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Coefficient estimates for a new subclass of bi-close-to-convex functions associated with the Horadam polynomials

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Abstract

In the present article, we introduce a new subclass of bi-close-to-convex functions in the open unit disk U defined by means of the Horadam polynomials. Estimates upper bounds for the coefficients $|a_2|$ and $|a_3|$ for functions belonging to this subclass are derived. Also, Fekete-Szegö inequalities of functions belonging to this subclass are also discussed. Further, several new special cases of our results are pointed out.

Keywords: *Analytic function, Univalent and bi-univalent functions, bi-close-to-convex functions, Fekete-Szegö problem , Horadam polynomials, Coefficient bounds, Subordination.*

1 Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit open disk $U = \{z : z \in C, |z| < 1\}$. Also, let $\mathcal{S}$ be the subclass of all functions in $\mathcal{A}$ which are univalent and normalized by the conditions

$$f(0) = 0 = f^{'}(0) - 1$$

in U.

If f_1 and f_2 are analytic in U, then we call that f_1 is subordinate to f_2, denoted by $f_1 \prec f_2$, if there exists Schwarz function

$$\varpi(z) = \sum_{n=1}^{\infty} \mathbf{c}_n z^n \quad (\varpi(0) = 0, |\varpi(z)| < 1), \tag{2}$$

analytic in U such that

$$f_1(z) = f_2(\varpi(z)) \qquad (z \in U). \tag{3}$$

It is known that $|\mathbf{c}_n| \leq 1$ (see [20]) for $\varpi(z)$.

Beside, it is known that

$$f(z) \prec g(z) \ (z \in U) \ \Leftrightarrow \ f(0) = g(0) \ \ and \ \ f(U) \subset g(U).$$

It is well known that every univalent function f has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \ \ (z \in U),$$

and

$$f^{-1}(f(w)) = w \ \ (|w| < r_0(f); r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(w) = w + a_2 w^2 + (2a_2^2 - 3a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{4}$$

If f and f^{-1} are univalent in U, then a function $f \in \mathcal{A}$ is called *bi-univalent.*

In 1967, the class Σ of bi-univalent functions was first discussed by Lewin [28] and that the bound $|a_2| < 1.51$ was obtained for $f(z)$. Brannan and Taha [19] also considered certain subclasses of bi-univalent functions, and derived estimates for the initial coefficients. In 2010, the work of Srivastava et al. [32] have actually revived the investigation of holomorphic and bi-univalent functions in recent year. Also, many researchers investigated and studied various subclasses of analytic and bi-univalent functions, one can refer to the works of [1], [2], [3], [4], [5], [6], [7], [8], [9], [12], [33], [34], [35], and [36].

By $S^*(\phi)$ and $C(\phi)$ we denote the following classes of functions

$$S^*(\phi) = \left\{ f: \ f \in \mathcal{A}, \ \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}, \ z \in U,$$

and

$$C(\phi) = \left\{ f: \ f \in \mathcal{A}, \ 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\}, \ z \in U,$$

where $S^*(\phi)$ and $C(\phi)$ are the class of starlike and convex functions, respectively, were investigated by Ma and Minda [30]. So, if $f(z) \in C(\phi)$, then $zf'(z) \in S^*(\phi)$.

A function $f \in \mathcal{A}$ belongs to K, the class of close-to-convex domain, if, and only if, there exist $0 \leq \delta \leq \pi$ and $g \in S^*$ such that for $z \in U$,

$$\Re\left(e^{i\delta}\frac{zf'(z)}{g(z)}\right) > 0 \quad z \in U.$$

The class K was defined and studied by Kaplan [25]. Note that, $C \subset S^* \subset K \subset S$.

Next, by follows the definition of Kaplan, Ozaki [31] considered functions in $\mathcal{A}$ satisfying the following condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \quad z \in U,$$

whose members are known to be close-to-convex, and therefore univalent. Recently, Kargar and Ebadian [26] considered the generalization of Ozaki's condition as the following.

Definition 1.1 *Let $f \in \mathcal{A}$ be locally univalent for $z \in U$ and let $-\frac{1}{2} \leq \lambda \leq 1$. Then $f \in F(\lambda)$ if and only if*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2} - \lambda \quad z \in U,$$

where $F(\lambda)$ is the class of locally univalent normalized analytic functions f in the unit disk U. It is clear that, for $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$, we have $F(\lambda) \subset K \subset S^*$.

By extending the class $F(\lambda)$, Allu et al. [11] defined new class $F(\lambda, \alpha)$ for strongly Ozaki-close-to-convex as follows.

Definition 1.2 *Let $f \in \mathcal{A}$. Then f is called strongly Ozaki-close-to-convex if and only if*

$$\left|\arg\left(\frac{2\lambda - 1}{2\lambda + 1} + \frac{2}{2\lambda + 1}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\right| < \frac{\alpha\pi}{2} \quad \left(z \in U,\ \frac{1}{2} \leq \lambda \leq 1,\ 0 < \alpha \leq 1\right).$$

The Horadam polynomials $h_n(x)$ are given by the following recurrence relation (see [23])

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad (n \in N > 2), \tag{5}$$

with $h_1(x) = a,\ h_2 = bx$, and $h_3 = pbx^2 + aq$ where a, b, p, q are some real constants.
The characteristic equation of recurrence relation (5) is

$$t^2 - pxt - q = 0. \tag{6}$$

This equation has two real roots;

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2},$$

and

$$\beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

Some particular cases regarding of Horadam polynomials sequence can be found in [12]. For more information related to Horadam polynomials see ([21], [22], [27], [29]).

Remark 1.3 *[22] The generating function of the Horadam polynomials $\Omega(x, z)$ is given by*

$$\Omega(x, z) = \frac{a + (b - ap)xt}{1 - pxt - qt^2} = \sum_{n=1}^{\infty} h_n(x) z^{n-1}. \tag{7}$$

In this paper, we introduce a new subclass of bi-close-to-convex functions by using the Horadam polynomials $h_n(x)$ and the generating function $\Omega(x, z)$. Moreover, we find the initial coefficients and the Fekete-Szegö inequality for functions belonging to the class $F(\lambda, \alpha, x)$. Several special cases were obtained to our results.

2 Coefficient Bounds for the Function Class $F(\lambda, \alpha, x)$

Definition 2.1 *A function $f \in \Sigma$ given by (1) is said to be in the class $F(\lambda, \alpha, x)$, if the following conditions are satisfied:*

$$\frac{2\lambda - 1}{2\lambda + 1} + \frac{2}{2\lambda + 1}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \Omega(x, z) + 1 - \alpha \tag{8}$$

and

$$\frac{2\lambda - 1}{2\lambda + 1} + \frac{2}{2\lambda + 1}\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \Omega(x, w) + 1 - \alpha \tag{9}$$

where the real constants a, b and q are as in (5) and $g(w) = f^{-1}(z)$ is given by (4).

We first state and prove the following result.

Theorem 2.2 *Let the function $f \in \Sigma$ given by (1) be in the class $F(\lambda, \alpha, x)$. Then*

$$|a_2| \leq \frac{(2\lambda+1)|bx|\sqrt{bx}}{\sqrt{|2[[(2\lambda+1)b-4p]bx^2-4aq]|}} \tag{10}$$

$$|a_3| \leq \frac{(2\lambda+1)|bx|}{12} + \frac{[(2\lambda+1)bx]^2}{16}, \tag{11}$$

and for some $\eta \in R$,

$$|a_3 - \eta a_2^2| = \begin{cases} \frac{(2\lambda+1)|bx|}{12} & , \ |\eta - 1| \leq \frac{1}{24} \\ \frac{(2\lambda+1)^2|bx|^3|1-\eta|}{|4(2\lambda+1)][h_2(x)]^2-16h_3(x)|} & , \ |\eta - 1| \geq \frac{1}{24}. \end{cases} \tag{12}$$

Proof. Let $f \in \Sigma$ be given by the Taylor-Maclaurin expansion (1). Then, for some analytic functions Ψ and Φ such that $\Psi(0) = \Phi(0) = 0$, $|\psi(z)| < 1$ and $|\Phi(w)| < 1$, $z, w \in U$ and using Definition 2.1, we can write

$$\frac{2\lambda-1}{2\lambda+1} + \frac{2}{2\lambda+1}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \omega(x, \Phi(z)) + 1 - \alpha$$

and

$$\frac{2\lambda-1}{2\lambda+1} + \frac{2}{2\lambda+1}\left(1 + \frac{wg''(w)}{g'(w)}\right) = \omega(x, \psi(w)) + 1 - \alpha$$

or, equivalently,

$$\frac{2\lambda-1}{2\lambda+1} + \frac{2}{2\lambda+1}\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \cdots \tag{13}$$

and

$$\frac{2\lambda-1}{2\lambda+1} + \frac{2}{2\lambda+1}\left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^3 + \cdots . \tag{14}$$

From (13) and (14), we obtain

$$\frac{2\lambda-1}{2\lambda+1} + \frac{2}{2\lambda+1}\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + h_2(x)p_1 z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \cdots \tag{15}$$

and

$$\frac{2\lambda-1}{2\lambda+1} + \frac{2}{2\lambda+1}\left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 + h_2(x)p_1 w + [h_2(x)q_2 + h_3(x)q_1^2]w^2 + \cdots . \tag{16}$$

Notice that if

$$|\Phi(z)| = |p_1 z + p_2 z^2 + p_3 z^3 + \cdots| < 1 \quad (z \in U)$$

and

$$|\psi(w)| = |q_1 w + q_2 w^2 + q_3 w^3 + \cdots| < 1 \quad (w \in U),$$

then

$$|p_i| \leq 1 \quad \textit{and} \quad |q_i| \leq 1 \quad (i \in N).$$

Thus, upon comparing the corresponding coefficients in (15) and (16), we have

$$\frac{4}{2\lambda+1} a_2 = h_2(x) p_1, \tag{17}$$

$$\frac{12}{2\lambda+1} a_3 - \frac{8}{2\lambda+1} a_2^2 = h_2(x) p_2 + h_3(x) p_1^2, \tag{18}$$

$$-\frac{4}{2\lambda+1} a_2 = h_2(x) q_1 \tag{19}$$

and

$$\frac{16}{2\lambda+1} a_2^2 - \frac{12}{2\lambda+1} a_3 = h_2(x) q_2 + h_3(x) q_1^2. \tag{20}$$

From (17) and (19), we find that

$$p_1 = -q_1 \tag{21}$$

and

$$\frac{32}{(2\lambda+1)^2} a_2^2 = h_2^2(x)(p_1^2 + q_1^2). \tag{22}$$

Also, by using (20) and (18), we obtain

$$\frac{8}{2\lambda+1} a_2^2 = h_2(x)(p_2 + q_2) + h_3(x)(p_1^2 + q_1^2). \tag{23}$$

By using (22) in (23), we get

$$\left[\frac{8}{2\lambda+1} - \frac{32 h_3(x)}{(2\lambda+1)^2 [h_2(x)]^2} \right] a_2^2 = h_2(x)(p_2 + q_2). \tag{24}$$

From (5), and (23), we have the desired inequality (10).
Next, in order to find the bound on $|a_3|$, by subtracting (20) from (18) and using (21) and (22), we get

$$a_3 = \frac{h_2(x)(p_2 - q_2)(2\lambda+1)}{24} + \frac{h_2(x)(p_1^2 + q_1^2)(2\lambda+1)^2}{32}. \tag{25}$$

Hence using (21) and applying (5), we get desired inequality (12).
Now, by using (23) and (25) for some $\eta \in R$, we get

$$a_3 - \eta a_2^2 = \frac{(2\lambda+1)^2[h_2(x)]^3(1-\eta)(p_2+q_2)}{8(2\lambda+1)][h_2(x)]^2 - 32h_3(x)} + \frac{(2\lambda+1)h_2(x)(p_2-q_2)}{24}$$

$$= (2\lambda+1)h_2(x)\left[\left(\Theta(\eta,x)+\frac{1}{24}\right)p_2 + \left(\Theta(\eta,x)-\frac{1}{24}\right)q_2\right],$$

where

$$\Theta(\eta,x) = \frac{(2\lambda+1)[h_2(x)]^2(1-\eta)}{8(2\lambda+1)][h_2(x)]^2 - 32h_3(x)}.$$

So, we conclude that

$$|a_3 - \eta a_2^2| = \begin{cases} \frac{(2\lambda+1)h_2(x)}{12} & , \ |\Theta(\eta,x)| \le \frac{1}{24} \\ 2(2\lambda+1)|h_2(x)||\Theta(\eta,x)| & , \ |\Theta(\eta,x)| \ge \frac{1}{24}. \end{cases}$$

This proves Theorem 2.2. For $\lambda = \frac{1}{2}$ the class $F(\lambda,\alpha,x)$ reduced to the class $F(\frac{1}{2},\alpha,x)$ as follows.

Corollary 2.3 *Let the function $f \in \Sigma$ given by (1) be in the class $F(\frac{1}{2},\alpha,x)$. Then*

$$|a_2| \le \frac{|bx|\sqrt{bx}}{\sqrt{|2(b-2p)bx^2 - 4aq|}}, \tag{26}$$

$$|a_3| \le \frac{|bx|}{6} + \frac{[bx]^2}{4}, \tag{27}$$

and for some $\eta \in R$,

$$|a_3 - \eta a_2^2| = \begin{cases} \frac{|bx|}{6} & , \ |\eta - 1| \le \frac{1}{24} \\ \frac{|bx|^3|1-\eta|}{2([h_2(x)]^2 - 2h_3(x))|} & , \ |\eta - 1| \ge \frac{1}{24}. \end{cases} \tag{28}$$

For $\lambda = 1$ the class $F(\lambda,\alpha,x)$ reduced to the class $F(1,\alpha,x)$ as follows.

Corollary 2.4 *Let the function $f \in \Sigma$ given by (1) be in the class $F(1,\alpha,x)$. Then*

$$|a_2| \le \frac{3|bx|\sqrt{bx}}{\sqrt{|2[(3b-4p)bx^2 - 4aq]|}}, \tag{29}$$

$$|a_3| \le \frac{|bx|}{4} + \frac{9[bx]^2}{16}, \tag{30}$$

and for some $\eta \in R$,

$$|a_3 - \eta a_2^2| = \begin{cases} \frac{|bx|}{4} & , \ |\eta - 1| \le \frac{1}{24} \\ \frac{9|bx|^3|1-\eta|}{4(3[h_2(x)]^2 - 4h_3(x))|} & , \ |\eta - 1| \ge \frac{1}{24}. \end{cases} \tag{31}$$

3 Conclusion and open problems

This research paper has introduced a new subclass of bi-close-to-convex functions associated with the Horadam Polynomials. For this subclass, coefficient bounds and Fekete-Szegö inequalities have been investigated. More investigation can be made on other types of polynomials, see [37],[16],[24],[13],[14],[10],[15],[17], and [18].

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