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Coefficient Bounds for a Class of Bi-Univalent Functions Defined by Chebyshev Polynomials

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Abstract

Using Chebyshev polynomials and $q-$differential operator, we define a new class of bi-univalent functions defined in the open unit disk. Initial coefficient bounds and Fekete-Szego inequalities for this functions class are obtained.

Keywords: *Analytic functions, bi-univalent functions, bi-starlike functions, coefficient bounds, Chebyshev polynomial, Fekete-Szego inequality, subordination, $q-$differential operator.*

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1 Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = \ + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

defined in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S} \subset \mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. For every $f \in \mathcal{S}$ $\exists$ an inverse function f^{-1} which is defined in some neighbourhood of the origin satisfying

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{D}),$$

and

$$f^{-1}(f(\omega)) = \omega, \quad (|\omega| < r_0(f);\ r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 + ... \tag{1.2}$$

A function $f \in \mathcal{A}$ is called bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Denote the class of bi-univalent functions by σ.

A function $f \in \mathcal{A}$ is said to be in the class $C_{\Re}(\alpha)$ of close-to-convex of order α ([21]), if there exist a function $\Re \in S^*$ such that

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha,\ (0 \leq \alpha < 1;\ z \in \mathbb{D}), \tag{1.3}$$

where S^* the class of starlike functions (see [18] and [4]).

It is known that the calculus without the notion of limits is called $q-$calculus which has influenced many scientific fields due to its important applications.The generalization of derivative in $q-$calculus that is $q-$derivative was defined and studied by Jackson [20]. He defined the $q-$difference (derivative) operator ∇_q for $f \in \mathcal{A},\ 0 < q < 1$, by (see also [3], [5–7], [10], [17], [25–27]);

$$\nabla_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & ,z\neq 0 \\ f'(0) & ,z=0 \end{cases},$$

that is

$$\nabla_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{1.4}$$

where

$$[j]_q = \frac{1-q^j}{1-q},\ [0]_q = 0. \tag{1.5}$$

As $q \to 1^-$, $[j]_q = j$ and $\nabla_q f(z) = f'(z)$.

The orthogonal polynomials are important for the contemporary mathematics. These polynomials play an essential role of complex functions theory and it occur in the theory of differential and integral equations (see [11, 14]).

A special case of orthogonal polynomials are Chebyshev polynomials.

The significance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. For a brief history of Chebyshev polynomials of the second kind $U_k(\iota)$ and applications one can refer [13, 15, 22]. The Chebyshev polynomials of the its second kinds is well known and they defined by

$$U_k(\iota) = \frac{\sin(k+1)\theta}{\sin\theta} \quad (-1 < \iota < 1), \tag{1.6}$$

where k denotes the polynomial degree and $\iota = \cos\theta$.

We note that if $\iota = \cos\alpha$, where $\alpha \in (-\pi/3, \pi/3)$, then

$$\phi(z,\iota) = \frac{1}{1 - 2\cos\alpha z + z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin(k+1)\alpha}{\sin\alpha} z^k.$$

Thus

$$\phi(z,\iota) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + ...$$

From [29], we can write

$$\phi(z,\iota) = 1 + U_1(\iota)z + U_2(\iota)z^2 + ... \quad (\iota \in (-1,1)),$$

where

$$U_{k-1} = \frac{\sin(k\cos^{-1}\iota)}{\sqrt{1-\iota^2}} \quad (k \in \mathbb{N}),$$

are the Chebyshev polynomials of the second kind,

$$U_k(\iota) = 2\iota U_{k-1}(\iota) - U_{k-2}(\iota),$$

and

$$U_1(\iota) = 2\iota,\ U_2(\iota) = 4\iota^2 - 1,\ U_3(\iota) = 8\iota^3 - 4\iota,\ U_4(\iota) = 16\iota^4 - 12\iota^2 + 1, ... \quad (1.7)$$

Now for $f \in \mathcal{A}$, we define $q-$differential operator $\mathcal{D}^{\zeta}_{\delta,\mu,q}$ for $\delta \geq \mu \geq 0$, $0 < q < 1$, by

$$\mathcal{D}^{0}_{\delta,\mu,q} f(z) = f(z),$$

$$\mathcal{D}^{1}_{\delta,\mu,q} f(z) = \mathcal{D}_{\delta,\mu,q} f(z) = (1-\mu)^{\delta} f(z) + (1 - (1-\mu)^{\delta}) z \nabla_q f(z),$$

$$\mathcal{D}^{2}_{\delta,\mu,q} f(z) = \mathcal{D}_{\delta,\mu,q}(\mathcal{D}_{\delta,\mu,q} f(z)),$$

and

$$\mathcal{D}^{\zeta}_{\delta,\mu,q} f(z) = \mathcal{D}_{\delta,\mu,q}(\mathcal{D}^{\zeta-1}_{\delta,\mu,q} f(z)),\ \zeta \in \mathbb{N} = \{1,2,3,\ldots\}.$$

For $f(z)$ given by (1.1), we have

$$\mathcal{D}^{\zeta}_{\delta,\mu,q} f(z) = z + \sum_{k=2}^{\infty} \left[1 + ([k]_q - 1)\, c^{\delta}_j(\mu)\right]^{\zeta} a_k z^k,\ \zeta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad (1.8)$$

where

$$c^{\delta}_j(\mu) = \sum_{j=1}^{\delta} \binom{\delta}{j} (-1)^{j+1} \mu^j. \quad (1.9)$$

Note that:
(i) $\lim_{q\to 1-} \mathcal{D}^{\zeta}_{\delta,\mu,q} f(z) = \mathcal{D}^{\zeta}_{\delta,\mu} f(z)$ (see Frasin [16]);
(ii) $\mathcal{D}^{\zeta}_{1,1,q} f(z) = \mathcal{D}^{\zeta}_{q} f(z)$ (see [19], [28] and [8]);
(iii) $\mathcal{D}^{\zeta}_{1,\mu,q} f(z) = \mathcal{D}^{\zeta}_{\mu,q} f(z)$ (see Aouf et al. [9]);
(iv) $\lim_{q\to 1-} \mathcal{D}^{\zeta}_{1,\mu,q} f(z) = \mathcal{D}^{\zeta}_{\mu} f(z)$ (see Al-Oboudi [1]).
(v) $\lim_{q\to 1-} \mathcal{D}^{\zeta}_{1,1,q} f(z) = \mathcal{D}^{\zeta} f(z)$ (see Sălăgean [24]).

Definition 1. *([12]) For f and g, analytic in $\mathbb{D}$, the function f is subordinate to g in $\mathbb{D}$ written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{D}$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$.*

Definition 2. *For $0 \leq \lambda \leq 1$, $\delta \geq \mu \geq 0$, $0 < q < 1$, $\zeta \in \mathbb{N}_0$ and $\iota \in (-1, 1)$, $f \in \sigma$ given by (1.1) for $z \in \mathbb{D}$, we say that $f \in \mathcal{B}^{\zeta}_{\sigma,q}(\delta, \mu, \lambda; \phi(z, \iota))$ if*

$$(1-\lambda)\frac{\mathcal{D}^{\zeta}_{\delta,\mu,q} f(z)}{z} + \lambda \nabla_q (\mathcal{D}^{\zeta}_{\delta,\mu,q} f(z)) \prec \phi(z, \iota), \tag{1.10}$$

and

$$(1-\lambda)\frac{\mathcal{D}^{\zeta}_{\delta,\mu,q} \Re(\omega)}{\omega} + \lambda \nabla_q (\mathcal{D}^{\zeta}_{\delta,\mu,q} \Re(\omega)) \prec \phi(\omega, \iota), \tag{1.11}$$

$\Re(\omega) = f^{-1}(\omega)$ is defined by (1.2).

For $\lambda = 1$, $\mathcal{B}^{\zeta}_{\sigma,q}(\delta, \mu, \lambda; \phi(z, \iota))$ reduces to the following class.

Definition 3. *A function $f \in \sigma$ given by (1.1), we say that $f \in \mathcal{B}^{\zeta}_{\sigma,q}(\delta, \mu; \phi(z, \iota))$ if*

$$\nabla_q (\mathcal{D}^{\zeta}_{\delta,\mu,q} f(z)) \prec \phi(z, \iota),$$

and

$$\nabla_q (\mathcal{D}^{\zeta}_{\delta,\mu,q} \Re(\omega)) \prec \phi(\omega, \iota),$$

where $z, \omega \in \mathbb{D}$ and $\Re(\omega) = f^{-1}(\omega)$ is defined by (1.2).

Note that:

(i) $\mathcal{B}^{\zeta}_{\sigma,q}(1, \mu, \lambda; \phi(z, \iota)) = \left\{ \begin{array}{c} f(z) : (1-\lambda)\frac{\mathcal{D}^{\zeta}_{\mu,q} f(z)}{z} + \lambda \nabla_q (\mathcal{D}^{\zeta}_{\mu,q} f(z)) \prec \phi(z, \iota) \\ \Re(\omega) : (1-\lambda)\frac{\mathcal{D}^{\zeta}_{\mu,q} \Re(\omega)}{\omega} + \lambda \nabla_q (\mathcal{D}^{\zeta}_{\mu,q} \Re(\omega)) \prec \phi(\omega, \iota) \end{array} \right\}$;

(ii) $\mathcal{B}^{\zeta}_{\sigma,q}(1, 1, \lambda; \phi(z, \iota)) = \left\{ \begin{array}{c} f(z) : (1-\lambda)\frac{\mathcal{D}^{\zeta}_{q} f(z)}{z} + \lambda \nabla_q (\mathcal{D}^{\zeta}_{q} f(z)) \prec \phi(z, \iota) \\ \Re(\omega) : (1-\lambda)\frac{\mathcal{D}^{\zeta}_{q} \Re(\omega)}{\omega} + \lambda \nabla_q (\mathcal{D}^{\zeta}_{q} \Re(\omega)) \prec \phi(\omega, \iota) \end{array} \right\}$;

(iii) $\lim_{q \to 1^-} \mathcal{B}^{\zeta}_{\sigma,q}(1, \mu, \lambda; \phi(z, \iota)) = \left\{ \begin{array}{c} f(z) : (1-\lambda)\frac{\mathcal{D}^{\zeta}_{\mu} f(z)}{z} + \lambda (\mathcal{D}^{\zeta}_{\mu} f(z))' \prec \phi(z, \iota) \\ \Re(\omega) : (1-\lambda)\frac{\mathcal{D}^{\zeta}_{\mu} \Re(\omega)}{\omega} + \lambda (\mathcal{D}^{\zeta}_{\mu} \Re(\omega))' \prec \phi(\omega, \iota) \end{array} \right\}$;

(iv) $\lim_{q \to 1^-} \mathcal{B}^{\zeta}_{\sigma,q}(1, 1, \lambda; \phi(z, \iota)) = \left\{ \begin{array}{c} f(z) : (1-\lambda)\frac{\mathcal{D}^{\zeta} f(z)}{z} + \lambda (\mathcal{D}^{\zeta} f(z))' \prec \phi(z, \iota) \\ \Re(\omega) : (1-\lambda)\frac{\mathcal{D}^{\zeta} \Re(\omega)}{\omega} + \lambda (\mathcal{D}^{\zeta} \Re(\omega))' \prec \phi(\omega, \iota) \end{array} \right\}$;

(v) $\lim_{q \to 1^-} \mathcal{B}^{0}_{\sigma,q}(\delta, \mu, \lambda; \phi(z, \iota)) = \left\{ \begin{array}{c} f(z) : (1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \phi(z, \iota) \\ \Re(\omega) : (1-\lambda)\frac{\Re(\omega)}{\omega} + \lambda g'(\omega) \prec \phi(\omega, \iota) \end{array} \right\}$.

In the following obtain coefficient bounds for the function class $\mathcal{B}^{\zeta}_{\sigma,q}(\delta, \mu, \lambda; \phi(z, \iota))$ and some of its special classes.

2 Main Results

Unless indicated, we assume that $0 \leq \lambda \leq 1$, $\delta \geq \mu \geq 0$, $0 < q < 1$, $\zeta \in \mathbb{N}_0$, $\iota \in (-1,1)$ and $f(z) \in \sigma$ given by (1.1).

Theorem 2.1. *Let* $f \in \mathcal{B}^{\zeta}_{\sigma,q}(\delta,\mu,\lambda;\phi(z,\iota))$ *and* $\iota \in (0,1)$. *Then*

$$|a_2| \leq \frac{2\iota\sqrt{2\iota}}{\sqrt{\left|\begin{array}{c}\left\{\left[1+([3]_q-1)\,c_j^{\delta}(\mu)\right]^{\zeta}[1+\lambda([3]_q-1)]-\left[1+qc_j^{\delta}(\mu)\right]^{2\zeta}(1+\lambda q)^2\right\}4\iota^2\\ +\left[1+qc_j^{\delta}(\mu)\right]^{2\zeta}(1+\lambda q)^2\end{array}\right|}}, \tag{2.1}$$

and

$$|a_3| \leq \frac{4\iota^2}{\left[1+qc_j^{\delta}(\mu)\right]^{2\zeta}(1+\lambda q)^2}+\frac{2\iota}{\left[1+([3]_q-1)\,c_j^{\delta}(\mu)\right]^{\zeta}[1+\lambda([3]_q-1)]}, \tag{2.2}$$

where $\iota \neq 1/\sqrt{2}$.

Proof. Let $\mathcal{B}^{\zeta}_{\sigma,q}(\delta,\mu,\lambda;\phi(z,\iota))$ and $\Re = f^{-1}$. Considering (1.10) and (1.11), we have

$$(1-\lambda)\frac{\mathcal{D}^{\zeta}_{\delta,\mu,q}f(z)}{z}+\lambda\nabla_q(\mathcal{D}^{\zeta}_{\delta,\mu,q}f(z))=\phi(z,\iota), \tag{2.3}$$

$$(1-\lambda)\frac{\mathcal{D}^{\zeta}_{\delta,\mu,q}\Re(\omega)}{\omega}+\lambda\nabla_q(\mathcal{D}^{\zeta}_{\delta,\mu,q}\Re(\omega))=\phi(\omega,\iota), \tag{2.4}$$

for some analytic functions

$$p(z)=c_1z+c_2z^2+c_3z^3+\ldots \quad (z\in\mathbb{D}), \tag{2.5}$$

and

$$q(\omega)=d_1\omega+d_2\omega^2+d_3\omega^3+\ldots \quad (\omega\in\mathbb{D}), \tag{2.6}$$

such that $|p(z)|<1$ $(z\in\mathbb{D})$ and $|q(\omega)|<1$ $(\omega\in\mathbb{D})$, hence

$$|c_j|\leq 1 \text{ and } |d_j|\leq 1 \text{ for all } j\in\mathbb{N}. \tag{2.7}$$

From (2.3), (2.4), (2.5) and (2.6), we have

$$(1-\lambda)\frac{\mathcal{D}^{\zeta}_{\delta,\mu,q}f(z)}{z}+\lambda\nabla_q(\mathcal{D}^{\zeta}_{\delta,\mu,q}f(z))=1+U_1(\iota)c_1z+\left[U_1(\iota)c_2+U_2(\iota)c_1^2\right]z^2+\ldots \tag{2.8}$$

and

$$(1-\lambda)\frac{\mathcal{D}^{\zeta}_{\delta,\mu,q}\Re(\omega)}{\omega}+\lambda\nabla_q(\mathcal{D}^{\zeta}_{\delta,\mu,q}\Re(\omega))=1+U_1(\iota)d_1\omega+\left[U_1(\iota)d_2+U_2(\iota)d_1^2\right]\omega^2+\ldots. \tag{2.9}$$

Equating the coefficients in (2.8) and (2.9), we get

$$\left[1+qc_j^{\delta}(\mu)\right]^{\zeta}(1+\lambda q)\,a_2=U_1(\iota)c_1, \tag{2.10}$$

$$\left[1+([3]_q-1)\,c_j^{\delta}(\mu)\right]^{\zeta}\left[1+\lambda([3]_q-1)\right]a_3=U_1(\iota)c_2+U_2(\iota)c_1^2, \tag{2.11}$$

$$-\left[1+qc_j^{\delta}(\mu)\right]^{\zeta}(1+\lambda q)\,a_2=U_1(\iota)d_1, \tag{2.12}$$

and

$$\left[1+([3]_q-1)\,c_j^{\delta}(\mu)\right]^{\zeta}\left[1+\lambda([3]_q-1)\right](2a_2^2-a_3)=U_1(\iota)d_2+U_2(\iota)d_1^2. \tag{2.13}$$

From (2.10) and (2.12), we obtain

$$c_1=-d_1, \tag{2.14}$$

and

$$2\left[1+qc_j^{\delta}(\mu)\right]^{2\zeta}(1+\lambda q)^2\,a_2^2=U_1^2(\iota)(c_1^2+d_1^2). \tag{2.15}$$

Also, by using (2.11) and (2.13), we obtain

$$2\left[1+([3]_q-1)\,c_j^{\delta}(\mu)\right]^{\zeta}\left[1+\lambda([3]_q-1)\right]a_2^2=U_1(\iota)(c_2+d_2)+U_2(\iota)(c_1^2+d_1^2). \tag{2.16}$$

By using (2.15) in (2.16), we get

$$\begin{aligned}&\left\{2\left[1+([3]_q-1)\,c_j^{\delta}(\mu)\right]^{\zeta}\left[1+\lambda([3]_q-1)\right]-\frac{2U_2(\iota)}{U_1^2(\iota)}\left[1+qc_j^{\delta}(\mu)\right]^{2\zeta}(1+\lambda q)^2\right\}a_2^2\\ =\;& U_1(\iota)(c_2+d_2).\end{aligned} \tag{2.17}$$

From (1.7), (2.7) and (2.17), we have (2.1).

Next, by subtracting (2.13) from (2.11), we have

$$\begin{aligned}&2\left[1+([3]_q-1)\,c_j^{\delta}(\mu)\right]^{\zeta}\left[1+\lambda([3]_q-1)\right]a_3-2\left[1+([3]_q-1)\,c_j^{\delta}(\mu)\right]^{\zeta}\left[1+\lambda([3]_q-1)\right]a_2^2\\ =\;& U_1(\iota)(c_2-d_2)+U_2(\iota)(c_1^2-d_1^2).\end{aligned} \tag{2.18}$$

Further, in view of (2.14), we obtain

$$a_3=a_2^2+\frac{U_1(\iota)}{2\left[1+([3]_q-1)\,c_j^{\delta}(\mu)\right]^{\zeta}\left[1+\lambda([3]_q-1)\right]}(c_2-d_2). \tag{2.19}$$

Hence using (2.15) and applying (1.7), we get (2.2).

Theorem 2.2. *Let* $f\in\mathcal{B}_{\sigma,q}^{\zeta}(\delta,\mu,\lambda;\phi(z,\iota))$ *and* $\eta\in\mathbb{R}$. *Then*

$$\left|a_3-\eta a_2^2\right|\le\begin{cases}\dfrac{2\iota}{\left[1+([3]_q-1)c_j^{\delta}(\mu)\right]^{\zeta}\left[1+\lambda([3]_q-1)\right]} & ,0\le|h(\eta)|\le\dfrac{1}{2\left[1+([3]_q-1)c_j^{\delta}(\mu)\right]^{\zeta}\left[1+\lambda([3]_q-1)\right]}\\ 4\iota\,|h(\eta)| & ,|h(\eta)|\ge\dfrac{1}{2\left[1+([3]_q-1)c_j^{\delta}(\mu)\right]^{\zeta}\left[1+\lambda([3]_q-1)\right]}\end{cases}. \tag{2.20}$$

Proof. —By using (2.17) and (2.19) for some $\eta \in \mathbb{R}$, we get

$$\begin{aligned} a_3 - \eta a_2^2 &= (1-\eta)\left\{ \frac{U_1^3(\iota)(c_2+d_2)}{2\left\{ \begin{array}{c} \left[1+([3]_q-1)\, c_j^\delta(\mu)\right]^\zeta [1+\lambda([3]_q-1)]\, U_1^2(\iota) \\ -\left[1+qc_j^\delta(\mu)\right]^{2\zeta} (1+\lambda q)^2\, U_2(\iota) \end{array} \right\}} \right\} \\ &\quad + \frac{U_1(\iota)(c_2-d_2)}{2\left[1+([3]_q-1)\, c_j^\delta(\mu)\right]^\zeta [1+\lambda([3]_q-1)]} \\ &= U_1(\iota)\left[\begin{array}{c} \left(h(\eta)+\frac{1}{2\left[1+([3]_q-1)c_j^\delta(\mu)\right]^\zeta [1+\lambda([3]_q-1)]}\right) c_2 \\ +\left(h(\eta)-\frac{1}{2\left[1+([3]_q-1)c_j^\delta(\mu)\right]^\zeta [1+\lambda([3]_q-1)]}\right) d_2 \end{array} \right], \qquad (2.21) \end{aligned}$$

where

$$h(\eta) = \frac{U_1^2(\iota)(1-\eta)}{2\left\{\left[1+([3]_q-1)\, c_j^\delta(\mu)\right]^\zeta [1+\lambda([3]_q-1)]\, U_1^2(\iota) - \left[1+qc_j^\delta(\mu)\right]^{2\zeta} (1+\lambda q)^2\, U_2(\iota)\right\}}. \qquad (2.22)$$

So, we conclude (2.20).

For $\eta = 1$, we have

Corollary 2.1. *If* $f \in \mathcal{B}_{\sigma,q}^{\zeta}(\delta,\mu,\lambda;\phi(z,\iota))$*, then*

$$\left|a_3 - a_2^2\right| \le \frac{2\iota}{\left[1+([3]_q-1)\, c_j^\delta(\mu)\right]^\zeta [1+\lambda([3]_q-1)]}. \qquad (2.23)$$

Corollary 2.2. *If* $f \in \mathcal{B}_{\sigma,q}^{\zeta}(\delta,\mu;\phi(z,\iota))$*, then*

$$\left|a_3 - \eta a_2^2\right| \le \left\{ \begin{array}{ll} \frac{2\iota}{[3]_q\left[1+([3]_q-1)c_j^\delta(\mu)\right]^\zeta} & , 0 \le |h(\eta)| \le \frac{1}{2[3]_q\left[1+([3]_q-1)c_j^\delta(\mu)\right]^\zeta} \\ 4\iota\, |h(\eta)| & , |h(\eta)| \ge \frac{1}{2[3]_q\left[1+([3]_q-1)c_j^\delta(\mu)\right]^\zeta} \end{array} \right., \qquad (2.24)$$

where

$$h(\eta) = \frac{U_1^2(\iota)(1-\eta)}{2\left\{[3]_q\left[1+([3]_q-1)\, c_j^\delta(\mu)\right]^\zeta U_1^2(\iota) - \left[1+qc_j^\delta(\mu)\right]^{2\zeta}(1+q)^2\, U_2(\iota)\right\}}. \qquad (2.25)$$

Corollary 2.3. *If* $f \in \mathcal{B}_{\sigma,q}^{0}(\delta,\mu;\phi(z,\iota))$*, then*

$$\left|a_3 - \eta a_2^2\right| \le \left\{ \begin{array}{ll} \frac{2\iota}{[1+\lambda([3]_q-1)]} & , 0 \le |h(\eta)| \le \frac{1}{2[1+\lambda([3]_q-1)]} \\ 4\iota\, |h(\eta)| & , |h(\eta)| \ge \frac{1}{2[1+\lambda([3]_q-1)]} \end{array} \right., \qquad (2.26)$$

where

$$h(\eta) = \frac{U_1^2(\iota)(1-\eta)}{2\left\{[1+\lambda([3]_q-1)]\, U_1^2(\iota) - (1+\lambda q)^2\, U_2(\iota)\right\}}. \qquad (2.27)$$

3 Open Problem

The authors suggest finding the hankel determinant for the class

$$\Re\left\{(1-\lambda)\frac{\mathcal{D}^{\zeta}_{\delta,\mu,q}f(z)}{z}+\lambda\nabla_q(\mathcal{D}^{\zeta}_{\delta,\mu,q}f(z))\right\}>0, \tag{3.1}$$

where

$$\mathcal{D}^{\zeta}_{\delta,\mu,q}f(z)=z+\sum_{k=2}^{\infty}\left[1+([k]_q-1)\,c_j^{\delta}(\mu)\right]^{\zeta}a_kz^k,\ \zeta\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\,, \tag{3.2}$$

$$c_j^{\delta}(\mu)=\sum_{j=1}^{\delta}\left(\begin{array}{c}\delta\\ j\end{array}\right)(-1)^{j+1}\,\mu^j. \tag{3.3}$$

is the $q-$differential operator in [16]. Also, further investigation of other classes of orthogonal polynomials are suggested (see [2]).

4 Conclusion

By means of Chebyshev polynomials and $q-$ differential operator in [16], in this paper, we defined a class of bi-univalent functions and its special classes and obtained coefficient bounds for it.

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