),\end{aligned} \tag{19}$$

with $p = |x| \leq 1$. Furthermore,

$$\begin{aligned}G^{'}(p) \leq\ & \frac{c^2(4-c^2)([2]_q\zeta(1+[3]_q) + [2]_q + [3]_q - 2)}{2(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta + ([4]_q - 1))([2]_q\zeta + ([2]_q - 1))^2([2]_q[3]_q\zeta + ([3]_q - 1))} \\ & + \frac{2c^2(4-c^2) - 2pc^2(4-c^2) - 4(4-c^2)p}{4(1+3\delta)^m(1+\delta)^m([3]_q[4]_q\zeta + ([4]_q - 1))([2]_q\zeta + ([2]_q - 1))} \\ & + \frac{c^2(4-c^2)}{(1+2\delta)^{2m}([2]_q[3]_q\zeta + ([3]_q - 1))^2([2]_q\zeta + ([2]_q - 1))} \\ & + \frac{2c^2(4-c^2) + 2p(4-c^2)^2}{4(1+2\delta)^{2m}([2]_q[3]_q\zeta + ([3]_q - 1))^2}.\end{aligned} \tag{20}$$

By elementary calculations, we can show that $G^{'}(p) \geq 0$ for $p > 0$, which implies that G is an increasing function and thus the upper bound for (17) corresponds to $p = 1$ & $c = 0$, we have (10).

Theorem 2 Let $F(z) \in S_q^m(\delta, \zeta)$ then,

$$\left|a_3-\mu a_2^2\right| \leq \frac{2}{(1+2\delta)^m([2]_q[3]_q\zeta+([3]_q-1))}\max \{1; \left|1+\frac{2}{([2]_q\zeta+([2]_q-1))}(1-\frac{(1+2\delta)^m([2]_q[3]_q\zeta+([3]_q-1))}{(1+\delta)^{2m}([2]_q\zeta+([2]_q-1))}\mu)\right|\}. \tag{21}$$

Proof. Since if $F(z)\in S_q^m(\delta,\zeta)$, then a_2 and a_3 are given by (14) and (15), we have

$$a_3-\mu a_2^2 = \frac{c_2}{(1+2\delta)^m([2]_q[3]_q\zeta+([3]_q-1))} + \frac{c_1^2}{(1+2\delta)^m([2]_q\zeta+([2]_q-1))([2]_q[3]_q\zeta+([3]_q-1))} - \mu\frac{c_1^2}{(1+\delta)^{2m}([2]_q\zeta+([2]_q-1))^2}. \tag{22}$$

Therefore,

$$\left|a_3-\mu a_2^2\right| = \left|\frac{1}{(1+2\delta)^m([2]_q[3]_q\zeta+([3]_q-1))}\{c_2-\nu c_1^2\}\right|, \tag{23}$$

where

$$\nu=\frac{1}{([2]_q\zeta+([2]_q-1))}[\frac{(1+2\delta)^m([2]_q[3]_q\zeta+([3]_q-1))}{(1+\delta)^{2m}([2]_q\zeta+([2]_q-1))}\mu-1]. \tag{24}$$

Our result now follows by an application of Lemma 3.

This completes the proof of Theorem 2.

Remarks

(i) Letting $q\to 1-$ in Theorem 1, we have the results obtained by [18];

(ii) For different values δ and ζ, we obtain results for the classes mentioned in the introduction.

3 Open Problem

The authors suggest to fined upper bounds for class

$$\mathrm{Re}\left\{\frac{z\Delta_q(\mathbb{R}_q^{\varkappa}f(z))}{\mathbb{R}_q^{\varkappa}f(z)}+\zeta\frac{z^2\Delta_q(\Delta_q\mathbb{R}_q^{\varkappa}f(z))}{\mathbb{R}_q^{\varkappa}f(z)}\right\}>0, \tag{25}$$

where

$$\mathbb{R}_q^{\varkappa} f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\varkappa-1]_q!}{[\varkappa]_q![k-1]_q!} a_k z^k, \tag{26}$$

is the $q-$analogue of Ruscheweyh operator.

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