Int. J. Open Problems Complex Analysis, Vol. 14, No. 1, June 2022
ISSN 2074-2827;
www.i-csrs.org

Univalence Criteria For Analytic Functions Defined By Differential Operator

K. Narender, D. Madhava Krishna and P. Thirupathi Reddy

Department of Mathematics, Governament Junior College,
Bela- 504 001, Adilabad Dist, T.S. India.
e-mail: kallemnareshreddy1978@gmail.com

Department of Mathematics, Governament Degree College(W) (N),
Asifabad- 504 295, Kumuram Bheem Dist, T.S. India.
e-mail: madhavadkrishna@gmail.com

Department of Mathematics, School of Engineering,
NNRESGI - 500 088, Medichal Dist, Telangana, India.
e-mail: reddypt2@gmail.com

Received 21 May 2022; Accepted 28 June 2022

Abstract

In this paper we obtain sufficient condition for univalence of analytic functions defined by differential operator.

Keywords: *analytic, univalent, differential operator.*
2020 Mathematical Subject Classification: 30C45.

1 Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$.

Let S denote the subclass of A, which consists of functions of the form (1) that are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in E.

In geometric function theory, the univalence of complex functions is an important property, but it is difficult, and in many cases impossible, to show directly that a certain complex function is univalent. For this reason, many authors found different types of sufficient conditions of univalence. One of the most important of these conditions of univalence in the domains E and the exterior of a closed unit disc is the well-known criterion of Becker [5]. Becker's work depends upon a clever use of the theory of Loewner chains and the generalized Loewner differential equation. Extensions of this criterion were given by Deniz and Orhan [8, 9, 10].

Let f be a function in the class A. We define the following differential operator introduced by Deniz and Ozkan [11].

$$\begin{aligned}
&D_\lambda^0 f(z) = f(z)\\
&D_\lambda^1 f(z) = D_\lambda f(z) = \lambda z^3 f'''(z) + (2\lambda + 1)z^2 f''(z) + zf'(z)\\
&D_\lambda^2 f(z) = D_\lambda(D_\lambda^1 f(z))\\
&\quad\quad\quad .\\
&\quad\quad\quad .\\
&\quad\quad\quad .\\
&D_\lambda^m f(z) = D_\lambda(D_\lambda^{m-1} f(z))
\end{aligned}$$

where $\lambda \geq 0$ and $m \in N_0 = N \cup \{0\}$. If f is given by (1), then from the definition of the operator $D_\lambda^m f(z)$, it is to see that

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty} \phi^m(\lambda, n) a_n z^n \tag{2}$$

where

$$\phi^m(\lambda, n) = n^{2m}[\lambda(n-1) + 1]^m \tag{3}$$

Many differential operators studied by various authors can be seen in the literature (see [1, 2, 4, 6, 13]).

In this paper we derive sufficient conditions of univalence for the generalized operator $D_\lambda^m f(z)$. Also, a number of known univalent conditions would follow upon specialization the parameters involved. In order to prove our results wee need the following Lemmas.

Lemma 1.1 *[5] Let $f \in A$. If for all $z \in E$*

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{4}$$

then the function f is univalent in E.

Lemma 1.2 *[14] Let $f \in A$. If for all $z \in E$*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq 1 \tag{5}$$

then the function f is univalent in E.

Lemma 1.3 *[18] Let μ be a real number $\mu > \frac{1}{2}$ and $f \in A$. If for all $z \in E$*

$$(1 - |z|^{2\mu})\left|\frac{zf''(z)}{f'(z)} + 1 - \mu\right| \leq \mu \tag{6}$$

then the function f is univalent in E.

Lemma 1.4 *[12] If $f \in S$ (the class of univalent functions) and*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \tag{7}$$

then $\sum\limits_{n=1}^{\infty}(n-1)|b_n|^2 \leq 1$.

Lemma 1.5 *[15] Let $\nu \in C, Re\{\nu\} \geq 0$ and $f \in A$. If for all $z \in E$*

$$\tfrac{1-|z|^{2Re(\nu)}}{Re(\nu)}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{8}$$

then the function

$$F_\nu(z) = \left(\nu \int_0^z u^{\nu-1} f'(u)du\right)^{\frac{1}{\nu}}$$

is univalent in E.

2 Main Results

In this section, we establish the sufficient conditions to obtain a univalence for analytic functions involving the differential operator.

Theorem 2.1 *Let $f \in A$. If for all $z \in E$*

$$\sum_{n=1}^{\infty} \phi^m(\lambda, n)[n(2n-1)]|a_n| \leq 1 \tag{9}$$

then $D_\lambda^m f(z)$ is univalent in E.

Proof. Let $f \in A$. Then for all $z \in E$, we have

$$(1-|z|^2)\left|\frac{z(D_\lambda^m f(z))''}{(D_\lambda^m f(z))'}\right| \leq (1+|z|^2)\left|\frac{z(D_\lambda^m f(z))''}{(D_\lambda^m f(z))'}\right| \leq \frac{2\sum\limits_{n=2}^{\infty} n(n-1)\phi^m(\lambda,n)|a_n|}{1-\sum\limits_{n=2}^{\infty} n\phi^m(\lambda,n)|a_n|}$$

the last inequality is less than 1 if the assertion (9) is hold. Thus is view of Lemma 1.1, $D_\lambda^m f(z)$ is univalent in E.

Theorem 2.2 *Let $f \in A$. If for all $z \in E$*

$$\phi^m(\lambda,n)|a_n| \leq \frac{1}{\sqrt{7}} \tag{10}$$

then $D_\lambda^m f(z)$ is univalent in E.

Proof. Let $f \in A$. It sufficient to show that

$$\left|\frac{z^2(D_{\lambda\mu}^m f(z))'}{2(D_{\lambda\mu}^m f(z))^2}\right| \leq 1.$$

Now

$$\left|\frac{z^2(D_{\lambda\mu}^m f(z))'}{2(D_{\lambda\mu}^m f(z))^2}\right| \leq \frac{1+\sum\limits_{n=2}^{\infty} nB_n(\lambda,\mu,m)|a_n|}{2(1-2\sum\limits_{n=2}^{\infty}[B_n(\lambda,\mu,m)])^m|a_n| - (\sum\limits_{n=2}^{\infty} B_n(\lambda,\mu,m)|a_n|^2)}.$$

The last inequality is less than 1 if the assertion (10) is hold. Thus in view of Lemma 1.2, $D_{\lambda\mu}^m f(z)$ is univalent in E.

Theorem 2.3 *Let $f \in A$. If for all $z \in E$*

$$\sum_{n=1}^{\infty} n[2(n-1)+(2\mu-1)]\phi^m(\lambda,n)|a_n| \leq 2\mu-1,\ \ \mu > \frac{1}{2} \tag{11}$$

then $D_\lambda^m f(z)$ is univalent in E.

Proof. Let $f \in A$. Then for all $z \in E$, we have

$$(1-|z|^{2\mu})\left|\frac{z(D_\lambda^m f(z))''}{(D_\lambda^m f(z))'}+1-\mu\right| \leq (1+|z|^2)\left|\frac{z(D_\lambda^m f(z))''}{(D_\lambda^m f(z))'}\right| + |1-\mu| \leq \frac{2\sum\limits_{n=2}^{\infty} \phi^m(\lambda,n)[n(n-1)]|a_n|}{1-\sum\limits_{n=2}^{\infty} n\phi^m(\lambda,n)|a_n|} + |1-\mu|$$

the last inequality is less than μ if the assertion (11) is hold. Thus is view of Lemma 1.3, $D_\lambda^m f(z)$ is univalent in E.

As applications of Theorems 2.1, 2.2 and 2.3, we have the following Theorem.

Theorem 2.4 *Let $f \in A$. If for all $z \in E$ one of the inequality (9-11) holds then*

$$\sum_{n=1}^{\infty}(n-1)|b_n|^2 \leq 1, \tag{12}$$

where $\frac{z}{D_\lambda^m f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n$.

Proof. Let $f \in A$. Then in view of Theorems 2.1, 2.2 or 2.3, $D_\lambda^m f(z)$ is univalent in E.

Hence by Lemma 1.4, we obtain the result.

Theorem 2.5 *Let $f \in A$. If for all $z \in E$*

$$\sum_{n=1}^{\infty} n[2(n-1) + Re(v)]\phi^m(\lambda, n)|a_n| \leq Re(v), \;\; Re(v) > 0 \tag{13}$$

then

$$G_v(z) = \left(v \int_0^z u^{v-1}[D_\lambda^m f(z)]' du \right)^{\frac{1}{v}}$$

is univalent in E.

Proof. Let $f \in A$. Then for all $z \in E$,

$$\begin{aligned} \frac{1-|z|^{2Re(v)}}{Re(v)} \left| \frac{z(D_\lambda^m f(z))''}{(D_\lambda^m f(z))'} \right| &\leq \frac{1+|z|^{2Re(v)}}{Re(v)} \left| \frac{z(D_\lambda^m f(z))''}{(D_\lambda^m f(z))'} \right| \\ &\leq \frac{2\sum_{n=2}^{\infty} n(n-1)\phi^m(\lambda, n)|a_n|}{1 - \sum_{n=2}^{\infty} n\phi^m(\lambda, n)|a_n|} \end{aligned}$$

the least inequality is less than 1 if the assertion (13) is hold. Thus is view of Lemma 1.5, $G_v(z)$ is univalent in E.

3 Open problems

One can define another class by using another linear operator or an integral operator the same way as in this paper and hence new results can be obtained.

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