Int. J. Open Problems Complex Analysis, Vol. 14, No. 1, June 2022
ISSN 2074-2827; Copyright ©ICSRS Publication, 2022
www.i-csrs.org

Some Geometric Properties for Certain Subclasses of p-valent Functions Involving Differ-Integral Operator

W. Y. Kota and R. M. El-Ashwah

Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt
e-mail: wafaa_kota@yahoo.com
e-mail: r_elashwah@yahoo.com

Received 15 March 2022; Accepted 20 May 2022

Abstract

In the present paper, we aim at proving such results as inclusion relationships and convolution properties for the class $\mathfrak{S}_{p,\mu}^{(j)}(a,c;\alpha;\phi)$. Then we study the integral properties for the class $\mathfrak{S}_{p,\mu}^{(j)}(a,c;\tau)$. Also, we investigate majorization properties for subclass of analytic functions defined by differ-integral operator.

Keywords: *Analytic functions, Erdélyi-Kober type integral operator, inclusion properties, Hadamard product, majorization.*
2010 Mathematical Subject Classification: 30C45.

1 Introduction

Let $\mathcal{A}_p$ be the class of analytic and $p-$valent functions in the open unit disc $\triangle = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ which denote by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (p \in \mathbb{N} = \{1, 2, ...\}). \tag{1.1}$$

We note that, $\mathcal{A}_1 = \mathcal{A}$ is the class of univalent and analytic functions in $\triangle$.

Also, let $\mathcal{P}$ denote the class of functions of the form:

$$\mathcal{P}(z) = 1 + \sum_{k=1}^{\infty} \mathcal{P}_k z^k, \qquad (z \in \triangle),$$

which are analytic and convex in $\triangle$ and satisfy the following inequality:

$$\mathrm{Re}\{\mathcal{P}(z)\} > 0.$$

Let $f, g \in \mathcal{A}_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p},$$

then Hadmard product (or convolution) of the functions f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z).$$

Definition 1.1 *[6] For two functions f and g, analytic in $\triangle$, we say that the function f is subordinate to g in $\triangle$, written $f \prec g$, if there exists a Schwarz function $\omega(z)$ which is analytic in $\triangle$, satisfying the following conditions:*

$$\omega(0) = 0 \qquad \textit{and} \qquad |\omega(z)| < 1, \quad (z \in \triangle),$$

such that

$$f(z) = g(\omega(z)), \qquad (z \in \triangle).$$

In particular, if the function g is univalent in $\triangle$, we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \triangle) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \textit{and} \quad f(\triangle) \subset g(\triangle).$$

Definition 1.2 *[15] For two functions f and g, analytic in $\triangle$, we say that the function f is majorized by g in $\triangle$, written $f << g$ $(z \in \triangle)$, if there exists a function $\varphi(z)$ which is analytic in $\triangle$, such that*

$$|\varphi(z)| < 1 \qquad \textit{and} \qquad f(z) = \varphi(z) g(z); \quad (z \in \triangle), \tag{1.2}$$

Taking $\mu > 0$, a, $c \in \mathbb{C}$ such that $\mathrm{Re}(c-a) \geq 0, \mathrm{Re}(a) \geq -\mu p \, (p \in \mathbb{N})$ and $f(z) \in \mathcal{A}_p$ is given by (1.1), El-Ashwah and Drbuk [9, with $m = 0$] introduced the differ-integral operator $\mathfrak{d}_{p,\mu}^{a,c} : \mathcal{A}_p \to \mathcal{A}_p$ as follows:

- For $Re(c-a) > 0$ by

$$\mathfrak{d}_{p,\mu}^{a,c} f(z) = \frac{\Gamma(c+\mu p)}{\Gamma(a+\mu p)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} f(zt^\mu) dt; \tag{1.3}$$

- For $a = c$ by

$$\mathfrak{d}_{p,\mu}^{a,a} f(z) = f(z). \tag{1.4}$$

- For $a=\gamma$, $c=\gamma+1$ and $\mu=1$, we obtain a familiar integral operator $\mathcal{H}_{\gamma,p}$ defined by [22] as follows

$$\begin{aligned}\mathcal{H}_{\gamma,p}f(z) &= \frac{p+\gamma}{z^{\gamma}}\int_0^z t^{\gamma-1}f(t)dt \quad (\gamma>-p,\, p\in\mathbb{N}) \\ &= z^p+\sum_{k=1}^{\infty}\left(\frac{p+\gamma}{p+k+\gamma}\right)a_{k+p}z^{k+p},\end{aligned} \tag{1.5}$$

It is readily verified from (1.5) that

$$z\left[\mathfrak{d}_{p,\mu}^{a,c}\mathcal{H}_{\gamma,p}f(z)\right]'=(\gamma+p)\mathfrak{d}_{p,\mu}^{a,c}f(z)-\gamma\mathfrak{d}_{p,\mu}^{a,c}\mathcal{H}_{\gamma,p}f(z). \tag{1.6}$$

Using (1.3), the operator $\mathfrak{d}_{p,\mu}^{a,c}f(z)$ can be expressed as follows:

$$\mathfrak{d}_{p,\mu}^{a,c}f(z)=z^p+\frac{\Gamma(c+\mu p)}{\Gamma(a+\mu p)}\sum_{k=1}^{\infty}\frac{\Gamma(a+\mu(k+p))}{\Gamma(c+\mu(k+p))}a_{k+p}z^{k+p}, \tag{1.7}$$

where $\mu>0,\ a,\ c\in\mathbb{C}, \mathrm{Re}(c-a)\geq 0, \mathrm{Re}(a)\geq -\mu p\,(p\in\mathbb{N})$.
It is readily verified from (1.7) that

$$z(\mathfrak{d}_{p,\mu}^{a,c}f)'(z)=\left(\frac{a+\mu p}{\mu}\right)\left(\mathfrak{d}_{p,\mu}^{a+1,c}f\right)(z)-\left(\frac{a}{\mu}\right)\left(\mathfrak{d}_{p,\mu}^{a,c}f\right)(z). \tag{1.8}$$

We also note that the operator $\mathfrak{d}_{p,\mu}^{a,c}f(z)$ generalizes several previously studied familiar operators, and we will mention some of the interesting particular cases as follows:

(i) For $a=\beta$, $c=\alpha+\beta-\gamma+1$ and $\mu=1$, we obtain the operator $\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f(z)$ $(\gamma>0;\ \alpha\geq\gamma-1;\ \beta>-p)$ which studied by Aouf et al. [1];

(ii) For $a=\beta$, $c=\alpha+\beta$ and $\mu=1$, we obtain the operator $Q_{\beta,p}^{\alpha}f(z)$ $(\alpha\geq 0;\ \beta>-p)$ which studied by Liu and Owa [13];

(iii) For $p=1$, we obtain the operator $\check{I}_{\mu}^{a,c}f(z)$ which studied by Raina and Sharma [20];

(iv) For $p=1$, $a=\beta$, $c=\alpha+\beta$ and $\mu=1$, we obtain the operator $Q_{\beta}^{\alpha}f(z)$ $(\alpha\geq 0,\ \beta>-1)$ which studied by Jung et al. [11];

(v) For $p=1$, $a=\alpha-1$, $c=\beta-1$ and $\mu=1$, we obtain the operator $L(\alpha,\beta)f(z)$ $(\alpha,\ \beta\in\mathbb{C}\setminus\mathbb{Z}_0,\ \mathbb{Z}_0=\{0,-1,-2,...\})$ which studied by Carlson and Shaffer [3];

(vi) For $p=1$, $a=\nu-1$, $c=v$ and $\mu=1$, we obtain the operator $I_{\nu,v}f(z)$ $(\nu>0;\ v>-1)$ which studied by Choi et al. [5];

(vii) For $p = 1$, $a = \alpha$, $c = 0$ and $\mu = 1$, we obtain the operator $D^{\alpha}f(z)$ $(\alpha > -1)$ which studied by Ruscheweyh [21];

(viii) For $p = 1$, $a = 1$, $c = n$ and $\mu = 1$, we obtain the operator $I_n f(z)$ $(n \in \mathbb{N})$ which studied by Noor [17];

(ix) For $p = 1$, $a = \beta$, $c = \beta + 1$, and $\mu = 1$, we obtain the integral operator J_β which studied by Bernardi [2];

(x) For $p = 1$, $a = 1$, $c = 2$, and $\mu = 1$, we obtain the integral operator J which studied by Libera [12] and Livingston [14].

Note that

$$f^{(j)}(z) = \delta(p,j)z^{(p-j)} + \sum_{k=1}^{\infty} \delta(k+p,j)a_{k+p}z^{k+p-j},$$

where

$$\delta(p,j) = p(p-1)(p-2)...(p-j+1).$$

By making use of the operator $\mathfrak{d}_{p,\mu}^{a,c}$ and the above mentioned principle of subordination between analytic functions, we introduce and investigate the following subclass of the class $\mathcal{A}_p$ as follows:

Definition 1.3 *A function $f \in \mathcal{A}_p$ is said to be in the class $\mathfrak{S}_{p,\mu}^{(j)}(a,c;\alpha;\phi)$ if it satisfies the following subordination condition:*

$$\frac{z[(1-\alpha)(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j+1)}(z) + \alpha(\mathfrak{d}_{p,\mu}^{a+1,c}f)^{(j+1)}]}{(1-\alpha)(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j)}(z) + \alpha(\mathfrak{d}_{p,\mu}^{a+1,c}f)^{(j)}} \prec (p-j)\phi(z) \quad (z \in \triangle), \tag{1.9}$$

for some $\alpha\,(\alpha \geq 0)$ and $j\,(j \in \{0,1,...,p-1\})$ where $\phi \in \mathcal{P}$.
For simplicity, we write

$$\begin{aligned} \mathfrak{S}_{p,\mu}^{(j)}(a,c;0;\phi) &= \mathfrak{S}_{p,\mu}^{(j)}(a,c;\phi), \\ \mathfrak{S}_{p,\mu}^{(j)}\left(a,c;0;\frac{1+Az}{1+Bz}\right) &= \mathfrak{S}_{p,\mu}^{(j)}(a,c;A,B) \ (-1 \leq B < A \leq 1), \end{aligned}$$

and

$$\mathfrak{S}_{p,\mu}^{(j)}\left(a,c;0;\frac{1+(1-2\tau)z}{1-z}\right) = \mathfrak{S}_{p,\mu}^{(j)}(a,c;\tau), \qquad (0 \leq \tau < 1).$$

Remark 1.4 (i) *Putting $a = c$, $\alpha = 0$, $\mu = 1$ and $\phi = \frac{1+(1-2\tau)z}{1-z}$, $(0 \leq \tau < p-j)$, the class $\mathfrak{S}_{p,\mu}^{(j)}(a,c;\alpha;\phi)$ reduces to the class $S(p,j,\tau)$ which studied by Chen et al. [4];*

(ii) *Putting* $a = c$, $\mu = 1$, $\alpha = j = 0$ *and* $\phi = \frac{1+(1-2\tau)z}{1-z}$, $(0 \leq \tau < p)$, *the class* $\mathfrak{S}_{p,\mu}^{(j)}(a, c; \alpha; \phi)$ *reduces to the class* $S_p(\tau)$ *which studied by Patel and Thakare [19];*
(iii) *Putting* $a = c = j = 0$, $\mu = \alpha = 1$ *and* $\phi = \frac{1+(1-2\tau)z}{1-z}$, $(0 \leq \tau < p)$, *the class* $\mathfrak{S}_{p,\mu}^{(j)}(a, c; \alpha; \phi)$ *reduces to the class* $\mathcal{K}_p(\tau)$ *which studied by Owa [18].*

In order to establish our main results, we shall also make use of the following lemmas:

Lemma 1.5 *[7] Let* $\beta, \delta \in \mathbb{C}$. *Suppose that* $\phi(z)$ *is convex and univalent in* $\triangle$ *with*

$$\phi(0) = 1 \quad \text{and} \quad Re(\beta\phi(z) + \delta) > 0 \qquad (z \in \triangle).$$

If $\mathcal{P}(z)$ *is analytic in* $\triangle$ *with* $\mathcal{P}(0) = 1$, *then the following subordination:*

$$\mathcal{P}(z) + \frac{z\mathcal{P}'(z)}{\beta\mathcal{P}(z) + \delta} \prec \phi(z) \qquad (z \in \triangle)$$

implies that

$$\mathcal{P}(z) \prec \phi(z) \qquad (z \in \triangle).$$

Lemma 1.6 *[10] Let* $w(z)$ *is analytic function in* $\mathbb{U}$, *with* $w(0) = 0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r < 1$ *at a point* $z_0 \in \mathbb{U}$, *then* $z_0 w'(z_0) = \zeta w(z_0)$, *where* ζ *is a real number and* $\zeta \geq 1$.

In the present paper, we aim at proving such results as inclusion relationships and convolution properties for the class $\mathfrak{S}_{p,\mu}^{(j)}(a, c; \alpha; \phi)$. Then we study the integral properties for the class $\mathfrak{S}_{p,\mu}^{(j)}(a, c; \tau)$. Also, we investigate majorization properties for subclass of analytic functions defined by differ-integral operator.

Unless otherwise mentioned, we shall assume throughout the paper that $\mu > 0$, a, $c \in \mathbb{R}$ such that $(c - a) \geq 0$, $a \geq -\mu p \, (p \in \mathbb{N})$, $-1 \leq B < A \leq 1$ and $\alpha \geq 0$.

2 A set of inclusion relationships

We prove some inclusion relationships for the class $\mathfrak{S}_{p,\mu}^{(j)}(a, c; \alpha; \phi)$, which was given in the previous section.

Theorem 2.1 *Let* $\phi \in \mathcal{P}$ *with*

$$Re\left((p-j)\phi(z) + \frac{a+\mu p}{\alpha\mu} - p + j\right) > 0 \qquad (\alpha > 0; j \in \{0, 1, ..., p-1\}; z \in \triangle),$$

then

$$\mathfrak{S}_{p,\mu}^{(j)}(a, c; \alpha; \phi) \subset \mathfrak{S}_{p,\mu}^{(j)}(a, c; \phi).$$

Proof. Let $f \in \mathfrak{S}^{(j)}_{p,\mu}(a, c; \alpha; \phi)$ and suppose that

$$\eta(z) = \frac{z(\mathfrak{d}^{a,c}_{p,\mu} f)^{(j+1)}(z)}{(p - j)(\mathfrak{d}^{a,c}_{p,\mu} f)^{(j)}(z)} \qquad (z \in \triangle). \tag{2.1}$$

The function η is analytic in $\triangle$ and $\eta(0) = 1$. By using (1.8), we obtain

$$z(\mathfrak{d}^{a,c}_{p,\mu} f)^{(j+1)}(z) = \left(\frac{a + \mu p}{\mu}\right) \left(\mathfrak{d}^{a+1,c}_{p,\mu} f\right)^{(j)} (z) - \left(\frac{a}{\mu} + j\right) \left(\mathfrak{d}^{a,c}_{p,\mu} f\right)^{(j)} (z) \qquad (j \in \{0, 1, ..., p - 1\}). \tag{2.2}$$

It follows from (2.2) and (2.1) that

$$\frac{a}{\mu} + j + (p - j)\eta(z) = \left(\frac{a + \mu p}{\mu}\right) \frac{(\mathfrak{d}^{a+1,c}_{p,\mu} f)^{(j)}(z)}{(\mathfrak{d}^{a,c}_{p,\mu} f)^{(j)}(z)}. \tag{2.3}$$

From (2.1) and (2.3), we can find that

$$\begin{aligned} z(\mathfrak{d}^{a+1,c}_{p,\mu} f)^{(j+1)}(z) \\ = \frac{\mu(p - j)}{a + \mu p} \left[z\eta'(z) + \left\{\frac{a}{\mu} + j + (p - j)\eta(z)\right\} \eta(z)\right] (\mathfrak{d}^{a,c}_{p,\mu} f)^{(j)}(z). \end{aligned} \tag{2.4}$$

It now follows from (2.2), (2.1), (2.3) and (2.4) that

$$\begin{aligned} &\frac{z\left[(1 - \alpha)(\mathfrak{d}^{a,c}_{p,\mu} f)^{(j+1)}(z) + \alpha(\mathfrak{d}^{a+1,c}_{p,\mu} f)^{(j+1)}(z)\right]}{(p - j)\left[(1 - \alpha)(\mathfrak{d}^{a,c}_{p,\mu} f)^{(j)}(z) + \alpha(\mathfrak{d}^{a+1,c}_{p,\mu} f)^{(j)}(z)\right]} \\ = &\frac{(1 - \alpha)\eta(z) + \frac{\alpha\mu}{a+\mu p}\left[z\eta'(z) + \left\{\frac{a}{\mu} + j + (p - j)\eta(z)\right\}\eta(z)\right]}{(1 - \alpha) + \frac{\alpha\mu}{a+\mu p}\left[\frac{a}{\mu} + j + (p - j)\eta(z)\right]} \\ = &\frac{\frac{\alpha\mu}{a+\mu p} z\eta'(z) + \left[(1 - \alpha) + \frac{\alpha\mu}{a+\mu p}\left\{\frac{a}{\mu} + j + (p - j)\eta(z)\right\}\right]\eta(z)}{(1 - \alpha) + \frac{\alpha\mu}{a+\mu p}\left[\frac{a}{\mu} + j + (p - j)\eta(z)\right]} \\ = &\eta(z) + \frac{z\eta'(z)}{\frac{a+\mu p}{\alpha\mu} - p + j + (p - j)\eta(z)} \prec \phi(z) \quad (z \in \triangle). \end{aligned} \tag{2.5}$$

Moreover, since

$$\mathrm{Re}\left((p - j)\phi(z) + \frac{a + \mu p}{\alpha\mu} - p + j\right) > 0 \qquad (\alpha > 0; j \in \{0, 1, ..., p-1\}; z \in \triangle),$$

by Lemma 1.5 and (2.5), we have

$$\eta(z) = \frac{z(\mathfrak{d}^{a,c}_{p,\mu} f)^{(j+1)}(z)}{(p - j)(\mathfrak{d}^{a,c}_{p,\mu} f)^{(j)}(z)} \prec \phi(z),$$

that is, $f \in \mathfrak{S}^{(j)}_{p,\mu}(a,c;\phi)$. This implies that

$$\mathfrak{S}^{(j)}_{p,\mu}(a,c;\alpha;\phi) \subset \mathfrak{S}^{(j)}_{p,\mu}(a,c;\phi).$$

The proof of Theorem 2.1 is completed. ■

Theorem 2.2 *Let $\phi \in \mathcal{P}$ with*

$$Re\left((p-j)\phi(z)+\frac{a}{\mu}+j\right)>0 \qquad (j \in \{0,1,...,p-1\}; z \in \triangle),$$

then

$$\mathfrak{S}^{(j)}_{p,\mu}(a+1,c;\phi) \subset \mathfrak{S}^{(j)}_{p,\mu}(a,c;\phi).$$

Proof. Let $f \in \mathfrak{S}^{(j)}_{p,\mu}(a+1,c;\phi)$, then we obtain

$$\frac{z(\mathfrak{d}^{a+1,c}_{p,\mu}f)^{(j+1)}(z)}{(p-j)(\mathfrak{d}^{a+1,c}_{p,\mu}f)^{(j)}(z)} \prec \phi(z) \qquad (z \in \triangle). \tag{2.6}$$

Differentiating both sides of (2.3) with respect to z logarithmically and using (2.1), we obtain

$$\eta(z)+\frac{z\eta'(z)}{\frac{a}{\mu}+j+(p-j)\eta(z)}=\frac{z(\mathfrak{d}^{a+1,c}_{p,\mu}f)^{(j+1)}(z)}{(p-j)(\mathfrak{d}^{a+1,c}_{p,\mu}f)^{(j)}(z)} \quad (z \in \triangle). \tag{2.7}$$

From (2.6) and (2.7), we have

$$\eta(z)+\frac{z\eta'(z)}{\frac{a}{\mu}+j+(p-j)\eta(z)} \prec \phi(z) \quad (z \in \triangle). \tag{2.8}$$

Moreover, since

$$\mathrm{Re}\left((p-j)\phi(z)+\frac{a}{\mu}+j\right)>0 \qquad (z \in \triangle),$$

by Lemma 1.5 and (2.8), we know that

$$\eta(z)=\frac{z(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j+1)}(z)}{(p-j)(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j)}(z)} \prec \phi(z),$$

that is, $f \in \mathfrak{S}^{(j)}_{p,\mu}(a,c;\phi)$. This implies that

$$\mathfrak{S}^{(j)}_{p,\mu}(a+1,c;\phi) \subset \mathfrak{S}^{(j)}_{p,\mu}(a,c;\phi).$$

The proof of Theorem 2.2 is completed. ■

3 Convolution properties

In this section, we introduce some convolution properties for the class $\mathfrak{S}^{(j)}_{p,\mu}(a,c;\phi)$.

Theorem 3.1 *Let* $f \in \mathfrak{S}^{(j)}_{p,\mu}(a,c;\phi)$. *Then*

$$f^{(j)}(z) = \left(z^{p-j} exp\left((p-j)\int_0^z \frac{\phi(\omega(z))-1}{\zeta}d\zeta\right)\right) * \left(\sum_{k=0}^{\infty}\frac{\Gamma(a+\mu p)\Gamma(c+\mu(k+p))}{\Gamma(c+\mu p)\Gamma(a+\mu(k+p))}z^{k+p-j}\right)$$
$$(j \in \{0,1,...,p-1\}; z \in \triangle), \tag{3.1}$$

where ω *is analytic in* $\triangle$ *with* $\omega(0)=0$ *and* $|\omega(z)|<1$.

Proof. Suppose that $f \in \mathfrak{S}^{(j)}_{p,\mu}(a,c;\phi)$ and from (1.9) with $(\alpha=0)$ we have

$$\frac{z(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j+1)}(z)}{(p-j)(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j)}(z)} = \phi(\omega(z)) \qquad (z \in \triangle), \tag{3.2}$$

where ω is analytic in $\triangle$ with $\omega(0)=0$ and $|\omega(z)|<1$. We can easily find that

$$\frac{(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j+1)}(z)}{(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j)}(z)} - \frac{p-j}{z} = (p-j)\frac{\phi(\omega(z))-1}{z} \qquad (z \in \triangle), \tag{3.3}$$

upon integrating (3.3), we have

$$(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j)}(z) = z^{p-j}.exp\left((p-j)\int_0^z \frac{\phi(\omega(\zeta))-1}{\zeta}d\zeta\right). \tag{3.4}$$

On the other hand, we know from (1.7) that

$$(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j)}(z) = \left(\sum_{k=0}^{\infty}\frac{\Gamma(c+\mu p)\Gamma(a+\mu(k+p))}{\Gamma(a+\mu p)\Gamma(c+\mu(k+p))}z^{k+p-j}\right) * f^{(j)}(z). \tag{3.5}$$

The assertion (3.1) of Theorem 3.1 can now easily be derived from (3.4) and (3.5). ■

Theorem 3.2 *The function* $f \in \mathfrak{S}^{(j)}_{p,\mu}(a,c;\phi)$ *if and only if*

$$\frac{1}{z^{p-j}}\left[f^{(j)}(z) * \left(\sum_{k=0}^{\infty}\frac{\Gamma(c+\mu p)\Gamma(a+\mu(k+p))}{\Gamma(a+\mu p)\Gamma(c+\mu(k+p))}(k+p-j-(p-j)\phi(e^{i\theta}))z^{k+p-j}\right)\right] \neq 0$$
$$(j \in \{0,1,...,p-1\}; z \in \triangle; 0 \le \theta < 2\pi). \tag{3.6}$$

Proof. Suppose that $f \in \mathfrak{S}_{p,\mu}^{(j)}(a,c;\phi)$ and from (1.9) with $(\alpha = 0)$ we have

$$\frac{z(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j+1)}(z)}{(p-j)(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j)}(z)} \prec \phi(z) \qquad (z \in \triangle), \tag{3.7}$$

is equivalent to

$$\frac{z(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j+1)}(z)}{(p-j)(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j)}(z)} \neq \phi(e^{i\theta}) \qquad (z \in \triangle; 0 \leq \theta < 2\pi). \tag{3.8}$$

The condition (3.8) can be written as follows:

$$\frac{1}{z^{p-j}}\left[z(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j+1)}(z) - (p-j)(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j)}(z)\phi(e^{i\theta})\right] \neq 0 \qquad (z \in \triangle; 0 \leq \theta < 2\pi). \tag{3.9}$$

On the other hand, we know that

$$z(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j+1)}(z) = \left(\sum_{k=0}^{\infty} \frac{\Gamma(c+\mu p)\Gamma(a+\mu(k+p))}{\Gamma(a+\mu p)\Gamma(c+\mu(k+p))}(k+p-j)z^{k+p-j}\right) * f^{(j)}(z). \tag{3.10}$$

Upon substituting (3.5) and (3.10) into (3.9), we can easily get the convolution property (3.6). The proof of Theorem 3.2 is completed. ■

4 A set of integral preserving properties

In this section, obtain integral preserving properties involving the integral operator $\mathcal{H}_{\gamma,p}$ which given by (1.5). It is readily verified from (1.6) that

$$z\left[\mathfrak{d}_{p,\mu}^{a,c}\mathcal{H}_{\gamma,p}f\right]^{(j+1)}(z) = (\gamma+p)(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j)}(z) - (\gamma+j)(\mathfrak{d}_{p,\mu}^{a,c}\mathcal{H}_{\gamma,p}f)^{(j)}(z). \tag{4.1}$$

Theorem 4.1 *If* $f \in \mathfrak{S}_{p,\mu}^{(j)}(a,c;\tau)$, *then* $\mathcal{H}_{\gamma,p}f(z) \in \mathfrak{S}_{p,\mu}^{(j)}(a,c;\tau)$, *where* $\mathcal{H}_{\gamma,p}f(z)$ *is defined by (1.5).*

Proof. Suppose that $f \in \mathfrak{S}_{p,\mu}^{(j)}(a,c;\tau)$ and set

$$\frac{z(\mathfrak{d}_{p,\mu}^{a,c}\mathcal{H}_{\gamma,p}f)^{(j+1)}(z)}{(p-j)(\mathfrak{d}_{p,\mu}^{a,c}\mathcal{H}_{\gamma,p}f)^{(j)}(z)} = \frac{1+(1-2\tau)w(z)}{1-w(z)}, \tag{4.2}$$

where $w(0) = 0$. Then, by using (4.1) in (4.2), we obtain

$$\frac{(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j)}(z)}{(p-j)(\mathfrak{d}_{p,\mu}^{a,c}\mathcal{H}_{\gamma,p}f)^{(j)}(z)} = \frac{(\gamma+p)+[(p-j)(1-2\tau)-(\gamma+j)]\,w(z)}{(p-j)(\gamma+p)(1-w(z))}. \tag{4.3}$$

Differentiating (4.3) with respect to z, we obtain

$$\frac{z(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j+1)}(z)}{(p-j)(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j)}(z)} = \frac{1+(1-2\tau)w(z)}{1-w(z)} + \frac{[(p-j)(1-2\tau)-(\gamma+j)]zw'(z)}{(p-j)\left[(\gamma+p)+[(p-j)(1-2\tau)-(\gamma+j)]w(z)\right]} + \frac{zw'(z)}{(p-j)(1-w(z))}.$$

So that

$$\frac{z(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j+1)}(z)}{(p-j)(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j)}(z)} - \tau = (1-\tau)\frac{1+w(z)}{1-w(z)} + \frac{[(p-j)(1-2\tau)-(\gamma+j)]zw'(z)}{(p-j)\left[(\gamma+p)+[(p-j)(1-2\tau)-(\gamma+j)]w(z)\right]} + \frac{zw'(z)}{(p-j)(1-w(z))}.$$

Now, assuming that $\max_{|z|\le|z_0|}|w(z)| = |w(z_0)| = 1$ and applying Jack's lemma, we obtain

$$z_0w'(z_0) = \zeta w(z_0) \qquad (\zeta\in\mathbb{R}, \zeta\ge 1). \tag{4.4}$$

If we set $w(z_0) = e^{i\theta}$ $(\theta\in\mathbb{R})$ in (4.4) and observe that

$$\mathrm{Re}\left((1-\tau)\frac{1+w(z_0)}{1-w(z_0)}\right) = 0,$$

then we have

$$\begin{aligned}
&\mathrm{Re}\left(\frac{z(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j+1)}(z)}{(p-j)(\mathfrak{d}_{p,\mu}^{a,c}f)^{(j)}(z)} - \tau\right)\\
&= \frac{1}{p-j}\mathrm{Re}\left(\frac{[(p-j)(1-2\tau)-(\gamma+j)]z_0w'(z_0)}{(\gamma+p)+[(p-j)(1-2\tau)-(\gamma+j)]w(z_0)} + \frac{z_0w'(z_0)}{1-w(z_0)}\right)\\
&= \frac{1}{p-j}\mathrm{Re}\left(\frac{[(p-j)(1-2\tau)-(\gamma+j)]\zeta e^{i\theta}}{(\gamma+p)+[(p-j)(1-2\tau)-(\gamma+j)]e^{i\theta}} + \frac{\zeta e^{i\theta}}{1-e^{i\theta}}\right)\\
&= \frac{-\zeta}{2(p-j)}\frac{\tau(p-j)+(\gamma+j)}{(p-j)(1-\tau)} < 0,
\end{aligned}$$

which obviously contradicts the hypothesis f belongs to $\mathfrak{S}_{p,\mu}^{(j)}(a,c;\tau)$. The proof of Theorem 4.1 is completed. ■

5 Majorization properties for subclass of analytic functions

In this section, we investigate the majorization properties of subclass of analytic $p-$valent functions defined by differ-integral operator.

Theorem 5.1 *Let $f \in \mathcal{A}_p$ and suppose that $g \in \mathfrak{S}^{(j)}_{p,\mu}(a,c;A,B)$ and $|\frac{a+\mu p}{\mu}| > |(A-B)+(\frac{a+\mu p}{\mu})B|$. If $(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j)}(z)$ is majorized by $(\mathfrak{d}^{a,c}_{p,\mu}g)^{(j)}(z)$ in $\triangle$, then*

$$\left|(\mathfrak{d}^{a+1,c}_{p,\mu}f)^{(j)}(z)\right| \leq \left|(\mathfrak{d}^{a+1,c}_{p,\mu}g)^{(j)}(z)\right| \text{ for } |z| \leq r_0, \tag{5.1}$$

where $r_0 = r_0(a,c,A,B,\mu,p)$ is the smallest positive real root of the equation

$$\left|(A-B)+(\frac{a+\mu p}{\mu})B\right| r^3 - \left(\left|\frac{a+\mu p}{\mu}\right| + 2|B|\right) r^2 - \left(\left|(A-B)+\left(\frac{a+\mu p}{\mu}\right)B\right| + 2\right) r + \left|\frac{a+\mu p}{\mu}\right| = 0. \tag{5.2}$$

Proof. Since $g \in \mathfrak{S}^{(j)}_{p,\mu}(a,c;A,B)$, we have

$$1 + \frac{z(\mathfrak{d}^{a,c}_{p,\mu}g)^{(j+1)}(z)}{(\mathfrak{d}^{a,c}_{p,\mu}g)^{(j)}(z)} - (p-j) = \frac{1+Aw(z)}{1+Bw(z)}, \tag{5.3}$$

where $w(z)$ is analytic in $\triangle$ with $w(0)=0$ and $|w(z)| < |z| (z \in \triangle)$. From (5.3) and using (2.2), we get

$$|(\mathfrak{d}^{a,c}_{p,\mu}g)^{(j)}(z)| \leq \frac{(1+|B||z|)\left|\frac{a+\mu p}{\mu}\right|}{\left|\frac{a+\mu p}{\mu}\right| - \left|\left(\frac{a+\mu p}{\mu}\right)B + (A-B)\right||z|} \left|(\mathfrak{d}^{a+1,c}_{p,\mu}g)^{(j)}(z)\right|. \tag{5.4}$$

Next, since $(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j)}(z)$ is majorized by $(\mathfrak{d}^{a,c}_{p,\mu}g)^{(j)}(z)$ in $\triangle$, we have

$$(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j)}(z) = \varphi(z)(\mathfrak{d}^{a,c}_{p,\mu}g)^{(j)}(z).$$

Differentiating it with respect to z and multiplying by z, we get

$$z(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j+1)}(z) = z\varphi'(z)(\mathfrak{d}^{a,c}_{p,\mu}g)^{(j)}(z) + z\varphi(z)(\mathfrak{d}^{a,c}_{p,\mu}g)^{(j+1)}(z).$$

Using (2.2) in the last equation, it yields

$$(\mathfrak{d}^{a+1,c}_{p,\mu}f)^{(j)}(z) = \left(\frac{\mu}{a+\mu p}\right) z\varphi'(z)(\mathfrak{d}^{a,c}_{p,\mu}g)^{(j)}(z) + \varphi(z)(\mathfrak{d}^{a+1,c}_{p,\mu}g)^{(j)}(z). \tag{5.5}$$

Thus, noting that $\varphi(z) \in \mathcal{P}$ satisfies the inequality (see [16])

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2} \qquad (z \in \triangle), \tag{5.6}$$

and making use of (5.4) and (5.6) in (5.5), we get

$$|(\mathfrak{d}_{p,\mu}^{a+1,c} f)^{(j)}(z)| \leq \left[|\varphi(z)| + \left(\frac{1-|\varphi(z)|^2}{1-|z|^2} \right) \left(\frac{(1+|B||z|)|z|}{\left|\frac{a+\mu p}{\mu}\right| - \left|\left(\frac{a+\mu p}{\mu}\right) B + (A-B)\right| |z|} \right) \right] \left|(\mathfrak{d}_{p,\mu}^{a+1,c} g)^{(j)}(z)\right| . \tag{5.7}$$

which upon putting $|z| = r$ and $|\varphi(z)| = \varrho$ $(0 \leq \varrho \leq 1)$ leads to the inequality

$$|(\mathfrak{d}_{p,\mu}^{a+1,c} f)^{(j)}(z)| \leq \Upsilon(r, \varrho) \left|(\mathfrak{d}_{p,\mu}^{a+1,c} g)^{(j)}(z)\right| ,$$

where

$$\Upsilon(r, \varrho) = \frac{-r(1+|B|r)\varrho^2 + (1-r^2)\left(\left|\frac{a+\mu p}{\mu}\right| - \left|\left(\frac{a+\mu p}{\mu}\right) B + (A-B)\right| r\right) \varrho + r(1+|B|r)}{(1-r^2)\left(\left|\frac{a+\mu p}{\mu}\right| - \left|\left(\frac{a+\mu p}{\mu}\right) B + (A-B)\right| r\right)} .$$

In order to determine r_0, we note that

$$\begin{aligned} r_0 &= \max\{r \in [0,1] : \Upsilon(r, \varrho) \leq 1 \ \forall \varrho \in [0,1]\} \\ &= \max\{r \in [0,1] : \Psi(r, \varrho) \geq 0 \ \forall \varrho \in [0,1]\}, \end{aligned}$$

where

$$\begin{aligned} \Psi(r, \varrho) = (1-r^2) &\left(\left|\frac{a+\mu p}{\mu}\right| - \left|\left(\frac{a+\mu p}{\mu}\right) B + (A-B)\right| r \right) \\ &- (1-r^2)\varrho \left[\left|\frac{a+\mu p}{\mu}\right| - \left|\left(\frac{a+\mu p}{\mu}\right) B + (A-B)\right| r \right] - (1-\varrho^2) r (1+|B|r). \end{aligned}$$

A simple calculation shows that the inequality $\Psi(r, \varrho) \geq 0$ is equivalent to

$$v(r, \varrho) = (1-r^2)\left(\left|\frac{a+\mu p}{\mu}\right| - \left|\left(\frac{a+\mu p}{\mu}\right) B + (A-B)\right| r \right) - (1+\varrho) r (1+|B|r) \geq 0.$$

Obviously the function $v(r, \varrho)$ takes its minimum value at $\varrho = 1$, we conclude that (5.1) holds true for $|z| \leq r_0 = r_0(a, c, A, B, \mu, p)$ where $r_0(a, c, A, B, \mu, p)$ is the smallest positive real root of (5.2). The proof of Theorem 5.1 is completed. ■

Setting $A = 1 - 2\tau$ and $B = -1$ in Theorem 5.1, we will get the following result:

Corollary 5.2 *Let $f \in \mathcal{A}_p$ and suppose that $g \in \mathfrak{S}^{(j)}_{p,\mu}(a,c;1-2\tau,-1)$. If $(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j)}(z)$ is majorized by $(\mathfrak{d}^{a,c}_{p,\mu}g)^{(j)}(z)$ in $\triangle$, then*

$$\left|(\mathfrak{d}^{a+1,c}_{p,\mu}f)^{(j)}(z)\right| \le \left|(\mathfrak{d}^{a+1,c}_{p,\mu}g)^{(j)}(z)\right| \text{ for } |z| \le r_1,$$

where $r_1 = r_1(a,c,1-2\tau,-1,\mu,p)$ is the smallest positive real root of the equation

$$\left|-\left(\frac{a+\mu p}{\mu}\right)+2(1-\tau)\right|r^3-\left(\left|\frac{a+\mu p}{\mu}\right|+2\right)r^2-\left(\left|-\left(\frac{a+\mu p}{\mu}\right)+2(1-\tau)\right|+2\right)r + \left|\frac{a+\mu p}{\mu}\right| = 0.$$

Setting $\tau = 0$ in Corollary 5.2, we will get the following result:

Corollary 5.3 *Let $f \in \mathcal{A}_p$ and suppose that $g \in \mathfrak{S}^{(j)}_{p,\mu}(a,c;1,-1)$. If $(\mathfrak{d}^{a,c}_{p,\mu}f)^{(j)}(z)$ is majorized by $(\mathfrak{d}^{a,c}_{p,\mu}g)^{(j)}(z)$ in $\triangle$, then*

$$\left|(\mathfrak{d}^{a+1,c}_{p,\mu}f)^{(j)}(z)\right| \le \left|(\mathfrak{d}^{a+1,c}_{p,\mu}g)^{(j)}(z)\right| \text{ for } |z| \le r_2,$$

where $r_2 = r_2(a,c,\mu,p)$ is the smallest positive real root of the equation

$$r_2(a,c,\mu,p) = \frac{\kappa - \sqrt{\kappa^2 - 4|\nu||2-\nu|}}{2|2-\nu|},$$

where $\nu = \frac{a+\mu p}{\mu}$, $\kappa = |2+\nu| + |2-\nu|$, $p \in \mathbb{N}$.

Remark 5.4

- Putting $a = c = 0$, $\mu = 1$ and $j = 0$ in Corollary 5.3, we obtain the results which obtained by El-Ashwah and Aouf. [8, Corollary 2.4 with $\gamma = 1$];
- Putting $a = c = 0$, $\mu = 1$, $j = 0$ and $p = 1$ in Corollary 5.3, we obtain the results which obtained by MacGregor [15].

6 Open problem

Discussing some results as inclusion relationships and convolution properties for the class $\mathfrak{S}^{(j)}_{p,\mu}(a,c;\alpha;A,B)$, $(-1 \le B < A \le 1$, $j \in \{0,1,...,p-1\}$, $\alpha \ge 0$, $\mu > 0$, $a,c \in \mathbb{R}$, $(c-a) \ge 0$, $a \ge -\mu p$, $p \in \mathbb{N})$.

References

[1] M. K. Aouf, R. M. El-Ashwah and A. M. Abd-Eltawab, Some inclusion relationships of certain subclasses of p-valent functions associated with a family of integral operators, ISRN Math. Anal., Article ID. 384170, (2013), 8 pages.

[2] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc., 135 (1969), 429-446.

[3] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike Hypergeometric functions, SIAM J. Math. Anal., 15 (1984), no. 4, 737-745.

[4] M. C. Chen, H. Irmak and H. M. Srivastava, Some multivalent functions with negative coefficients defined by using a differential operator, Panamerican Math. J., 6 (1996), no. 2, 55-64.

[5] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl., 276 (2002), 432-445.

[6] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259 springer-Verlag, NewYork, Berlin, Heidelberg and Tokyo, 1983.

[7] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, On a Briot-Bouquet differential subordination, in General Mathematics 3, International Series of Numerical Mathematics, Vol. 64, Birkhäuser Verlag, Basel, 1983, 339-348.

[8] R. M. El-Ashwah and M. K. Aouf, Majorization properties for subclasses of analytic p-valent functions defined by convolution, Kyungpook Math. J., 53 (2013), 615-624.

[9] R. M. El-Ashwah and M. E. Drbuk, Subordination properties of p-valent functions defined by integral operators, Biritish J. Math. Comp. Sci., 21 (2014), no. 4, 3000-3013.

[10] I. S. Jack, Functions starlike and convex of order α, London Math. Soc., 3 (1971), no. 2, 469-474.

[11] I. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl., 176 (1993), 138-147.

[12] R. J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc., 16 (1965), 755-758.

[13] J. L. Liu and S. Owa, Properties of certain integral operator, Int. J. Math. Sci., 3 (2004), no. 1, 69-75.

[14] A. E. Livingston, On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc., 17 (1966), 352-357.

[15] T. H. MacGregor, Majorization by univalent of functions, Duke. Math. J., 34 (1967), 95-102.

[16] Z. Nehari, Conformal mappings, MacGraw-Hill Book Company, New York, Toronto and London, 1952.

[17] K. I. Noor, On new classes of integral operators, J. Natural Geometry, 16 (1999), 71-80.

[18] S. Owa, On certain classes of $p-$valent functions with negative coefficients, Simon Stevin, 59 (1985), 385-402.

[19] J. Patel and N. K. Thakare, On convex hulls and extreme points of $p-$valent starlike and convex classes with applications, Bull. Math. Soc. Math. R. S. Roumaine (N. S.), 27(75) (1983), 487-500.

[20] R. K. Raina and P. Sharma, Subordination preserving properties associated with a class of operators, Le Matematiche, 68 (2013), 217-228.

[21] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc., 49 (1975), no. 1, 109-115.

[22] H. Saitoh, S. Owa, T. Sekine, M. Nunokawa and R. Yamakawa, An application of certain integral operator, Appl. Math. Lett., 5 (1992), 21-24.