Int. J. Open Problems Complex Analysis, Vol. 14, No. 1, June 2022
ISSN 2074-2827; Copyright ©ICSRS Publication, 2022
www.i-csrs.org

Local growth and oscillation of solutions of a class of linear differential equations in a punctured disc

Said Mazouz and Saada Hamouda

Laboratory of Pure and Applied Mathematics,
Department of Mathematics,
University Abdelhamid Ibn Badis of Mostaganem, Algeria
e-mail: said.mazouz.etu@univ-mosta.dz
e-mail: saada.hamouda@univ-mosta.dz

Received 20 March 2022; Accepted 5 June 2022

Abstract

In this paper, we investigate the local growth and oscillation, near the singular point $z = 0$, ***of solutions to the differential equation***

$$f''+\left(A(z)\exp\left\{\frac{a}{z^n}\right\}+A_0(z)\right)f'+\left(B(z)\exp\left\{\frac{b}{z^n}\right\}+B_0(z)\right)f=H(z),$$

where $A(z), A_0(z), B(z), B_0(z), H(z)$ ***are analytic functions in***

$$D(0,R)=\{z\in\mathbb{C}:0<|z|<R\}$$

and a, b ***are non-zero complex constants.***

Keywords: *Growth and oscillation of solutions, linear differential equations, Nevanlinna theory, singular point.*
2010 Mathematical Subject Classification: 34M10, 30D35.

1 Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental results of the Nevanlinna value distribution theory of meromorphic function f in the complex plane $\mathbb{C}$, in particular the definitions and the standard notations $N(r,f), m(r,f), T(r,f), \sigma(f)$, etc., (see [14, 26, 19]). The importance of this theory has inspired many authors to find modifications and generalizations to different domains. Extensions of some results of Nevanlinna Theory to annuli have been made by [3, 16, 17, 20]. Linear ordinary differential equations with singular points represents a rich and classical field for which the symbolic computation of the solutions is a challenge for the capabilities of Mathematics. Only the simplest differential equations admit solutions given by explicit formula; however, some properties of solutions of a given differential equation may be determined without finding their exact form. The idea to study the growth of solutions of the linear differential equations near a finite singular point by using the Nevanlinna theory has began by the paper [10]; then after some publications have followed, see [12, 6, 7, 8]. The principal tools used in these investigations is the estimates of the logarithmic derivative $\left|\frac{f^{(k)}(z)}{f(z)}\right|$ for a meromorphic function f in $\overline{\mathbb{C}}\setminus\{z_0\}$, $\left(\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}\right)$. A question was asked in [10, 12] about if we can get similar estimates near z_0 of $\left|\frac{f^{(k)}(z)}{f(z)}\right|$ where f is a meromorphic function in a region of the form $D_{z_0}(0,R)=\{z\in\mathbb{C}:0<|z-z_0|<R\}$. This question is answered in [13] with some applications.

First we recall the appropriate definitions for this paper [10, 20]. Suppose that $f(z)$ is meromorphic in $D(0,+\infty]=\overline{\mathbb{C}}\setminus\{0\}$. Define the counting function near 0 by

$$N_0(r,f)=\int_r^{\infty}\frac{n(t,f)-n(\infty,f)}{t}dt-n(\infty,f)\log r, \tag{1.1}$$

where $n(t,f)$ counts the number of poles of $f(z)$ in the region $\{z\in\mathbb{C}:t\le|z|\}\cup\{\infty\}$ each pole according to its multiplicity; and the proximity function by

$$m_0(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\ln^+\left|f\left(re^{i\varphi}\right)\right|d\varphi. \tag{1.2}$$

The characteristic function of f is defined by

$$T_0(r,f)=m_0(r,f)+N_0(r,f). \tag{1.3}$$

For a meromorphic function $f(z)$ in $D(0,R)=\{z\in\mathbb{C}:0<|z|<R\}$, we define the counting function near 0 by

$$N_0(r,R',f)=\int_r^{R'}\frac{n(t,f)}{t}dt, \tag{1.4}$$

where $n(t,f)$ counts the number of poles of $f(z)$ in the region $\{z \in \mathbb{C} : t \leq |z| \leq R'\}$ $(0 < R' < R)$, each pole according to its multiplicity; and the proximity function near the singular point 0 by (1.2). The characteristic function of f is defined in the usual manner by

$$T_0(r, R', f) = m_0(r, f) + N_0(r, R', f). \tag{1.5}$$

In addition, the order of growth of a meromorphic function $f(z)$ near 0 is defined by

$$\sigma_T(f, 0) = \limsup_{r \to 0} \frac{\log^+ T_0(r, R', f)}{-\log r}. \tag{1.6}$$

For an analytic function $f(z)$ in $D(0, R)$, we have also the definition

$$\sigma_M(f, 0) = \limsup_{r \to 0} \frac{\log^+ \log^+ M_0(r, f)}{-\log r}, \tag{1.7}$$

where $M_0(r, f) = \max\{|f(z)| : |z| = r\}$.

By the usual manner, we define the hyper order near 0 as follows:

$$\sigma_{2,T}(f, 0) = \limsup_{r \to 0} \frac{\log^+ \log^+ T_0(r, f)}{-\log r}, \tag{1.8}$$

$$\sigma_{2,M}(f, 0) = \limsup_{r \to 0} \frac{\log^+ \log^+ \log^+ M_0(r, f)}{-\log r}. \tag{1.9}$$

We will use $\lambda(f, 0)$, (resp. $\overline{\lambda}(f, 0)$) to denote the exponent of convergence of the zero-sequence (resp. the exponent of convergence of the distinct zero-sequence) of the meromorphic function $f(z)$ in $D(0, R)$ and $\lambda_2(f, 0)$, (resp. $\overline{\lambda}_2(f, 0)$) to denote the hyper-exponent of convergence of the zero-sequence (resp. the hyper-exponent of convergence of the distinct zero-sequence) of $f(z)$, which are defined as follows:

$$\lambda(f, 0) = \limsup_{r \to 0} \frac{\log^+ N_0\left(r, R', \frac{1}{f}\right)}{-\log r},$$

$$\overline{\lambda}(f, 0) = \limsup_{r \to 0} \frac{\log^+ \overline{N}_0\left(r, R', \frac{1}{f}\right)}{-\log r},$$

$$\lambda_2(f, 0) = \limsup_{r \to 0} \frac{\log^+ \log^+ N_0\left(r, R', \frac{1}{f}\right)}{-\log r},$$

$$\overline{\lambda}_2(f, 0) = \limsup_{r \to 0} \frac{\log^+ \log^+ \overline{N}_0\left(r, R', \frac{1}{f}\right)}{-\log r},$$

where $\overline{N}_0\left(r, R', \frac{1}{f}\right)$ is defined as $N_0\left(r, R', \frac{1}{f}\right)$ in (1.4) but instead of $n(t, f)$ we use $\overline{n}(t, f)$ which counts the number of distinct poles without multiplicity.

Remark 1.1 *The choice of R' in (1.1) does not have any influence in the values* $\sigma_T(f,0), \sigma_{2,T}(f,0), \lambda(f,0), \lambda_2(f,0), \overline{\lambda}(f,0), \overline{\lambda}_2(f,0)$. *In fact, if we take two values of R', namely $0 < R'_1 < R'_2 < R$, then we have*

$$\int_{R'_1}^{R'_2} \frac{n(t,f)}{t} dt = p \log \frac{R'_2}{R'_1},$$

where p designates the number of poles of $f(z)$ in the region $\{z \in \mathbb{C} : R'_1 \leq |z| \leq R'_2\}$ which is bounded. Thus, $N_0(r, R'_1, f) = N_0(r, R'_2, f) + C$; and then $T_0(r, R'_1, f) = T_0(r, R'_2, f) + C$ where C is a real constant. So, we can write briefly $T_0(r,f)$ instead of $T_0(r, R', f)$.

Remark 1.2 *It is shown in [10] that $\sigma_M(f,0) = \sigma_T(f,0)$, $\sigma_{2,T}(f,0) = \sigma_{2,M}(f,0)$. So, we can use the notations $\sigma(f,0)$, $\sigma_2(f,0)$ without any ambiguity.*

Example 1.3 *Consider the function $f(z) = \exp\left\{\frac{1}{z^2}\right\}$. We have*

$$T_0(r,f) = m_0(r,f) = \frac{1}{\pi r^2},$$

then $\sigma_T(f,0) = 2$. Also we have

$$M_0(r,f) = \exp\left\{\frac{1}{r^2}\right\},$$

then $\sigma_M(f,0) = 2$.

Example 1.4 *For the function $f(z) = \exp\exp\left\{\frac{1}{z^3}\right\}$, we have*

$$M_0(r,f) = \exp\exp\left\{\frac{1}{r^3}\right\},$$

and then $\sigma(f,0) = +\infty$, $\sigma_2(f,0) = 3$.

The linear differential equation

$$f'' + A(z) e^{az} f' + B(z) e^{bz} f = H(z),$$

where $A(z), B(z)$ and $H(z)$ are entire functions, is investigated by many authors; see [1, 2, 4, 5, 11, 18, 15, 23]. In [10], Fettouch and Hamouda studied the local growth near the singular point z_0 of solutions of the linear differential equation

$$f'' + A(z)\exp\left\{\frac{a}{(z_0 - z)^n}\right\} f' + B(z)\exp\left\{\frac{b}{(z_0 - z)^n}\right\} f = 0,$$

where $A(z), B(z) \not\equiv 0$ are analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. The case $c > 1$ has been completed recently by Cherief and Hamouda in [6]. The question which arises here is how about the case when the coefficients are analytic only in a punctured disc $D(0, R)$? In this paper we will deal with this question.

2 Main results

In this work, we will investigate the order of growth and the exponent of convergence of the zero-sequence of solutions of certain class of second order linear differential equations where the coefficients are analytic in $D(0, R)$. In fact, we will prove the following results.

Theorem 2.1 *Let $A(z) \not\equiv 0, B(z) \not\equiv 0, F(z)$ be analytic functions in $D(0, R)$ such that $\max\{\sigma(A, 0), \sigma(B, 0), \sigma(F, 0)\} < n, n \in \mathbb{N} \setminus \{0\}$; let a, b be complex constants such that $ab \neq 0$ and $a \neq b$. Then, every solution $f(z) \not\equiv 0$ of the differential equation*

$$f'' + A(z) \exp\left\{\frac{a}{z^n}\right\} f' + B(z) \exp\left\{\frac{b}{z^n}\right\} f = F(z), \tag{2.1}$$

satisfies $\sigma(f, 0) = \infty$. Further, if $F(z) \not\equiv 0$, we have

$$\bar{\lambda}(f, 0) = \lambda(f, 0) = \sigma(f, 0) = +\infty, \ \bar{\lambda}_2(f, 0) = \lambda_2(f, 0) = \sigma_2(f, 0) \leq n.$$

Theorem 2.2 *Let $A(z) \not\equiv 0, A_0(z), B(z) \not\equiv 0, B_0(z), F(z)$ be analytic functions in $D(0, R)$ such that*

$$\max\{\sigma(A_0, 0), \sigma(B_0, 0), \sigma(A, 0), \sigma(B, 0), \sigma(F, 0)\} < n, \ n \in \mathbb{N} \setminus \{0\};$$

let a, b be complex constants such that $ab \neq 0$ and $a = cb, \ c < 0$. Then, every solution $f(z) \not\equiv 0$ of the differential equation

$$f'' + \left(A(z) \exp\left\{\frac{a}{z^n}\right\} + A_0(z)\right) f' + \left(B(z) \exp\left\{\frac{b}{z^n}\right\} + B_0(z)\right) f = F(z), \tag{2.2}$$

satisfies $\sigma(f, 0) = \infty$. Further, if $F(z) \not\equiv 0$, we have

$$\bar{\lambda}(f, 0) = \lambda(f, 0) = \sigma(f, 0) = +\infty, \ \bar{\lambda}_2(f, 0) = \lambda_2(f, 0) = \sigma_2(f, 0) \leq n.$$

Theorem 2.3 *Let $A(z) \not\equiv 0, B(z) \not\equiv 0, F(z) \not\equiv 0$ be analytic functions in $D(0, R)$ such that $\max\{\rho(A, 0), \rho(B, 0), \rho(F, 0)\} < n, \ n \in \mathbb{N} \setminus \{0\}$ and*

$P(z) \not\equiv 0, Q(z) \not\equiv 0$ are polynomials. Let a, b be complex numbers such that $ab \neq 0, a \neq b$. Then, every solution f of the differential equations

$$f'' + P(z)\exp\{\frac{a}{z^n}\}f' + B(z)\exp\{\frac{b}{z^n}\}f = F(z)\exp\{\frac{a}{z^n}\}, \tag{2.3}$$

$$f'' + A(z)\exp\{\frac{a}{z^n}\}f' + Q(z)\exp\{\frac{b}{z^n}\}f = F(z)\exp\{\frac{b}{z^n}\} \tag{2.4}$$

satisfies

$$\bar{\lambda}(f,0) = \lambda(f,0) = \sigma(f,0) = +\infty,\ \bar{\lambda}_2(f,0) = \lambda_2(f,0) = \sigma_2(f,0) \leq n.$$

If some conditions of the previous theorems are not satisfied, the equations (2.1), (2.2), (2.3) and (2.4) may admit a solutions of finite order as shown in the following examples.

Example 2.4 *The function $g(z) = \exp\left\{\frac{1}{z}\right\}$ of order $\sigma(g,0) = 1$ satisfies the differential equations*

$$f'' - \exp\left\{\frac{-1}{z}\right\}f' - \frac{1}{z^2}\exp\left\{\frac{-1}{z}\right\}f = \left(\frac{2}{z^3} + \frac{1}{z^4}\right)\exp\left\{\frac{-1}{z}\right\},$$

$$f'' - \exp\left\{\frac{-1}{z}\right\}f' - \left(\frac{2}{z^3} + \frac{1}{z^4}\right)f = \frac{1}{z^2};$$

$$f'' + \exp\left\{\frac{-1}{z}\right\}f' + \left(\frac{1}{z^2}\exp\left\{\frac{-1}{z}\right\} - \frac{2}{z^3} - \frac{1}{z^4}\right)f = 0.$$

Example 2.5 *The function $h(z) = \frac{1}{z}$ of order $\sigma(h,0) = 0$ satisfies the differential equation*

$$f'' - \exp\{\frac{a}{z^n}\}f' - \frac{1}{z^2}\exp\{\frac{b}{z^n}\}f = \frac{1}{z^2}\exp\{\frac{a}{z^n}\} - \frac{1}{z^3}\exp\{\frac{b}{z^n}\} + \frac{2}{z^3},$$

where a, b $(ab \neq 0)$ are arbitrary complex numbers.

3 Preliminary lemmas

To prove these results we need the following lemmas.

Lemma 3.1 *[13] Let f be a non-constant meromorphic function in $D(0,R)$ with a singular point at the origin of finite order $\sigma(f,0) = \sigma < \infty$; let $\varepsilon > 0$ be a given constant and k be a positive integer. Then the following two statements hold.*

i) There exists a set $F \subset (0, R')$ that has finite logarithmic measure such that for all $r = |z|$ satisfying $r \in (0, R') \backslash F$, we have

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq \frac{1}{r^{k(\sigma+1)+\varepsilon}}. \tag{3.1}$$

ii) There exists a set $E \subset [0, 2\pi)$ that has a linear measure zero such that for all $\theta \in [0, 2\pi) \backslash E$ there exists a constant $r_0 = r_0(\theta) > 0$ such that for all z satisfying $\arg(z) \in [0, 2\pi) \backslash E$ and $r = |z| < r_0$ the inequality (3.1) holds.

Lemma 3.2 *[13] Let $A(z)$ be a non-constant analytic function in $D(0, R)$ with $\sigma(A, 0) < n$. Set $g(z) = A(z) \exp\left\{ \frac{a}{z^n} \right\}$, $(n \geq 1$ is an integer$)$, $a = \alpha + i\beta \neq 0$, $z = re^{i\varphi}$, $\delta_a(\varphi) = \alpha \cos(n\varphi) + \beta \sin(n\varphi)$, and $E = \{\varphi \in [0, 2\pi) : \delta_a(\varphi) = 0\}$, (obviously, E is of linear measure zero). Then for any given $\varepsilon > 0$ and for any $\varphi \in [0, 2\pi) \backslash E$, there exists $r_0 > 0$ such that for $0 < r < r_0$, the two following statements hold.*
(i) If $\delta_a(\varphi) > 0$, then

$$\exp\left\{ (1-\varepsilon) \delta_a(\varphi) \frac{1}{r^n} \right\} \leq |g(z)| \leq \exp\left\{ (1+\varepsilon) \delta_a(\varphi) \frac{1}{r^n} \right\}.$$

(ii) If $\delta_a(\varphi) < 0$, then

$$\exp\left\{ (1+\varepsilon) \delta_a(\varphi) \frac{1}{r^n} \right\} \leq |g(z)| \leq \exp\left\{ (1-\varepsilon) \delta_a(\varphi) \frac{1}{r^n} \right\}.$$

Lemma 3.3 *Let $f(z)$ be analytic function in $D(0, R)$ and suppose that*

$$G(z) := |z^\rho| \log^+ \left| f^{(k)}(z) \right|$$

is unbounded as $z \to 0$ on some ray $\arg z = \theta$, where $\rho > 0$. Then there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m \geq 1)$, $r_m \to 0$, such that $G(z_m) \to +\infty$ and

$$\left| \frac{f^{(j)}(z_m)}{f^{(k)}(z_m)} \right| \leq M, \ (M > 0) (j = 0, 1, ..., k-1),$$

as $m \to +\infty$.

Proof. Let $M(r, \theta, G)$ denotes the maximum modulus of $G(z)$ on the line segment $\left[r_1 e^{i\theta}, re^{i\theta} \right]$. Clearly, we may construct a sequence of points $z_m = r_m e^{i\theta}$ $(m \geq 1)$, $r_m \to 0$, such that $M(r, \theta, G) = G(z_m) \to +\infty$. Since $G(z_m) \to +\infty$ as $r_m \to 0$, we see immediately that $\left| f^{(k)}(z_m) \right| \to +\infty$. For each m, by $(k-j)$-fold iteration integration along the line segment $[z_1, z_m]$ we have

$$f^{(j)}(z_m) = f^{(j)}(z_1) + f^{(j+1)}(z_1)(z_m - z_1) + ..$$

$$.. + \frac{1}{(k-j-1)!} f^{(k-1)}(z_1)(z_m - z_1)^{k-j-1} + \int_{z_1}^{z_m} ... \int_{z_1}^{y} f^{(k)}(x)\, dxdy...dt;$$

and by an elementary triangle inequality estimate we obtain

$$\left|f^{(j)}(z_m)\right| \leq \left|f^{(j)}(z_1)\right| + \left|f^{(j+1)}(z_1)\right| |(z_m - z_1)| + ...$$
$$+ \frac{1}{(k-j-1)!} \left|f^{(k-1)}(z_1)\right| |(z_m - z_1)|^{k-j-1} +$$
$$\frac{1}{(k-j)!} \left|f^{(k)}(z_m)\right| |(z_m - z_1)|^{k-j}. \quad (3.2)$$

From (3.2) and taking account that when $m \to +\infty$, $f^{(k)}(z_m) \to +\infty$, $z_m \to 0$, we obtain

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \leq M, \ (M > 0).$$

Lemma 3.4 *Let $f(z)$ be a non constant meromorphic function in $D(0,R)$. Then $\sigma\left(f^{(j)}, 0\right) = \sigma(f, 0)$, $(j = 1, 2, ...)$*

Proof. We have just to show that $\sigma(f', 0) = \sigma(f, 0)$. By Valiron's decomposition lemma, we have $f(z) = z^m \phi(z) \mu(z)$, where
a) The poles and zeros of f in $D(0, R')$ are precisely the poles and zeros of $\phi(z)$. The poles and zeros of f in $D(R', R)$ are precisely the poles and zeros of $\mu(z)$.
b) $\phi(z)$ is meromorphic in $D(0, \infty]$ and analytic and nonzero in $D[R', \infty]$.
c) $\mu(z)$ is meromorphic in $D(R) = \{z \in \mathbb{C} : |z| < R\}$ and analytic and nonzero in $D(R')$.
d) $m \in \mathbb{Z}$.

Set $\tilde{\phi}(z) = z^m \phi(z)$. Since $\mu(z)$ is analytic at zero, it is immediate to see that $T_0(r, f) = T_0\left(r, \tilde{\phi}\right) + O(1)$; and then $\sigma(f, 0) = \sigma\left(\tilde{\phi}, 0\right)$. Since $\tilde{\phi}(z)$ is meromorphic in $D(0, \infty]$, the function $g(w) = \tilde{\phi}\left(\frac{1}{w}\right)$ is meromorphic in $\mathbb{C}$ and $\sigma(g) = \sigma\left(\tilde{\phi}, 0\right)$. It is well known that for a meromorphic function in $\mathbb{C}$ we have $\sigma(g') = \sigma(g)$, (see [25, 21]). We have $\tilde{\phi}'(z) = -w^2 g'(w)$. Obviously, we have $\sigma(-w^2 g'(w)) = \sigma(g')$, and then $\sigma(g') = \sigma\left(\tilde{\phi}', 0\right)$. So, we get $\sigma\left(\tilde{\phi}', 0\right) = \sigma\left(\tilde{\phi}, 0\right)$. In the other hand, we have

$$f'(z) = \tilde{\phi}'(z)\mu(z) + \tilde{\phi}(z)\mu'(z). \quad (3.3)$$

Since $\mu(z)$ is analytic at zero, we have $\sigma(\mu, 0) = 0$. By (3.3) and since $\sigma\left(\tilde{\phi}', 0\right) = \sigma\left(\tilde{\phi}, 0\right)$, we get

$$\sigma(f', 0) \leq \sigma\left(\tilde{\phi}', 0\right).$$

For the inverse inequality, we have

$$\tilde{\phi}'(z) = \frac{f'(z)\,\mu(z) - f(z)\,\mu'(z)}{\mu^2(z)};$$

and then

$$\sigma\left(\tilde{\phi}', 0\right) \le \max\left\{\sigma(f', 0), \sigma(f, 0)\right\};$$

and by taking account that $\sigma(f,0) = \sigma\left(\tilde{\phi}, 0\right) = \sigma\left(\tilde{\phi}', 0\right)$, we obtain

$$\sigma\left(\tilde{\phi}', 0\right) \le \sigma(f', 0).$$

Thus, we conclude that

$$\sigma(f', 0) = \sigma(f, 0).$$

Lemma 3.5 *Let f be an analytic function in $D(0,R)$ with finite order $\sigma(f,0) = \sigma$. Suppose that there exists a set $E \subset [0, 2\pi)$ that has a linear measure zero such that*

$$\log^+\left|f\left(re^{i\theta}\right)\right| \le \frac{M}{r^\alpha}$$

for any $\theta \in [0, 2\pi)\setminus E$ where M is a positive constant depending on θ, while α is a positive constant independent of θ. Then $\sigma(f,0) \le \alpha$.

Proof. By Valiron's decomposition lemma [22, 20], we have $f(z) = z^m\phi(z)\,\mu(z)$ with the properties a)-d) cited in the proof of Lemma 3.4. Set $\tilde{\phi}(z) = z^m\phi(z)$. As in the proof of Lemma 3.4, we have $\sigma(f,0) = \sigma\left(\tilde{\phi}, 0\right)$. If $\sigma(f,0) = 0$ there is nothing to prove; so we may assume that $\sigma(f,0) = \sigma > 0$; and then $\left|f\left(re^{i\theta}\right)\right| > 1$ for r small enough. We have

$$\log\left|f\left(re^{i\theta}\right)\right| = \log\left|\tilde{\phi}\left(re^{i\theta}\right)\right| + \log\left|\mu\left(re^{i\theta}\right)\right| \le \frac{M}{r^\alpha}. \tag{3.4}$$

Since $\mu(z)$ is analytic and nonzero in $D(R')$, $\log\left|\mu\left(re^{i\theta}\right)\right|$ is bounded near zero; and then by (3.4), for any $\theta \in [0, 2\pi)\setminus E$ there exists $M' > 0$, such that

$$\log\left|\tilde{\phi}\left(re^{i\theta}\right)\right| \le \frac{M'}{r^\alpha}. \tag{3.5}$$

Since $\tilde{\phi}(z)$ is analytic in $D(0,\infty]$, by the change of variable $z = \frac{1}{w}$ the function $g(w) = \tilde{\phi}\left(\frac{1}{w}\right)$ is entire and $\sigma(g) = \sigma\left(\tilde{\phi}, 0\right) = \sigma$. From (3.5), we have

$$\log\left|g\left(Re^{i\varphi}\right)\right| \le M'R^\alpha.$$

By [24, Lemma 2.6.], we deduce that $\sigma \le \alpha$.

Lemma 3.6 *Let $A_0(z), A_1(z), ..., A_{k-1}(z), H(z)$ be analytic functions in $D(0,R)$ such that*

$$\max\{\sigma(A_0,0), ..., \sigma(A_{k-1},0), \sigma(H,0)\} = \alpha < \infty. \tag{3.6}$$

If f is a solution of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = H(z), \tag{3.7}$$

then $\sigma_2(f,0) \leq \alpha$.

Proof. By Valiron's decomposition lemma [22, 20], we have $f(z) = z^m \phi(z) \mu(z)$ with the properties a)-d) cited in the proof of Lemma 3.4. Set $\tilde{\phi}(z) = z^m \phi(z)$. As in the proof of Lemma 3.4, we have $\sigma(f,0) = \sigma\left(\tilde{\phi},0\right)$. Since $f(z)$ is analytic function in $D(0,R)$, $\tilde{\phi}(z)$ is analytic in $D(0,\infty]$. By [13, Theorem 8], there exists a set $E \subset (0,1)$ that has finite logarithmic measure, such that for all $j = 0, 1, ..., k$, we have

$$\frac{\tilde{\phi}^{(j)}(z_r)}{\tilde{\phi}(z_r)} = (1 + o(1)) \left(\frac{V_0(r)}{z_r}\right)^j, \tag{3.8}$$

as $r \to 0$, $r \notin E$, where $V_0(r)$ is the central index of $\tilde{\phi}$ near the singular point 0, z_r is a point in the circle $|z| = r$ that satisfies $\left|\tilde{\phi}(z_r)\right| = \max_{|z|=r} \left|\tilde{\phi}(z)\right|$. Since $\mu(z)$ is analytic and non zero in $D(R')$, we have

$$\left|\frac{\mu^{(j)}(z)}{\mu(z)}\right| \leq M, \ (j = 1, ..., k). \tag{3.9}$$

We have $f(z) = \tilde{\phi}(z) \mu(z)$, and then

$$\frac{f^{(j)}(z)}{f(z)} = \sum_{i=0}^{i=j} \begin{pmatrix} j \\ i \end{pmatrix} \frac{\tilde{\phi}^{(j-i)}(z)}{\tilde{\phi}(z)} \frac{\mu^{(i)}(z)}{\mu(z)}, \ j = 1, ..., k, \tag{3.10}$$

where $\begin{pmatrix} j \\ i \end{pmatrix} = \frac{j!}{i!(j-i)!}$ is the binomial coefficient. From (3.7), we have

$$\frac{f^{(k)}(z)}{f(z)} = -A_{k-1}(z) \frac{f^{(k-1)}(z)}{f(z)} - \cdots - A_1(z) \frac{f'(z)}{f(z)} - A_0(z) + \frac{H(z)}{f(z)}. \tag{3.11}$$

If $\sigma(f,0) < \infty$, then the result is trivial: $\sigma_2(f,0) = 0 \leq \alpha$. So, we may assume that $\sigma(f,0) = \infty$. Since $\sigma(H,0) < \infty$, we have

$$\left|\frac{H(z_r)}{f(z_r)}\right| = o(1), \ r \to 0. \tag{3.12}$$

Set $M_0(r) = \max_{|z|=r} \{|A_j(z)| : j = 0, 1, ..., k-1\}$. By combining (3.8), (3.9), (3.10) and (3.12) in (3.11), we get

$$(V_0(r))^k \leq C (V_0(r))^{k-1} M_0(r), \ r \to 0,$$

where $C > 0$, and then

$$V_0(r) \leq C M_0(r). \tag{3.13}$$

By (3.13), we obtain $\sigma_2(f, 0) \leq \alpha$.

By the well known logarithmic derivative lemma of meromorphic functions in $\mathbb{C}$ we can prove its new version in $D(0, R)$ as the following.

Lemma 3.7 *Let f be a non constant meromorphic function in $D(0, R)$, and let $k \in \mathbb{N}$. Then*

$$m_0\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log T_0(r, f) + \log \frac{1}{r}\right),$$

for all $r \in (0, R) \setminus E$, where $\int_E \frac{dr}{r} < \infty$.

Proof. *By Valiron's decomposition lemma [22, 20], we have $f(z) = z^m \phi(z) \mu(z)$ with the properties a)-d) cited in the proof of Lemma 3.4. Set $\tilde{\phi}(z) = z^m \phi(z)$. By property b) the function $\tilde{\phi}(z)$ is meromorphic in $D(0, \infty]$. By [9, Lemma 13], we have*

$$m_0\left(r, \frac{\tilde{\phi}^{(k)}}{\tilde{\phi}}\right) = O\left(\log T_0\left(r, \tilde{\phi}\right) + \log \frac{1}{r}\right), \tag{3.14}$$

for all $r \in (0, R) \setminus E$, where $\int_E \frac{dr}{r} < \infty$. Since $\mu(z)$ analytic at zero, it is clear that

$$T_0(r, f) = T_0\left(r, \tilde{\phi}\right) + O(1). \tag{3.15}$$

By (3.10), (3.14) and (3.15), there exists a set E of finite logarithmic measure such that for all we $r \in (0, R) \setminus E$, we have

$$m_0\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log T_0(r, f) + \log \frac{1}{r}\right).$$

Lemma 3.8 *Let $A_0(z), A_1(z), ..., A_{k-1}(z), H(z) \not\equiv 0$ be meromorphic functions in $D(0, R)$ such that*

$$\max\{\sigma(A_0, 0), ..., \sigma(A_{k-1}, 0), \sigma(H, 0)\} = \alpha < \infty.$$

If $f(z)$ is meromorphic solution in $D(0, R)$ of (3.7) with $\sigma(f, 0) = \infty$ and $\sigma_2(f, 0) = \alpha$, then f satisfies

$$\bar{\lambda}(f, 0) = \lambda(f, 0) = \rho(f, 0) = +\infty, \ \bar{\lambda}_2(f, 0) = \lambda_2(f, 0) = \rho_2(f, 0) = \alpha.$$

Proof. From (3.7), we can write

$$\frac{1}{f}=\frac{1}{H}\left(\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+...+A_0\right). \tag{3.16}$$

If f has a zero at $z_0\in D(0,R)$ of order $\alpha>k$, then H has a zero at z_0 of order $\alpha-k$. Hence,

$$n_0\left(r,\frac{1}{f}\right)\leq k\overline{n}_0\left(r,\frac{1}{f}\right)+n_0\left(r,\frac{1}{H}\right)+\sum_{j=0}^{k-1}n_0\left(r,A_j\right)$$

and then

$$N_0\left(r,\frac{1}{f}\right)\leq k\overline{N}_0\left(r,\frac{1}{f}\right)+N_0\left(r,\frac{1}{H}\right)+\sum_{j=0}^{k-1}N_0\left(r,A_j\right). \tag{3.17}$$

By (3.16), we have

$$m_0\left(r,\frac{1}{f}\right)\leq\sum_{j=1}^{k}m_0\left(r,\frac{f^{(j)}}{f}\right)+\sum_{j=0}^{k-1}m_0\left(r,A_j\right)+m_0\left(r,\frac{1}{H}\right)+O\left(1\right). \tag{3.18}$$

By Lemma 3.7, we have

$$m_0\left(r,\frac{f^{(j)}}{f}\right)=O\left(\log T_0\left(r,f\right)+\log\frac{1}{r}\right)\quad\left(j=1,...,k-1\right) \tag{3.19}$$

holds for all $r\in(0,R)\setminus E$ where E is of finite logarithmic measure. By (3.17), (3.18) and (3.19), we get

$$\begin{aligned}T_0\left(r,f\right) &= T_0\left(r,\frac{1}{f}\right)+O\left(1\right)\\ &\leq k\overline{N}_0\left(r,\frac{1}{f}\right)+\sum_{j=0}^{k-1}T_0\left(r,A_j\right)+T_0\left(r,H\right)+\\ &\quad+O\left(\log T_0\left(r,f\right)+\log\frac{1}{r}\right),\ r\notin E\end{aligned} \tag{3.20}$$

By (3.20) and by taking account that $O\left(\log T_0\left(r,f\right)+\log\frac{1}{r}\right)\leq\frac{1}{2}T_0\left(r,f\right)$, we obtain

$$\frac{1}{2}T_0\left(r,f\right)\leq k\overline{N}_0\left(r,\frac{1}{f}\right)+\sum_{j=0}^{k-1}T_0\left(r,A_j\right)+T_0\left(r,H\right). \tag{3.21}$$

By (3.21), we have

$$\sigma_n\left(f,0\right)\leq\max\left\{\overline{\lambda}_n\left(f,0\right),\sigma_n\left(A_j,0\right),\sigma_n\left(H,0\right)\right\}\ \left(n=1,2\right).$$

Since

$$\max\{\sigma_n(H,0), \sigma_n(A_j,0); j=0,1,...,k-1\} < \sigma_n(f,0),$$

we get $\sigma_n(f,0) \le \overline{\lambda}_n(f,0)$ $(n=1,2)$. Therefore $\overline{\lambda}(f,0)=\lambda(f,0)=\sigma(f,0)=+\infty$ and $\overline{\lambda}_2(f,0)=\lambda_2(f,0)=\sigma_2(f,0)=\alpha$.

Lemma 3.9 *[6] Let $P(z)=a_nz^n+...+a_0$ with $a_n \neq 0$ be a polynomial and $A(z)=P\left(\frac{1}{z}\right)$. Then, for every $\varepsilon>0$, there exists $r_0>0$ such that for all $0<r=|z|\le r_0$, the inequalities*

$$(1-\varepsilon)\frac{|a_n|}{r^n} \le |A(z)| \le (1+\varepsilon)\frac{|a_n|}{r^n}$$

hold.

4 Proof of theorems

Proof of Theorem 2.1. It is clear that all solutions of (2.1) are analytic in $D(0,R)$. First we prove that every solution f of (2.3) satisfies $\sigma(f,0)\ge n$. We assume that $\sigma(f,0)<n$, and we prove that is failing. By Lemma 3.4, we have $\sigma(f',0)=\sigma(f'',0)=\sigma(f,0)<n$. From (2.1) we have

$$A_1(z)\exp\left\{\frac{a}{z^n}\right\}f'+A_0(z)\exp\left\{\frac{b}{z^n}\right\}f=F(z)-f'', \tag{4.1}$$

By the properties of the order of growth, we have

$$\sigma\left(A_1(z)\exp\left\{\frac{a}{z^n}\right\}f'+A_0(z)\exp\left\{\frac{b}{z^n}\right\}f,0\right)=n$$

and

$$\sigma(F(z)-f'',0)<n;$$

contradiction with (4.1). So $\sigma(f,0)\ge n$. Now, we prove that $\sigma(f,0)=+\infty$. We assume to the contrary that $\sigma(f,0)<+\infty$. Since $\sigma(F,0)=\alpha<n$ then for any given ε such that $0<2\varepsilon<n-\alpha$ and r small enough, we have

$$|F(z)|\le\exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}. \tag{4.2}$$

Since $a\neq b$, it is clear that the set E_1 of $\theta=\arg(z)\in[0,2\pi)$ such that $\delta_a(\theta)=0, \delta_b(\theta)=0$ and $\delta_a(\theta)=\delta_b(\theta)$ is of linear measure zero, where $\delta_a(\theta)$ is defined in Lemma 3.2. By Lemma 3.1, there exists a set $E_2\in[0,2\pi)$ of linear measure zero such that if $\theta\in[0,2\pi)\setminus E_2$, then there is a constant $r_0(\theta)<R'$ such that for all z satisfying $\arg(z)=\theta$ and $|z|<r_0(\theta)$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\le\frac{1}{r^{2\sigma+3}},\ (0\le j\le k\le 2). \tag{4.3}$$

Set $\delta_1 = \max\{\delta_a(\theta), \delta_b(\theta)\}$ and $\delta_2 = \min\{\delta_a(\theta), \delta_b(\theta)\}$. For any fixed $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$ there exist three cases:

Case 1. $\delta_1 = \delta_a(\theta) > 0$. By Lemma 3.2, for any given $\varepsilon > 0$, we get

$$\left| A(z) \exp\left\{\frac{a}{z^n}\right\}\right| \geq \exp\left\{\frac{(1-\varepsilon)\delta_1}{r^n}\right\} \tag{4.4}$$

Now we prove that $|z^{\alpha+\varepsilon}| \log^+ |f'(z)|$ is bounded on the ray $\arg(z) = \theta$. We assume to the contrary that $|z^{\alpha+\varepsilon}| \log^+ |f'(z)|$ is unbounded on the ray $\arg(z) = \theta$ and we prove that this leads to a contradiction. Then by Lemma 3.3, there is a sequence of points $z_m = r_m e^{i\theta}$ $(m \geq 1)$, $r_m \to 0$, such that

$$r_m^{\alpha+\varepsilon} \log^+ |f'(z_m)| \to +\infty \tag{4.5}$$

and

$$\left|\frac{f(z_m)}{f'(z_m)}\right| \leq M_1, \ (M_1 > 0), \tag{4.6}$$

as $m \to +\infty$. From (4.5) for any $c > 1$ we have

$$r_m^{\alpha+\varepsilon} \log^+ |f'(z_m)| > c;$$

then

$$|f'(z_m)| > \exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}, \ m \to +\infty. \tag{4.7}$$

From (4.2), and (4.7), we obtain

$$\left|\frac{F(z_m)}{f'(z_m)}\right| < \exp\left\{\frac{-1}{r_m^{\alpha+\varepsilon}}\right\} \to 0, \ m \to +\infty. \tag{4.8}$$

From (2.1), we can write

$$\left|A(z)\exp\left\{\frac{a}{z^n}\right\}\right| \leq \left|\frac{f''}{f'}\right| + \left|B(z)\exp\left\{\frac{b}{z^n}\right\}\right|\left|\frac{f}{f'}\right| + \left|\frac{F(z)}{f'}\right|. \tag{4.9}$$

Since $\delta_b(\theta) = \delta_2 < \delta_1$ and $\sigma(B, 0) < n$, for $0 < 2\varepsilon < \min\left\{1, 1 - \frac{\delta_2}{\delta_1}\right\}$, we have

$$\left|B(z)\exp\left\{\frac{b}{z^n}\right\}\right| \leq \exp\left\{\frac{(1-2\varepsilon)\delta_1}{r^n}\right\}, \ r \to 0. \tag{4.10}$$

Using (4.4), (4.6), (4.8), (4.3) and (4.10) into (4.9), we obtain

$$\exp\left\{\frac{(1-\varepsilon)\delta_1}{r_m^n}\right\} \leq \frac{M_1}{r_m^{2\sigma+3}} \exp\left\{\frac{(1-2\varepsilon)\delta_1}{r_m^n}\right\},$$

as $r \to 0$, where $M_1 > 0$ is a constant, and then

$$r_m^{2\sigma+3} \exp\left\{\frac{\varepsilon\delta_1}{r_m^n}\right\} \le M_1. \tag{4.11}$$

A contradiction in (4.11) as $m \to +\infty$. So $|z^{\alpha+\varepsilon}| \log^+ |f'(z)|$ is bounded on the ray $\arg(z) = \theta$ and we get

$$|f'(z)| \le \exp\left\{\frac{C_1}{r^{\alpha+\varepsilon}}\right\}, \ C_1 > 0. \tag{4.12}$$

By integration along the line segment $[z_0, z]$, where $\arg z_0 = \arg z = \theta$ and $0 < |z| < |z_0|$, we obtain

$$f(z) = f(z_0) + \int_{z_0}^{z} f'(u)\, du; \tag{4.13}$$

and by using (4.12), we get

$$|f(z)| \le |f(z_0)| + |z_0| \exp\left\{\frac{C_1}{r^{\alpha+\varepsilon}}\right\}, \ C_1 > 0. \tag{4.14}$$

By (4.14), as $r \to 0$ with $\arg z = \theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, we obtain

$$|f(z)| \le \exp\left\{\frac{C_1'}{r^{\alpha+\varepsilon}}\right\}, \ C_1' > C_1. \tag{4.15}$$

Case 2. $\delta_1 = \delta_b(\theta) > 0$. By Lemma 3.2, for any given $\varepsilon > 0$, we have

$$\left| B(z) \exp\left\{\frac{b}{z^n}\right\} \right| \ge \exp\left\{\frac{(1-\varepsilon)\delta_1}{r^n}\right\}. \tag{4.16}$$

Now we prove that $|z^{\alpha+\varepsilon}| \log^+ |f(z)|$ is bounded on the ray $\arg(z) = \theta$. We assume that $|z^{\alpha+\varepsilon}| \log^+ |f(z)|$ is unbounded on the ray $\arg(z) = \theta$; then, there is a sequence of points $z_m = r_m e^{i\theta}$ $(m \ge 1)$, $r_m \to 0$, such that

$$r_m^{\alpha+\varepsilon} \log^+ |f(z_m)| \to +\infty. \tag{4.17}$$

which implies that for any $c > 1$ we have

$$r_m^{\alpha+\varepsilon} \log^+ |f(z_m)| > c;$$

and then

$$|f(z_m)| > \exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}, \ m \to +\infty. \tag{4.18}$$

From (4.2) and (4.18), we get

$$\left|\frac{F(z_m)}{f(z_m)}\right| < \exp\left\{\frac{-1}{r_m^{\alpha+\varepsilon}}\right\} \to 0,\ m \to +\infty. \tag{4.19}$$

From (2.1), we can write

$$\left|B(z)\exp\left\{\frac{b}{z^n}\right\}\right| \leq \left|\frac{f''(z)}{f(z)}\right| + \left|A(z)\exp\left\{\frac{a}{z^n}\right\}\right|\left|\frac{f'(z)}{f(z)}\right| + \left|\frac{F(z)}{f(z)}\right|. \tag{4.20}$$

Since $\delta_a(\theta) = \delta_2 < \delta_1$, for $0 < 2\varepsilon < \min\left\{1, 1 - \frac{\delta_2}{\delta_1}\right\}$, we have

$$\left|\exp\left\{\frac{a}{z^n}\right\}\right| \leq \exp\left\{\frac{(1-2\varepsilon)\delta_1}{r^n}\right\},\ r \to 0. \tag{4.21}$$

Combining (4.16), (4.3), (4.19) and (4.21) with (4.20), we obtain

$$\exp\left\{\frac{(1-\varepsilon)\delta_1}{r_m^n}\right\} \leq \frac{M_2}{r^{2\sigma+3}}\exp\left\{\frac{(1-2\varepsilon)\delta_1}{r_m^n}\right\},$$

as $r \to 0$, where $M_2 > 0$ is a constant, and then

$$\exp\left\{\frac{\varepsilon\delta_1}{r_m^n}\right\} \leq \frac{M_2}{r^{2\sigma+3}}. \tag{4.22}$$

(4.22) leads to a contradiction as $m \to +\infty$. So $|z^{\alpha+\varepsilon}|\log^+|f(z)|$ is bounded on the ray $\arg(z) = \theta$ and then, when $r \to 0$ with $\arg z = \theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, we have

$$|f(z)| \leq \exp\left\{\frac{C_2}{r^{\alpha+\varepsilon}}\right\},\ C_2 > 0. \tag{4.23}$$

Case 3. $\delta_1 < 0$. From (2.1), we can write

$$1 \leq \left|A(z)\exp\left\{\frac{a}{z^n}\right\}\right|\left|\frac{f'(z)}{f''(z)}\right| + \left|B(z)\exp\left\{\frac{b}{z^n}\right\}\right|\left|\frac{f(z)}{f''(z)}\right| + \left|\frac{F(z)}{f''(z)}\right|. \tag{4.24}$$

By Lemma 3.2, for any given $0 < \varepsilon < 1$, we have

$$\left|B(z)\exp\left\{\frac{b}{z^n}\right\}\right| \leq \exp\left\{\frac{(1-\varepsilon)\delta_1}{r^n}\right\} \tag{4.25}$$

and

$$\left|\exp\left\{\frac{a}{z^n}\right\}\right| \leq \exp\left\{\frac{(1-\varepsilon)\delta_1}{r^n}\right\}. \tag{4.26}$$

Now we prove that $|z^{\alpha+\varepsilon}|\log^+|f''(z)|$ is bounded on the ray $\arg(z) = \theta$. We assume that $|z^{\alpha+\varepsilon}|\log^+|f''(z)|$ is unbounded on the ray $\arg(z) = \theta$; then by

Lemma 3.3, there is a sequence of points $z_m = r_m e^{i\theta}$ $(m \geq 1)$, $r_m \to 0$, such that

$$r_m^{\alpha+\varepsilon} \log^+ |f''(z_m)| \to +\infty, \tag{4.27}$$

and

$$\left| \frac{f^{(j)}(z_m)}{f''(z_m)} \right| \leq M_2, \ (M_2 > 0) \ (j = 0, 1). \tag{4.28}$$

as $m \to +\infty$. From (4.27), for any $c > 1$ we have

$$r_m^{\alpha+\varepsilon} \log^+ |f''(z_m)| > c;$$

and then

$$|f''(z_m)| > \exp\left\{ \frac{2}{r_m^{\alpha+\varepsilon}} \right\}, \ m \to +\infty. \tag{4.29}$$

From (4.2) and (4.29), we obtain

$$\left| \frac{F(z_m)}{f''(z_m)} \right| < \exp\left\{ \frac{-1}{r_m^{\alpha+\varepsilon}} \right\} \to 0, \ m \to +\infty. \tag{4.30}$$

By combining (4.3), (4.25), (4.26), (4.28) and (4.30) with (4.24), we obtain

$$1 \leq 2M_2 \exp\left\{ \frac{(1-\varepsilon)\delta_1}{r_m^n} \right\} + \exp\left\{ \frac{-1}{r_m^{\alpha+\varepsilon}} \right\} \to 0, \ m \to +\infty; \tag{4.31}$$

a contradiction; then $|z^{\alpha+\varepsilon}| \log^+ |f''(z)|$ is bounded on the ray $\arg(z) = \theta$. As above when $r \to 0$ with $\arg z = \theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, we obtain

$$|f(z)| \leq \exp\left\{ \frac{C_3}{r^{\alpha+\varepsilon}} \right\}, \ C_3 > 0. \tag{4.32}$$

Now, we proved (4.32) on any ray $\arg z = \theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$ as $|z| = r \to 0$. By Lemma 3.5, we obtain $\sigma(f, 0) \leq \alpha$; which is a contradiction with $\alpha < n$ and $\sigma(f, 0) \geq n$; so we conclude that every solution f of (2.1) is of infinite order. Now, we have

$$\max\left\{ \sigma\left(A \exp\{\frac{a}{z^n}\}, 0 \right), \sigma\left(B(z) \exp\{\frac{b}{z^n}\}, 0 \right), \sigma\left(F(z) \exp\{\frac{a}{z^n}\}, 0 \right) \right\} = n;$$

and by applying Lemma 3.6, we get $\sigma_2(f, 0) \leq n$. Since $F(z) \not\equiv 0$, by Lemma 3.8, we obtain

$$\bar{\lambda}(f, 0) = \lambda(f, 0) = \sigma(f, 0) = +\infty, \ \bar{\lambda}_2(f, 0) = \lambda_2(f, 0) = \sigma_2(f, 0) \leq n.$$

Proof of Theorem 2.2. First, we prove that every solution f of (2.2) satisfies $\sigma(f, 0) \geq n$. We assume that $\sigma(f, 0) < n$, and we prove that is failing.

By Lemma 3.4, we have $\sigma(f',0)=\sigma(f'',0)=\sigma(f,0)<n$. From (2.2) we can write

$$A(z)\exp\left\{\frac{a}{z^n}\right\}f'+B(z)\exp\left\{\frac{b}{z^n}\right\}f=F(z)-f''-A_0(z)f'-B_0(z)f \quad (4.33)$$

By the properties of the order of growth and since $a\neq b$, we have

$$\sigma\left(A(z)\exp\left\{\frac{a}{z^n}\right\}f'+B(z)\exp\left\{\frac{b}{z^n}\right\}f,0\right)=n$$

and

$$\sigma\left(F(z)-f''-A_0(z)f'-B_0(z)f,0\right)<n;$$

a contradiction in (4.33). So $\sigma(f,0)\geq n$. Now, we prove that $\sigma(f,0)=+\infty$. We suppose to the contrary that $\sigma(f,0)<+\infty$. Since $\sigma(B_0,0)=\sigma(A_0,0)=\alpha<n$ then for any given ε such that $0<2\varepsilon<n-\alpha$ and r small enough, we have

$$\max\{|A_0(z)|,|B_0(z)|\}\leq\exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}. \quad (4.34)$$

It is clear that the set E_3 of $\theta=\arg(z)\in[0,2\pi)$ such that $\delta_a(\theta)=0,\delta_b(\theta)=0$ is of linear measure zero. For any fixed $\theta\in[0,2\pi)\setminus(E_3\cup E_2)$ there exist two cases:

Case 1. $\delta=\delta_a(\theta)>0$. We will prove that $|z^{\alpha+\varepsilon}|\log^+|f'(z)|$ is bounded on the ray $\arg(z)=\theta$. We assume to the contrary that $|z^{\alpha+\varepsilon}|\log^+|f'(z)|$ is unbounded on the ray $\arg(z)=\theta$. Then by Lemma 3.3, there is a sequence of points $z_m=r_me^{i\theta}\ (m\geq1)$, $r_m\to0$, such that we have (4.5) and (4.6); and then, we have (4.8). From (2.2), we can write

$$\left|A(z)\exp\left\{\frac{a}{z^n}\right\}\right|\leq\left|\frac{f''}{f'}\right|+A_0(z)+\left|B(z)\exp\left\{\frac{b}{z^n}\right\}+B_0(z)\right|\left|\frac{f}{f'}\right|+\left|\frac{F(z)}{f'}\right|. \quad (4.35)$$

Since $\delta_b(\theta)=\frac{1}{c}\delta<0$ and $\sigma(B,0)<n$, by Lemma 3.2, for any $\varepsilon>0$, we have

$$\left|B(z)\exp\left\{\frac{b}{z^n}\right\}\right|\leq\exp\left\{\frac{(1+\varepsilon)\frac{1}{c}\delta}{r^n}\right\},\ r\to0. \quad (4.36)$$

Using (4.4), (4.6), (4.8), (4.3), (4.36) and (4.34) into (4.35), we obtain

$$\exp\left\{\frac{(1-\varepsilon)\delta}{r_m^n}\right\}\leq\frac{M_1}{r_m^{2\sigma+3}}\exp\left\{\frac{(1+\varepsilon)\frac{1}{c}\delta}{r_m^n}\right\}, \quad (4.37)$$

as $r\to0$, where $M_1>0$ is a constant; a contradiction by taking $0<\varepsilon<1$: the right side of (4.37) tends to 0 as $m\to+\infty$ while the left side tends to $+\infty$. So $|z^{\alpha+\varepsilon}|\log^+|f'(z)|$ is bounded on the ray $\arg(z)=\theta$ and we get

$$|f'(z)|\leq\exp\left\{\frac{C_1}{r^{\alpha+\varepsilon}}\right\},\ C_1>0;$$

and then, as above in the proof of Theorem 2.1, we get

$$|f(z)| \le \exp\left\{\frac{C}{r^{\alpha+\varepsilon}}\right\}, \ C > 0. \tag{4.38}$$

Case 2. $\delta_b(\theta) = \frac{1}{c}\delta > 0$; (in this case $\delta < 0$). We prove that $|z^{\alpha+\varepsilon}| \log^+ |f(z)|$ is bounded on the ray $\arg(z) = \theta$. We assume that $|z^{\alpha+\varepsilon}| \log^+ |f(z)|$ is unbounded on the ray $\arg(z) = \theta$. From (2.2), we can write

$$\left|B(z)\exp\left\{\frac{b}{z^n}\right\}\right| \le \left|\frac{f''(z)}{f(z)}\right| + \left|A(z)\exp\left\{\frac{a}{z^n}\right\} + A_0(z)\right| \left|\frac{f'(z)}{f(z)}\right| + B_0(z) + \left|\frac{F(z)}{f(z)}\right|. \tag{4.39}$$

By Lemma 3.2, for any given $\varepsilon > 0$, we have

$$\left|B(z)\exp\left\{\frac{b}{z^n}\right\}\right| \ge \exp\left\{\frac{(1-\varepsilon)\frac{1}{c}\delta}{r^n}\right\} \tag{4.40}$$

and

$$\left|B(z)\exp\left\{\frac{b}{z^n}\right\}\right| \le \exp\left\{\frac{(1+\varepsilon)\delta}{r^n}\right\}. \tag{4.41}$$

Combining (4.3), (4.19), (4.34), (4.40) and (4.41) with (4.20), we obtain

$$\exp\left\{\frac{(1-\varepsilon)\frac{1}{c}\delta}{r_m^n}\right\} \le \frac{M_2}{r^{2\sigma+3}} \exp\left\{\frac{(1+\varepsilon)\delta}{r_m^n}\right\}, \tag{4.42}$$

as $r \to 0$, where $M_2 > 0$ is a constant. Also (4.42) leads to a contradiction as $m \to +\infty$. So $|z^{\alpha+\varepsilon}| \log^+ |f(z)|$ is bounded on the ray $\arg(z) = \theta$ and then, when $r \to 0$ with $\arg z = \theta \in [0, 2\pi) \setminus (E_3 \cup E_2)$, we have

$$|f(z)| \le \exp\left\{\frac{C}{r^{\alpha+\varepsilon}}\right\}, \ C > 0. \tag{4.43}$$

We proved (4.43) on any ray $\arg z = \theta \in [0, 2\pi) \setminus (E_3 \cup E_2)$ as $|z| = r \to 0$. By Lemma 3.5, we obtain $\sigma(f, 0) \le \alpha$; which is a contradiction with $\alpha < n$ and $\sigma(f, 0) \ge n$; so we conclude that every solution f of (2.2) is of infinite order. Now, by applying Lemma 3.6 to the equation (2.2), we get $\sigma_2(f, 0) \le n$. Furthermore, since $F(z) \not\equiv 0$, by Lemma 3.8, we obtain

$$\bar{\lambda}(f,0) = \lambda(f,0) = \sigma(f,0) = +\infty, \ \bar{\lambda}_2(f,0) = \lambda_2(f,0) = \sigma_2(f,0) \le n.$$

Proof of Theorem 2.3. We prove the results for the solutions of (2.3) and we can use the same method for (2.4). First, we prove that every solution f of (2.3) satisfies $\sigma(f, 0) \ge n$. We assume that $\sigma(f, 0) < n$, and we prove that is failing. By Lemma 3.4, we have $\sigma(f',0) = \sigma(f'',0) = \sigma(f,0) < n$. From (2.3) we can write

$$\exp\left\{\frac{-a}{z^n}\right\} f'' + B(z)\exp\left\{\frac{b-a}{z^n}\right\} f = F(z) - P\left(\frac{1}{z}\right) f'. \tag{4.44}$$

By the properties of the order of growth and since $-a \neq b-a$, we have

$$\sigma\left(\exp\left\{\frac{-a}{z^n}\right\} f'' + B(z)\exp\left\{\frac{b-a}{z^n}\right\} f, 0\right) = n$$

and

$$\sigma\left(F(z) - P\left(\frac{1}{z}\right) f', 0\right) < n;$$

a contradiction with (4.44). So $\sigma(f,0) \geq n$. Now, we prove that $\sigma(f,0) = +\infty$. We suppose to the contrary that $\sigma(f,0) < +\infty$. Since $\sigma(F,0) = \alpha < n$ then for any given ε such that $0 < 2\varepsilon < n - \alpha$ and r small enough, we have

$$|F(z)| \leq \exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}. \tag{4.45}$$

Since $-a \neq b-a$, it is clear that the set E_1 of $\theta = \arg(z) \in [0,2\pi)$ such that $\delta_{-a}(\theta) = 0, \delta_{b-a}(\theta) = 0$ and $\delta_{-a}(\theta) = \delta_{b-a}(\theta)$ is of linear measure zero. By Lemma 3.1, there exists a set $E_2 \in [0,2\pi)$ of linear measure zero such that if $\theta \in [0,2\pi) \setminus E_2$, then there is a constant $r_0(\theta) < R'$ such that for all z satisfying $\arg(z) = \theta$ and $|z| < r_0(\theta)$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \frac{1}{r^{2\sigma+3}},\ (0 \leq j \leq k \leq 2). \tag{4.46}$$

Set $\delta_1 = \max\{\delta_{-a}(\theta), \delta_{b-a}(\theta)\}$ and $\delta_2 = \min\{\delta_{-a}(\theta), \delta_{b-a}(\theta)\}$. For any fixed $\theta \in [0,2\pi) \setminus (E_1 \cup E_2)$ there exist three cases:

Case 1. $\delta_1 = \delta_{-a}(\theta) > 0$. By Lemma 3.2, for any given $\varepsilon > 0$, we get

$$\left|\exp\left\{\frac{-a}{z^n}\right\}\right| \geq \exp\left\{\frac{(1-\varepsilon)\delta_1}{r^n}\right\}. \tag{4.47}$$

Now we prove that $|z^{\alpha+\varepsilon}|\log^+|f''(z)|$ is bounded on the ray $\arg(z) = \theta$. We assume to the contrary that $|z^{\alpha+\varepsilon}|\log^+|f''(z)|$ is unbounded on the ray $\arg(z) = \theta$ and we prove that this leads to a contradiction. Then by Lemma 3.3, there is a sequence of points $z_m = r_m e^{i\theta}$ $(m \geq 1)$, $r_m \to 0$, such that

$$r_m^{\alpha+\varepsilon}\log^+|f''(z_m)| \to +\infty \tag{4.48}$$

and

$$\left|\frac{f^{(j)}(z_m)}{f''(z_m)}\right| \leq M_1,\ (M_1 > 0)\ (j = 0,1), \tag{4.49}$$

as $m \to +\infty$. From (4.48) for any $c > 1$ we have

$$r_m^{\alpha+\varepsilon} \log^+ |f''(z_m)| > c;$$

then

$$|f''(z_m)| > \exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}, \ m \to +\infty. \tag{4.50}$$

From (4.45) and (4.50), we obtain

$$\left|\frac{F(z_m)}{f''(z_m)}\right| < \exp\left\{\frac{-1}{r_m^{\alpha+\varepsilon}}\right\}, \ m \to +\infty. \tag{4.51}$$

From (2.3), we can write

$$\left|\exp\left\{\frac{-a}{z^n}\right\}\right| \leq \left|P\left(\frac{1}{z}\right)\right|\left|\frac{f'(z)}{f''(z)}\right| + \left|B(z)\exp\left\{\frac{b-a}{z^n}\right\}\right|\left|\frac{f(z)}{f''(z)}\right| + \left|\frac{F(z)}{f''(z)}\right|. \tag{4.52}$$

Since $\delta_{b-a}(\theta) = \delta_2 < \delta_1$ and $\sigma(B, 0) < n$, for $0 < 2\varepsilon < \min\left\{1, 1 - \frac{\delta_2}{\delta_1}\right\}$, we have

$$\left|B(z)\exp\left\{\frac{b-a}{z^n}\right\}\right| \leq \exp\left\{\frac{(1-2\varepsilon)\delta_1}{r^n}\right\}, \ r \to 0. \tag{4.53}$$

By Lemma 3.9, there exists $\lambda > 0$ such that for r small enough, we have

$$\left|P\left(\frac{1}{z}\right)\right| \leq \frac{\lambda}{r_m^d}, \ d = \deg P. \tag{4.54}$$

Using (4.47), (4.49), (4.51), (4.53) and (4.54) into (4.52), we obtain

$$\exp\left\{\frac{(1-\varepsilon)\delta_1}{r_m^n}\right\} \leq M_1 \frac{\lambda}{r_m^d} \exp\left\{\frac{(1-2\varepsilon)\delta_1}{r_m^n}\right\},$$

as $r \to 0$; and then

$$r_m^d \exp\left\{\frac{\varepsilon\delta_1}{r_m^n}\right\} \leq M_1\lambda. \tag{4.55}$$

A contradiction in (4.55) as $m \to +\infty$. So $|z^{\alpha+\varepsilon}| \log^+ |f''(z)|$ is bounded on the ray $\arg(z) = \theta$ and we get

$$|f''(z)| \leq \exp\left\{\frac{C_1}{r^{\alpha+\varepsilon}}\right\}, \ C_1 > 0. \tag{4.56}$$

By two-fold iterated integration, along the line segment $[z_0, z]$, where $\arg z_0 = \arg z = \theta$ and $0 < |z| < |z_0|$, we obtain

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \int_{z_0}^{z}\int_{z_0}^{w} f''(u)\,du\,dw; \tag{4.57}$$

and then

$$|f(z)| \leq |f(z_0)| + |f'(z_0)| |(z - z_0)| + \int_{z_0}^{z} \int_{z_0}^{w} |f''(u)| \, dudw. \tag{4.58}$$

From (4.56) and (4.58), we get

$$|f(z)| \leq |f(z_0)| + |f'(z_0)| |z_0| + \frac{|z_0|^2}{2} \exp\left\{\frac{C_1}{r^{\alpha+\varepsilon}}\right\}, \ C_1 > 0. \tag{4.59}$$

By (4.59), as $r \to 0$ with $\arg z = \theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, we obtain

$$|f(z)| \leq \exp\left\{\frac{C_1'}{r^{\alpha+\varepsilon}}\right\}, \ C_1' > 0. \tag{4.60}$$

Case 2. $\delta_1 = \delta_{b-a}(\theta) > 0$. By Lemma 3.2, for any given $\varepsilon > 0$, we have

$$\left| B(z) \exp\left\{\frac{b-a}{z^n}\right\} \right| \geq \exp\left\{\frac{(1-\varepsilon)\delta_1}{r^n}\right\}. \tag{4.61}$$

Now we prove that $|z^{\alpha+\varepsilon}| \log^+ |f(z)|$ is bounded on the ray $\arg(z) = \theta$. We assume that $|z^{\alpha+\varepsilon}| \log^+ |f(z)|$ is unbounded on the ray $\arg(z) = \theta$; then, there is a sequence of points $z_m = r_m e^{i\theta} \ (m \geq 1)$, $r_m \to 0$, such that

$$r_m^{\alpha+\varepsilon} \log^+ |f(z_m)| \to +\infty. \tag{4.62}$$

which implies that for any $c > 1$ we have

$$r_m^{\alpha+\varepsilon} \log^+ |f(z_m)| > c;$$

and then

$$|f(z_m)| > \exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}, \ m \to +\infty. \tag{4.63}$$

From (4.45) and (4.63), we get

$$\left|\frac{F(z_m)}{f(z_m)}\right| < \exp\left\{\frac{-1}{r_m^{\alpha+\varepsilon}}\right\}, \ m \to +\infty. \tag{4.64}$$

From (2.3), we can write

$$\left| B(z) \exp\left\{\frac{b-a}{z^n}\right\} \right| \leq \left|\exp\left\{\frac{-a}{z^n}\right\}\right| \left|\frac{f''(z)}{f(z)}\right| + \left|P\left(\frac{1}{z}\right)\right| \left|\frac{f'(z)}{f(z)}\right| + \left|\frac{F(z)}{f(z)}\right|. \tag{4.65}$$

Since $\delta_{-a}(\theta) = \delta_2 < \delta_1$, for $0 < 2\varepsilon < \min\left\{1, 1 - \frac{\delta_2}{\delta_1}\right\}$, we have

$$\left|\exp\left\{\frac{-a}{z^n}\right\}\right| \leq \exp\left\{\frac{(1-2\varepsilon)\,\delta_1}{r^n}\right\},\ r \to 0. \tag{4.66}$$

Combining (4.61), (4.46), (4.64) and (4.66) with (4.65), we obtain

$$\exp\left\{\frac{(1-\varepsilon)\,\delta_1}{r_m^n}\right\} \leq \frac{M_2}{r^{d+2\sigma+3}} \exp\left\{\frac{(1-2\varepsilon)\,\delta_1}{r_m^n}\right\},$$

as $r \to 0$, where $M_2 > 0$ is a constant, and then

$$r^{d+2\sigma+3} \exp\left\{\frac{\varepsilon\delta_1}{r_m^n}\right\} \leq M_2. \tag{4.67}$$

(4.67) leads to a contradiction as $m \to +\infty$. So $|z^{\alpha+\varepsilon}| \log^+ |f(z)|$ is bounded on the ray $\arg(z) = \theta$ and we get

$$|f(z)| \leq \exp\left\{\frac{C_2}{r^{\alpha+\varepsilon}}\right\},\ C_2 > 0,$$

and then, when $r \to 0$ with $\arg z = \theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, we have

$$|f(z)| \leq \exp\left\{\frac{C_2'}{r^{\alpha+\varepsilon}}\right\},\ C_2' > 0. \tag{4.68}$$

Case 3. $\delta_1 < 0$. From (2.3), we can write

$$\left|P\left(\frac{1}{z}\right)\right| \leq \left|\exp\left\{\frac{-a}{z^n}\right\}\right| \left|\frac{f''(z)}{f'(z)}\right| + \left|B(z)\exp\left\{\frac{b-a}{z^n}\right\}\right| \left|\frac{f(z)}{f'(z)}\right| + \left|\frac{F(z)}{f'(z)}\right|. \tag{4.69}$$

By Lemma 3.2, for any given $\varepsilon > 0$, we have

$$\left|B(z)\exp\left\{\frac{b-a}{z^n}\right\}\right| \leq \exp\left\{\frac{(1-\varepsilon)\,\delta_1}{r^n}\right\} \tag{4.70}$$

and

$$\left|\exp\left\{\frac{-a}{z^n}\right\}\right| \leq \exp\left\{\frac{(1-\varepsilon)\,\delta_1}{r^n}\right\}. \tag{4.71}$$

By Lemma 3.9, there exists $\lambda' > 0$ such that for r small enough, we have

$$\frac{\lambda'}{r_m^d} \leq \left|P\left(\frac{1}{z}\right)\right| \tag{4.72}$$

Now we prove that $|z^{\alpha+\varepsilon}| \log^+ |f'(z)|$ is bounded on the ray $\arg(z) = \theta$. We assume that $|z^{\alpha+\varepsilon}| \log^+ |f'(z)|$ is unbounded on the ray $\arg(z) = \theta$; then by

Lemma 3.3, there is a sequence of points $z_m = r_m e^{i\theta}$ $(m \geq 1)$, $r_m \to 0$, such that

$$r_m^{\alpha+\varepsilon} \log^+ |f'(z_m)| \to +\infty, \tag{4.73}$$

and

$$\left| \frac{f(z_m)}{f'(z_m)} \right| \leq M_2, \ (M_2 > 0). \tag{4.74}$$

as $m \to +\infty$. From (4.73), for any $c > 1$ we have

$$r_m^{\alpha+\varepsilon} \log^+ |f'(z_m)| > c;$$

and then

$$|f'(z_m)| > \exp\left\{ \frac{2}{r_m^{\alpha+\varepsilon}} \right\}, \ m \to +\infty. \tag{4.75}$$

From (4.45) and (4.75), we obtain

$$\left| \frac{F(z_m)}{f'(z_m)} \right| < \exp\left\{ \frac{-1}{r_m^{\alpha+\varepsilon}} \right\}, \ m \to +\infty. \tag{4.76}$$

By combining (4.46), (4.70), (4.71), (4.72), (4.74) and (4.76) with (4.69), we obtain

$$\frac{\lambda'}{r_m^d} \leq \exp\left\{ \frac{(1-\varepsilon)\delta_1}{r_m^n} \right\} \left(\frac{1}{r_m^{2\sigma+3}} + M_2 \right) + \exp\left\{ \frac{-1}{r_m^{\alpha+\varepsilon}} \right\}. \tag{4.77}$$

Since the right side of (4.77) tends to zero as $m \to +\infty$, a contradiction follows and then $|z^{\alpha+\varepsilon}| \log^+ |f'(z)|$ is bounded on the ray $\arg(z) = \theta$. As above, as $r \to 0$ with $\arg z = \theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, we have

$$|f(z)| \leq \exp\left\{ \frac{C_3}{r^{\alpha+\varepsilon}} \right\}, \ C_3 > 0. \tag{4.78}$$

In all cases we proved

$$|f(z)| \leq \exp\left\{ \frac{C}{r^{\alpha+\varepsilon}} \right\}, \ C > 0$$

on any ray $\arg z = \theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$ as $|z| = r \to 0$. By Lemma 3.5, we obtain $\sigma(f, 0) \leq \alpha$; which is a contradiction with $\alpha < n$ and $\sigma(f, 0) \geq n$; so we conclude that every solution f of (2.3) is of infinite order. Now, the maximum of the order of growth near 0 of the three terms:

$$P\left(\frac{1}{z}\right) \exp\left\{ \frac{a}{z^n} \right\}, B(z) \exp\left\{ \frac{b}{z^n} \right\}, F(z) \exp\left\{ \frac{a}{z^n} \right\};$$

is equal to n; and by applying Lemma 3.6, we get $\sigma_2(f, 0) \leq n$. Since $F(z) \not\equiv 0$, by Lemma 3.8, we obtain

$$\bar{\lambda}(f, 0) = \lambda(f, 0) = \sigma(f, 0) = +\infty, \ \bar{\lambda}_2(f, 0) = \lambda_2(f, 0) = \sigma_2(f, 0) \leq n$$

5 Open Problem

In this work, the following questions remain open:
1) How about the case when $\sigma(F, 0) > n$?
2) How about the case when the coefficients are meromorphic in $D(0, R)$?
3) Can we generalize these results to the higher order linear differential equations?

Acknowledgment. This work is supported by Abdelhamid Ibn Badis University of Mostaganem and PRFU Project (Projet de Recherche Formation Universitaire, Code C00L03UN270120220005).

References

[1] I. Amemiya, M. Ozawa; Non-existence of finite order solutions of $w'' + e^{-z}w' + Q(z)w = 0$, Hokkaido Math. J., 10 (1981), 1-17.

[2] B. Belaidi; Differential polynomials generated by solutions of second order non-homogeneous linear differential equations, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. (to appear).

[3] L. Bieberbach; *Theorie der gewöhnlichen Differentialgleichungen,* Springer-Verlag, Berlin/ Heidelberg/ New York, 1965.

[4] Z. X. Chen; The growth of solutions of $f'' + e^{-z}f' + Q(z)f = 0$ where the order $(Q) = 1$, Sci. China Ser. A, 45 (2002), 290-300.

[5] Z. X. Chen, K. H. Shon; On the growth of solutions of a class of higher order linear differential equations, Acta. Mathematica Scientia, 24 B (1) (2004), 52-60.

[6] S. Cherief, S. Hamouda; Growth of solutions of a class of linear differential equations near a singular point, Kragujevac J. Math. Vol. 47(2) (2023), 187-201.

[7] S. Cherief, S. Hamouda; Linear differential equations with analytic coefficients having the same order near a singular point, Bull. Iranian Math. Soc. Vol 47 (2021), 1737–1749.

[8] S. Cherief, S. Hamouda; Finite and infinite order of growth of solutions to linear differential equations near a singular point, Math. Bohem., Vol. 145 (3) (2021), 1-18.

[9] S. Cherief, S. Hamouda; Exponent of convergence of solutions to linear differential equations near a singular point, Graduate J. Math. 6 (2021), 22-30.

[10] H. Fettouch, S. Hamouda; Growth of local solutions to linear differential equations around an isolated essential singularity, Electron. J. Differential Equations, Vol 2016 (2016), No. 226, 1-10.

[11] G. G. Gundersen; "On the question of whether $f'' + e^{-z}f' + B(z)f = 0$ can admit a solution $f \not\equiv 0$ of finite order", Proc. Roy. Soc. Edinburgh 102A (1986), 9-17.

[12] S. Hamouda; The possible orders of growth of solutions to certain linear differential equations near a singular point, J. Math. Anal. Appl. 458 (2018), 992–1008.

[13] S. Hamouda; Estimates of the logarithmic derivative near a singular point and applications, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. Vol. 24, 542 (2020), 99-116.

[14] W. K. Hayman; *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[15] W. P. Huang, J. L. Zhou, J. Tu, J. H. Ning; On the hyper-order of solutions of two class of complex linear differential equations, Adv. Difference Equ., (2015) 2015, No.234, 1-12.

[16] A.Ya. Khrystiyanyn, A. A. Kondratyuk; On the Nevanlinna theory for meromorphic functions on annuli, Matematychni Studii 23 (1) (2005) 19–30.

[17] A. A. Kondratyuk, I. Laine; Meromorphic functions in multiply connected domains, in: Fourier Series Methods in Complex Analysis, in: Univ. Joensuu Dept. Math. Rep. Ser., vol. 10, Univ. Joensuu, Joensuu, 2006, 9-111.

[18] K.H. Kwon; Nonexistence of finite order solutions of certain second order linear differential equations, Kodai Math. J. 19 (1996) 378–387.

[19] I. Laine; *Nevanlinna theory and complex differential equations,* W. de Gruyter, Berlin, 1993.

[20] E. L. Mark, Y. Zhuan; Logarithmic derivatives in annulus, J. Math. Anal. Appl. 356 (2009) 441-452.

[21] M. Tsuji; *Potential theory in modern function theory,* Chelsea, New York, 1975, reprint of the 1959 edition.

[22] G. Valiron; *Lectures on the General Theory of Integral Functions,* Chelsea Publishing Company, New York, 1949.

[23] J. Wang and I. Laine; Growth of solutions of second order linear differential equations, J. Math. Anal. Appl. 342 (2008) 39–51.

[24] J. Wang and I. Laine; Growth of solutions of nonhomogeneous linear differential equations, Abstract and Applied Analysis, Vol. 2009, ID 363927, 11 pages.

[25] J. M. Wittaker; The order of the derivative of a meromorphic function, J. London Math. Soc. 11 1936, 82-87, Jbuch 62, 357.

[26] L. Yang; *Value distribution theory,* Springer-Verlag Science Press, Berlin-Beijing. 1993.