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A Certain Subclass of Harmonic Meromorphic Functions with Respect to k-Symmetric Points

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Abstract

In this paper, we introduce a new subclass of harmonic meromorphic functions using a new differential operator associated with q-calculus. We obtain coefficient conditions, extreme points for functions f belong to this subclass. In addition, the convolution conditions, closure and convex combinations are also obtained.

Keywords: *Harmonic functions, Meromorphic functions, q-Starlike functions, k-Symmetric points, q-Calculus.*
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1 Introduction

The field of quantum calculus, also known as q-calculus, has garnered significant attention from scholars owing to its numerous applications across various domains of mathematics and physics, with a particular emphasis on geometric function theory. The utilization of q-calculus structure amplifies the efficacy of traditional complements in diverse modules of orthogonal polynomials and functions. The linkage between equilibriums of differential formulae (including equations, operators, and inequalities) and their solutions is a highly effective and meticulously crafted mechanism for scrutinizing the attributes of special functions in the domains of mathematical analysis and mathematical physics. The field of q-calculus was first introduced by prominent mathematicians Euler and Jacobi during the 18th century. The systematic development and initiation of q-calculus was carried out by Jackson [1, 2]. Aral and Gupta ([3, 4]) introduced a q-analogue of the Baskakov and Durmeyer operator that is contingent upon quantum calculus. Aral et al. [5] and Elhaddad et al. [6] have conducted research on additional uses of the q-operator. The scholarly literature has identified the harmonic variety of q-analogues calculus in several works, including [7]-[10]. Various issues pertaining to q-calculus have been observed in recent times, as documented in ([11]-[27]). It is anticipated that the derivation of operators on q-analogues within the category of harmonic functions will become increasingly significant in the coming years.

We present some notations and concepts of q-calculus that are used in this paper. For $0 < q < 1$, $\vartheta \in \mathbb{N}$ and any non-negative integer i, the q-binomial coefficients denoted by $\mathcal{C}_q(\vartheta, i)$ is defined by (see Gasper [28]) as follows:

$$\mathcal{C}_q(\vartheta, i) = \begin{bmatrix} \vartheta \\ i \end{bmatrix}_q = \frac{[\vartheta - 1 + i]_q!}{[\vartheta]_q![i-1]_q!} = \frac{[\vartheta]_q[\vartheta-1]_q \cdots [\vartheta - i + 1]_q}{[i]_q!} \tag{1}$$

where the q-analogue of $[i]_q!$ is defined by:

$$[i]_q! = \begin{cases} [i]_q[i-1]_q \cdots [2]_q[1]_q & , i = 2, 3, 4, \cdots \\ \\ 1 & , i = 1 \ , \end{cases} \tag{2}$$

where $[i]_q$ known as the q-number, defined by

$$[i]_q = \begin{cases} \frac{1-q^i}{1-q} & , \quad \text{if} \quad 0 < q < 1, i \in \mathbb{C}\backslash\{0\} \\ 1 & , \quad \text{if} \quad q \to 0^+, i \in \mathbb{C}\backslash\{0\} \\ i & , \quad \text{if} \quad q \to 1^-, i \in \mathbb{C}\backslash\{0\} \\ 1 + q + \cdots + q^{\kappa-1} = \sum\limits_{\ell=0}^{\kappa-1} q^\ell & , \quad \text{if} \quad 0 < q < 1, i = \kappa \in \mathbb{N}. \end{cases}$$

The q-derivative, also known as the q-difference operator, of a function f is defined by

$$\partial_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{z - qz}, & \text{if} \quad 0 < q < 1, z \neq 0, \\ \\ 1, & \text{if} \quad q \to 1^-, z = 0, \\ \\ f'(z), & \text{if} \quad q \to 1^-, z \neq 0. \end{cases} ,$$

as referenced in [29].

The harmonic function is very important function in geometric function theory. The first study of harmonic functions was conducted by Clunie and Sheil Small [31], then followed by several researchers whom include Jahangiri and Silverman [32] and Yadav [34]. Aldweby and Darus [11] introduced new class of harmonic meromorphic functions depends on q-calculus. More recent work, Abdulahi and Darus [35] introduced a new class of concave meromorphic harmonic functions using integral operator and others [36, 37].

Let $\mathbb{U}^* = \{z : 0 < |z| < 1\}$ denotes the punctured unit disk in $\mathbb{C}$, and let M_H denotes the family of meromorphic harmonic functions of the form $f = h + \overline{g}$ that are univalent and sense preserving such that $|h'(z)| > |g'(z)|$ in $\mathbb{U}^*$. Subsequently, we may express the analytic functions h and g in $\mathbb{U}^*$ and $\mathbb{U} = \mathbb{U}^* \cup \{0\}$, respectively, by

$$h(z) = \frac{1}{z} + \sum_{i=1}^{\infty} a_i z^i, \quad g(z) = \sum_{i=1}^{\infty} b_i z^i, \tag{3}$$

where $h(z)$ has a simple pole at $z = 0$. Note that if $g(z) = 0$, then the class M_H is reduced to Σ, the class of meromorphic functions which are analytic in $\mathbb{U}^*$. The harmonic meromorphic starlike functions has been studied by Jahangiri and Silverman [32], and Jahangiri [33]. Many other authors followed the same steps for different classes of functions (see for example: [34] and [37]).

The authors in [13] introduced a q-differential operator $D_q^{n,\vartheta} f(z) : \Sigma \to \Sigma$ by

$$D_q^{n,\vartheta} f(z) = \frac{(-1)^n}{z} + \sum_{i=1}^{\infty} q^n [i]_q^n \mathcal{C}_q(\vartheta, i) a_i z^i, \ \ (n, \vartheta \in \mathbb{N}_0). \tag{4}$$

For $f = h + \overline{g}$ as in (3), we define the operator $D_q^{n,\vartheta} f(z) : \mathcal{M_H} \to \mathcal{M_H}$, where $D_q^{n,\vartheta} f(z)$ as in (4) by

$$D_q^{n,\vartheta} f(z) = D_q^{n,\vartheta} h(z) + (-1)^n \overline{D_q^{n,\vartheta} g(z)}, z \in \mathbb{U}^*, \tag{5}$$

where

$$D_q^{n,\vartheta} h(z) = \frac{(-1)^n}{z} + \sum_{i=1}^{\infty} q^n [i]_q^n \mathcal{C}_q(\vartheta, i) a_i z^i, \tag{6}$$

and

$$D_q^{n,\vartheta} g(z) = \sum_{i=1}^{\infty} q^n [i]_q^n \mathcal{C}_q(\vartheta, i) b_i z^i. \tag{7}$$

Remarks

- When $n = 0$ and $q \to 1^-$, then $D_q^{n,\vartheta} f(z) := I_{n,\mu} f(z)$ was introduced by Yuan et al. [38].
- When $\vartheta = 0$ and $q \to 1^-$, then $D_q^{n,\vartheta} f(z) := D^n f(z)$ was introduced by Bostanci and Oztürk [30].

It is clear that

$$\partial_q D_q^{n,\vartheta} h(z) = \frac{(-1)^{n+1}}{qz^2} + \sum_{i=1}^{\infty} q^n [i]_q^{n+1} \mathcal{C}_q(\vartheta, i) a_i z^{i-1}, \tag{8}$$

and

$$\partial_q D_q^{n,\vartheta} g(z) = \sum_{i=1}^{\infty} q^n [i]_q^{n+1} \mathcal{C}_q(\vartheta, i) b_i z^{i-1}.$$

Next, we define $\mathcal{MHS}_s^{(k)}(n, \vartheta, q, \alpha)$ as a new subfamily of harmonic meromorphic functions using $D_q^{n,\vartheta} f(z)$ as follows:

Definition 1.1 *Let* $0 \leq \alpha < 1$, $k \geq 1$ *and* $n, \vartheta \in \mathbb{N}_0$. *A function* $f = h + \overline{g}$ *given by* (3) *belong the class of meromorphic starlike function of order* α *denoted by* $\mathcal{MHS}_s^{(k)}(n, \vartheta, q, \alpha)$. *If the following inequality holds true*

$$Re\left\{\frac{-qz\partial_q(D_q^{n,\vartheta} f(z))}{D_q^{n,\vartheta} f_k(z)}\right\} \geqslant \alpha, z \in \mathbb{U}^*, \tag{9}$$

where

$$D_q^{n,\vartheta} f_k(z) = D_q^{n,\vartheta} h_k(z) + (-1)^n \overline{D_q^{n,\vartheta} g_k(z)}, k \in \mathbb{N}, \ \ z \in \mathbb{U}^*, \tag{10}$$

$$h_k(z) = \frac{(-1)^n}{z} + \sum_{i=1}^{\infty} a_i \Delta_i z^i, \ \ g_k(z) = \sum_{i=1}^{\infty} b_i \Delta_i z^i, \tag{11}$$

and

$$\Delta_i = \frac{1}{k} \sum_{\iota=0}^{k-1} \xi^{(i-1)\iota}, \quad (\xi = e^{\frac{2\pi i}{k}}, \ \ k \geq 1). \tag{12}$$

Note that when $q \to 1^-$, then we have $\mathcal{MHS}_s^{(k)}(n, \vartheta, q, \alpha) := \mathcal{MHS}_s^{(k)}(n, \vartheta, \alpha)$ which was introduced by Alshaqsi and Darus [37].

Finally, Let $\overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$ denotes the subclass of $\mathcal{MHS}_s^{(k)}(n, \vartheta, q, \alpha)$ consisting of all functions $f_n(z) = h_n + \overline{g_n}$ where h_n and g_n are given by

$$h_n(z) = \frac{(-1)^n}{z} + \sum_{i=1}^{\infty} |a_i| z^i, \ \ g_n(z) = (-1)^n \sum_{i=1}^{\infty} |b_i| z^i, \ \ a_i, b_i \geq 0, \ \ (z \in \mathbb{U}^*). \tag{13}$$

Also, let $f_{k_n}(z) = h_{k_n} + \overline{g_{k_n}}$ where h_{k_n} and g_{k_n} is given by

$$h_{k_n}(z) = \frac{(-1)^n}{z} + \sum_{i=1}^{\infty} \Delta_i |a_i| z^i, \quad g_{k_n}(z) = (-1)^n \sum_{i=1}^{\infty} \Delta_i |b_i| z^i, \tag{14}$$

where Δ_i is given by (12).

In this paper, we obtain the sufficient coefficient conditions for functions f belong to the subclasses $\mathcal{MHS}_s^{(k)}(n, \vartheta, q, \alpha)$ and $\overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$, respectively. Furthermore, the extreme points, the convolution conditions, closure and convex combinations are also obtained for the subclass $\overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$.

2 Coefficient Bounds

In this section, we determine the sufficient coefficient bound for functions f in the classes $\mathcal{MHS}_s^{(k)}(n, \vartheta, q, \alpha)$ and $\overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$, respectively.

Theorem 2.1 *For $n, \vartheta \in \mathbb{N}_0$ and $0 \leq \alpha < 1$, If $f = h + \overline{g}$ and $f_k = h_k + \overline{g_k}$ defined in (3) and (10), respectively, and satisfies the condition*

$$\sum_{i=1}^{\infty} \Big[\frac{(q[(i-1)k+1]_q + \alpha)}{1-\alpha} |a_{(i-1)k+l}| + \frac{(q[(i-1)k+1]_q - \alpha)}{1-\alpha} |b_{(i-1)k+1}| \Big] \Gamma_q + \sum_{\substack{i=2 \\ i \neq lk+1}}^{\infty} \frac{q^{n+1}[i]_q^{n+1} \mathcal{C}_q(\vartheta, i)}{1-\alpha} \Big(|a_i| + |b_i| \Big) \leq 1 - \alpha, \tag{15}$$

where $\Gamma_q = \Gamma_q(n, \vartheta, i, k) = q^n[(i-1)k+1]_q^n \mathcal{C}_q(\vartheta, ik+1)$, then f is harmonic sense-preserving, univalent in $\mathbb{U}^*$ and $f \in \mathcal{MHS}_s^{(k)}(n, \vartheta, q, \alpha)$.

Proof. For $|z| = r \in (0,1)$, we have

$$\begin{aligned} q|\partial_q h(z)| &\geq \frac{1}{|z|^2} - \sum_{i=1}^{\infty} q[i]_q |a_i||z^{i-1}| \\ &= \frac{1}{r^2} - \sum_{i=1}^{\infty} q[i]_q |a_i| r^{i-1} > 1 - \sum_{i=1}^{\infty} q[i]_q |a_i| \\ &\geq 1 - \sum_{i=1}^{\infty} \Big[(q[(i-1)k+1]_q + \alpha)|a_{(i-1)k+l}| \Big] \times \Gamma_q - \sum_{\substack{i=2 \\ i \neq lk+1}}^{\infty} q^{n+1}[i]_q^{n+1} \mathcal{C}_q(\vartheta, i)|a_i| \\ &\geq \sum_{i=1}^{\infty} \Big[(q[(i-1)k+1]_q - \alpha)|b_{(i-1)k+l}| \Big] \times \Gamma_q + \sum_{\substack{i=2 \\ i \neq lk+1}}^{\infty} q^{n+1}[i]_q^{n+1} \mathcal{C}_q(\vartheta, i)|b_i| \\ &\geq \sum_{i=1}^{\infty} q[2i]_q |b_{2i}| + \sum_{i=1}^{\infty} q[2i-1]_q |b_{2i-1}| \\ &> \sum_{i=1}^{\infty} q[i]_q |b_i| r^{i-1} = q|\partial_q g(z)|. \end{aligned}$$

Therefore, $h'(z) = \lim_{q\to 1} |q\partial_q h(z)| > \lim_{q\to 1} |q\partial_q g(z)| = g'(z)$, then f is sense-preserving in $\mathbb{U}^*$.

To show that f is univalent in $\mathbb{U}^*$, for $0 < |z_1| < |z_2| < 1$, we want to show that $\mathcal{M} = |f(z_2) - f(z_1)| > 0$.

$$\begin{aligned} \mathcal{M} &\geq \frac{|z_2 - z_1|}{|z_1||z_2|} - \sum_{i=1}^{\infty} (|a_i| + |b_i|)|z_2^i - z_1^i| \\ &\geq \frac{|z_1 - z_2|}{|z_1 z_2|} \left(1 - |z_2|^2 \sum_{i=1}^{\infty} (|a_i| + |b_i|) \frac{|{z_2}^i - {z_1}^i|}{|z_2 - z_1|} \right) \\ &\geq \frac{|z_1 - z_2|}{|z_1 z_2|} \left(1 - \sum_{i=1}^{\infty} q[i]_q (|a_j| + |b_j|) \right) \\ &\geq \frac{|z_1 - z_2|}{|z_1 z_2|} \left(1 - \sum_{i=1}^{\infty} q[i]_q (|a_j| + |b_j|) - \sum_{i=1}^{\infty} (q[(i-1)k+1]_q) \Big[|a_{(i-1)k+l}| + |b_{(i-1)k+l}| \Big] \right) \\ &\geq \frac{|z_1 - z_2|}{|z_1 z_2|} \left(1 - \sum_{i=1}^{\infty} \Big[(q[(i-1)k+1]_q + \alpha)|a_{(i-1)k+l}| \right. \\ &\quad \left. + (q[(i-1)k+1]_q - \alpha)|b_{(i-1)k+l}| \Big] \times \Gamma_q - \sum_{\substack{i=2 \\ i \neq lk+1}}^{\infty} q^{n+1}[i]_q^{n+1} \mathcal{C}_q(\vartheta, i)(|a_i| + |b_i|) \right). \end{aligned}$$

The last expression is non-negative by (15), hence f is univalent in $\mathbb{U}^*$.
In order to show that f belong to $\mathcal{MHS}_s^{(k)}(n,\vartheta,q,\alpha)$, we must show that the inequality (9) holds, that is equivalent to

$$\left|1+\alpha+\frac{qz\partial_q(D_q^{n,\vartheta}f(z))}{D_q^{n,\vartheta}f_k(z)}\right| \le \left|1-\alpha-\frac{qz\partial_q(D_q^{n,\vartheta}f(z))}{D_q^{n,\vartheta}f_k(z)}\right|. \tag{16}$$

It suffices to show that

$$\left|(1-\alpha)D_q^{n,\vartheta}f_k(z)-qz\partial_q(D_q^{n,\vartheta}f(z))\right|-\left|(1+\alpha)D_q^{n,\vartheta}f_k(z)+qz\partial_q(D_q^{n,\vartheta}f(z))\right|\ge 0. \tag{17}$$

Substitute the value of $D_q^{n,\vartheta}f_k(z)$ and $\partial_q(D_q^{n,\vartheta}f(z))$ in (17) we have

$$\begin{aligned}\Omega &= \left|(1-\alpha)D_q^{n,\vartheta}f_k(z)-qz\partial_q(D_q^{n,\vartheta}f(z))\right|-\left|(1+\alpha)D_q^{n,\vartheta}f_k(z)+qz\partial_q(D_q^{n,\vartheta}f(z))\right|\\
&\left|\frac{(-1)^n(2-\alpha)}{z}+\sum_{i=1}^{\infty}q^n[i]_q^n\mathcal{C}_q(\vartheta,i)\left[\Big((1-\alpha)\Theta_i-q[i]_q\Big)a_iz^i+(-1)^n\Big((1-\alpha)\Theta_i+q[i]_q\Big)\overline{b_iz^i}\right]\right.\\
&\left.-\frac{\alpha(-1)^n}{z}+\sum_{i=1}^{\infty}q^n[i]_q^n\mathcal{C}_q(\vartheta,i)\left[\Big((1+\alpha)\Theta_i+q[i]_q\Big)a_iz^i+(-1)^n\Big((1+\alpha)\Delta_i-q[i]_q\Big)\overline{b_iz^i}\right]\right|\\
&\ge\frac{2(1-\alpha)}{|z|}\left\{1-\sum_{i=1}^{\infty}\frac{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q+\alpha\Delta_i)}{1-\alpha}|a_i||z|^{i+1}\right.\\
&\qquad\left.-\sum_{i=1}^{\infty}\frac{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q-\alpha\Delta_i)}{1-\alpha}|b_i||z|^{i+1}\right\}.\end{aligned}$$

For $|z|=r<1$, then we have

$$\Omega\ge 2(1-\alpha)\left\{1-\sum_{i=1}^{\infty}\frac{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q+\alpha\Delta_i)}{1-\alpha}|a_i|-\sum_{i=1}^{\infty}\frac{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q-\alpha\Delta_i)}{1-\alpha}|b_i|\right\}. \tag{18}$$

From the definition of Δ_i, we know that

$$\Delta_i=\begin{cases}1 & ,i=lk+1\\ o & ,i\ne lk+1\end{cases}\qquad (i\geqslant 2,k,l\geqslant 1), \tag{19}$$

Therefor, the expression (18), become

$$\begin{aligned}\Omega\ge 2(1-\alpha)&\left\{1-\sum_{i=1}^{\infty}\Big[\frac{(q[(i-1)k+1]_q+\alpha)}{1-\alpha}|a_{(i-1)k+l}|\right.\\
&\left.+\frac{(q[(i-1)k+1]_q-\alpha)}{1-\alpha}|b_{(i-1)k+1}|\Big]\Gamma_q-\sum_{\substack{i=2\\ i\ne lk+1}}^{\infty}\frac{q^{n+1}[i]_q^{n+1}\mathcal{C}_q(\vartheta,i)}{1-\alpha}\Big(|a_i|+|b_i|\Big)\right\}.\end{aligned}$$

This expression is positive by condition (15) and this completes the proof. Next, we show that the condition (15) is necessary and sufficient condition for the functions f in the class $\overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$.

Theorem 2.2 *If $f_n = h_n + \overline{g_n}$ where h_n and g_n are of the form (12), and $f_{kn} = h_{kn}+\overline{g_{kn}}$ where h_{kn} and g_{kn} of the form (14), then $f_n \in \overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$, if and only if the condition (15) holds.*

Proof. Since $\overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha) \subset \mathcal{MHS}_s^{(k)}(n,\vartheta,q,\alpha)$ then the "if part" holds by Theorem 2.1 It is enough to prove the 'only if' part.
We assume the condition (15) does not hold, since $f_n \in \overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$ then

$$Re\left\{\frac{-qz\partial_q(D_q^{n,\vartheta}f_n(z))}{D_q^{n,\vartheta}f_{k_n}(z)}\right\} \geqslant \alpha, \qquad (z\in\mathbb{U}^*)\,.$$

This equivalent to

$$\frac{\frac{1-\alpha}{r} - \sum_{i=1}^{\infty} q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q+\alpha\Delta_i)|a_i|r^i + \sum_{i=1}^{\infty} q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q-\alpha\Delta_i)|b_i|r^i}{\frac{1}{r} + \sum_{i=1}^{\infty} q^n[i]_q^n\mathcal{C}_q(\vartheta,i)\Delta_i|a_i|r^i + \sum_{i=1}^{\infty} q^n[i]_q^n\mathcal{C}_q(\vartheta,i)\Delta_i|b_i|z^i}. \tag{20}$$

For sufficiently r close to 1^-, then the numerator of last equation is negative. This meaning there exist $r_1 \in (0,1)$ for which (20) is negative and this is contradicts with assumption for $f_n \in \overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$ and this complete the proof.

3 Extreme points and distortion bounds

Throughout this section, we provide extreme points and obtain distortion bounds for the class $\overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$.

Theorem 3.1 *For $|z| = r \in (0,1)$ and if $f_n = h_n+\overline{g_n} \in \overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$, then*

$$\frac{1}{r} - \frac{r(1-\alpha)}{q^n[2]_q^n[\vartheta+1]_q(q[2]_q-\alpha)} \leq \left|f_n(z)\right| \leq \frac{1}{r} + \frac{r(1-\alpha)}{q^n[2]_q^n[\vartheta+1]_q(q[2]_q-\alpha)}. \tag{21}$$

Proof. It is enough to prove the right sides, we omit the proof of the left side because it is similar to the right. Let $f_n = h_n + \overline{g_n} \in \overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$.

Taking the absolute value of f_n, we have

$$\begin{aligned}
\Big|f_n(z)\Big| &= \left|\frac{1}{z} + \sum_{i=1}^{\infty} a_i z^i + (-1)^n \sum_{i=1}^{\infty} \overline{b_i z^i}\right| \\
&\leq \frac{1}{r} + \sum_{i=1}^{\infty} |a_i + b_i| r^i \leq \frac{1}{r} + r \sum_{i=1}^{\infty} |a_i + b_i| \\
&\leq \frac{1}{r} + \frac{r(1-\alpha)}{q^n[2]_q^n[\vartheta+1]_q(q[2]_q - \alpha\Delta_2)} \sum_{i=1}^{\infty} \frac{q^n[2]_q^n[\vartheta+1]_q(q[2]_q - \alpha\Delta_2)}{1-\alpha}\Big|a_i + b_i\Big| \\
&\leq \frac{1}{r} + \frac{r(1-\alpha)}{q^n[2]_q^n[\vartheta+1]_q(q[2]_q - \alpha)} \sum_{i=1}^{\infty} \Bigg(\frac{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q + \alpha\Delta_i)}{1-\alpha}\Big|a_i\Big| \\
&\qquad + \frac{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q - \alpha\Delta_i)}{1-\alpha}\Big|b_i\Big|\Bigg) \\
&\leq \frac{1}{r} + \frac{r(1-\alpha)}{q^n[2]_q^n[\vartheta+1]_q(q[2]_q - \alpha)}.
\end{aligned}$$

Thus, the proof is complete.

Corollary 3.2 *If $f_n = h_n + \overline{g_n} \in \overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$ then*

$$f_n(\mathbb{U}^*) \subset \left\{\frac{q^n[2]_q^n[\vartheta+1]_q(q[2]_q - \alpha) - (1-\alpha)}{q^n[2]_q^n[\vartheta+1]_q(q[2]_q - \alpha)}\right\}. \tag{22}$$

Next, we provide the extreme points of the class of the closed convex halls of $\overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$ denoted by $clco\overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$.

Theorem 3.3 *Let $f_n = h_n + \overline{g_n}$ where h_n and g_n are of the form (13), then $f_n \in \overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$ if and only if f_{n_i} can be written as*

$$f_{n_i} = \sum_{i=0}^{\infty} \Big(\Phi_i h_{n_i}(z) + \Psi_i g_{n_i}(z)\Big)$$

where $\sum_{i=0}^{\infty} \Phi_i + \Psi_i = 1$, $(\Phi_i, \Psi_i \geq 0)$ and

$$h_{n_0} = \frac{(-1)^n}{z},\ \ h_{n_i} = \frac{(-1)^n}{z} + \Big(\frac{1-\alpha}{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q + \alpha\Delta_i)}\Big)z^i, \quad i = 1,2,3,\cdots,$$

$$g_{n_0} = \frac{(-1)^n}{z},\ \ g_{n_i} = \frac{(-1)^n}{z} + (-1)^n\Big(\frac{1-\alpha}{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q - \alpha\Delta_i)}\Big)\overline{z^i}, \quad i = 1,2,3,\cdots,$$

where h_{n_i} and g_{n_i} are the extreme points of $\overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$.

Proof. Let $f_n \in clco\overline{\mathcal{MHS}_s^{(k)}}(n,\vartheta,q,\alpha)$. Set for $i = 1, 2, 3, \cdots$

$$\Phi_i = \frac{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q+\alpha\Delta_i)}{1-\alpha}|a_i|, \quad 0 \le \Phi_i \le 1,$$

and

$$\Psi_i = \frac{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q-\alpha\Delta_i)}{1-\alpha}|b_i|, \quad 0 \le \Psi_i \le 1. \tag{23}$$

Then f_{n_i} can be written as

$$\begin{aligned}
f_{n_i} &= \frac{(-1)^n}{z} + \sum_{i=1}^{\infty}|a_i|z^i + (-1)^n\sum_{i=1}^{\infty}|b_i|\overline{z^i} \\
&= \frac{(-1)^n}{z} + \sum_{i=1}^{\infty}\frac{(1-\alpha)\Phi_i}{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q+\alpha\Delta_i)}z^i + (-1)^n\sum_{i=1}^{\infty}\frac{(1-\alpha)\Psi_i}{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q-\alpha\Delta_i)}\overline{z^i} \\
&= \sum_{i=1}^{\infty}\left(\Phi_i h_{n_i} + \Psi_i g_{n_i}\right) + \frac{(-1)^n}{z}\left(1 - \sum_{i=1}^{\infty}\left(\Phi_i + \Psi_i\right)\right) \\
&= \sum_{i=1}^{\infty}\left(\Phi_i h_{n_i} + \Psi_i g_{n_i}\right) + \frac{(-1)^n}{z}\left(\Phi_0 + \Psi_0\right) = \sum_{i=0}^{\infty}\left(\Phi_i h_{n_i} + \Psi_i g_{n_i}\right).
\end{aligned}$$

Conversely, for $f_n = h_n + \overline{g_n}$ as in (13), we have

$$\begin{aligned}
f_{n_i} &= \sum_{i=0}^{\infty}\Big(\Phi_i h_{n_i}(z) + \Psi_i g_{n_i}(z)\Big) \\
&= \Phi_0 h_{n_0}(z) + \Psi_0 g_{n_0}(z) + \sum_{i=1}^{\infty}\Big(\Phi_i h_{n_i}(z) + \Psi_i g_{n_i}(z)\Big) \\
&= (\Phi_0+\Psi_0)\frac{(-1)^n}{z} + \sum_{i=1}^{\infty}\frac{(-1)^n}{z}\Phi_i + \sum_{i=1}^{\infty}\left(\frac{1-\alpha}{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q+\alpha\Delta_i)}\right)\Phi_i z^i \\
&+ \sum_{i=1}^{\infty}\frac{(-1)^n}{z}\Psi_i + (-1)^n\sum_{i=1}^{\infty}\left(\frac{1-\alpha}{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q-\alpha\Delta_i)}\right)\Psi_i\overline{z^i} \\
&= \frac{(-1)^n}{z} + \sum_{i=1}^{\infty}\left(\frac{1-\alpha}{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q+\alpha\Delta_i)}\right)\Phi_i z^i \\
&\qquad + (-1)^n\sum_{i=1}^{\infty}\left(\frac{1-\alpha}{q^n[i]_q^n\mathcal{C}_q(\vartheta,i)(q[i]_q-\alpha\Delta_i)}\right)\Psi_i\overline{z^i}.
\end{aligned}$$

since $f_{n_i} \in \overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$, and applying Theorem 2.2, we have

$$\sum_{i=1}^{\infty} \left(q^n [i]_q^n \mathcal{C}_q(\vartheta, i)(q[i]_q + \alpha\Delta_i) \frac{1-\alpha}{q^n [i]_q^n \mathcal{C}_q(\vartheta, i)(q[i]_q + \alpha\Delta_i)} \right) \Phi_i$$
$$+ \sum_{i=1}^{\infty} \left(q^n [i]_q^n \mathcal{C}_q(\vartheta, i)(q[i]_q - \alpha\Delta_i) \frac{1-\alpha}{q^n [i]_q^n \mathcal{C}_q(\vartheta, i)(q[i]_q - \alpha\Delta_i)} \right) \Psi_i$$
$$= (1-\alpha) \sum_{i=1}^{\infty} \Phi_i + \Psi_i \leq 1 - \alpha.$$

4 Convolutions and Convex combinations

In this section, we show that the class $\overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$ is closed under convolution and convex combination of its member. For the harmonic meromorphic functions

$$f_n(z) = \frac{(-1)^n}{z} + \sum_{i=1}^{\infty} |a_i| z^i + (-1)^n \sum_{i=1}^{\infty} |b_i| \overline{z^i}$$

and

$$\beta_n(z) = \frac{(-1)^n}{z} + \sum_{i=1}^{\infty} |\eta_i| z^i + (-1)^n \sum_{i=1}^{\infty} |\nu_i| \overline{z^i},$$

then, the Hadamard product (or convolution) of $f_n(z)$ and $\beta_n(z)$ is given by

$$(f_n * \beta_n)(z) = (\beta_n * f_n)(z) = \frac{(-1)^n}{z} + \sum_{j=1}^{\infty} |a_i||\eta_i| z^i + (-1)^n \sum_{i=1}^{\infty} |b_i||\nu_i| \overline{z^i}.$$

Theorem 4.1 *For $0 \leq \gamma \leq \alpha < 1$, let $f \in \overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$ and $\beta \in \overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \gamma)$, then $f_n * \beta_n \in \overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha) \subset \overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \gamma)$.*

Proof. It is enough to show that $f_n * \beta_n$ satisfies the condition of Theorem 2.2. Since $\beta_n \in \overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$ and $|\eta_i| \leq 1$, $|\nu_i| \leq 1$, we have

$$\sum_{i=1}^{\infty} \Big(\frac{q^n [i]_q^n \mathcal{C}_q(\vartheta, i)(q[i]_q + \alpha\Delta_i)}{1-\gamma} \Big) |a_i||\eta_i| + \sum_{i=1}^{\infty} \Big(\frac{q^n [i]_q^n \mathcal{C}_q(\vartheta, i)(q[i]_q - \alpha\Delta_i)}{1-\gamma} \Big) |b_i||\nu_i|$$
$$\leq \sum_{i=1}^{\infty} \Big(\frac{q^n [i]_q^n \mathcal{C}_q(\vartheta, i)(q[i]_q + \alpha\Delta_i)}{1-\gamma} \Big) |a_i| + \sum_{i=1}^{\infty} \Big(\frac{q^n [i]_q^n \mathcal{C}_q(\vartheta, i)(q[i]_q - \alpha\Delta_i)}{1-\gamma} \Big) |b_i|$$
$$\leq \sum_{i=1}^{\infty} \Big(\frac{q^n [i]_q^n \mathcal{C}_q(\vartheta, i)(q[i]_q + \alpha\Delta_i)}{1-\alpha} \Big) |a_i| + \sum_{i=1}^{\infty} \Big(\frac{q^n [i]_q^n \mathcal{C}_q(\vartheta, i)(q[i]_q - \alpha\Delta_i)}{1-\alpha} \Big) |b_i|$$
$$\leq 1 - \alpha,$$

for $0 \leq \gamma \leq \alpha < 1$ and $f_n \in \overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$, therefore, $f_n * \beta_n \in \overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha) \subset \overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \gamma)$.

In the last theorem, we examine the convex combination of the class $\overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$.

Theorem 4.2 *let* $f_{n_\chi} \in \overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$ *for every* $\chi = 1, 2, \cdots, \varrho$, *where* f_{n_χ} *defined by*

$$f_{n_\chi} = \frac{(-1)^n}{z} + \sum_{i=1}^{\infty} |a_{n_\chi}| z^i + (-1)^n \sum_{i=1}^{\infty} |b_{n_\chi}| \overline{z^i}, \chi = 1, 2, \cdots, \varrho. \tag{24}$$

Then, the function

$$\Omega_\chi(z) = \sum_{\chi=1}^{\varrho} \zeta_\chi f_{n_\chi}, \quad ,(0 \leq \zeta_\chi \leq 1), \tag{25}$$

are also in the class $\overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$, *where* $\sum_{\chi=1}^{\varrho} \zeta_\chi = 1$.

Proof. According to the given $\Omega_\chi(z)$, we have

$$\begin{aligned} \Omega_\chi(z) &= \sum_{\chi=1}^{\varrho} \zeta_\chi f_{n_\chi} \\ &= \frac{(-1)^n}{z} + \sum_{i=1}^{\infty} \left[\sum_{\chi=1}^{\varrho} \zeta_\chi a_{n_\chi} \right] z^i + (-1)^n \sum_{i=1}^{\infty} \left[\sum_{\chi=1}^{\varrho} \zeta_\chi b_{n_\chi} \right] \overline{z^i}. \end{aligned} \tag{26}$$

For every $\chi = 1, 2, \cdots, \varrho$, we have $f_{n_\chi} \in \overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$, then by (18), we get

$$\begin{aligned} &\sum_{i=1}^{\infty} \frac{q^n [i]_q^n \mathcal{C}_q(\vartheta, i)}{1-\alpha} \left[(q[i]_q + \alpha\Delta_i) \Big(\sum_{\chi=1}^{\varrho} \zeta_\chi a_{n_\chi} \Big) + (q[i]_q - \alpha\Delta_i) \Big(\sum_{\chi=1}^{\varrho} \zeta_\chi b_{n_\chi} \Big) \right] \\ &= \sum_{\chi=1}^{\varrho} \zeta_\chi \left[\sum_{i=1}^{\infty} \frac{q^n [i]_q^n \mathcal{C}_q(\vartheta, i)}{1-\alpha} \left\{ (q[i]_q + \alpha\Delta_i) a_{n_\chi} + (q[i]_q - \alpha\Delta_i) b_{n_\chi} \right\} \right] \\ &\leq \sum_{\chi=1}^{\varrho} \zeta_\chi (1) \leq 1. \end{aligned}$$

Hence, the proof is complete.

Corollary 4.3 *The class* $\overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$ *is closed under convex linear combination.*

Proof. Let the functions $f_{n_\chi}(z)$, $(\chi = 1, 2)$ defined by (24) be in the class $\overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$, then the function $\Upsilon(z)$ defined by

$$\Upsilon(z) = \rho f_{n_1}(z) + (1-\rho) f_{n_2}(z), \quad (0 \leq \rho \leq 1), \tag{27}$$

is in the class $\overline{\mathcal{MHS}_s^{(k)}}(n, \vartheta, q, \alpha)$. Also, by taking $\varrho = 2$, $\chi_1 = \rho$ and $\chi_2 = 1-\rho$ in Theorem 4.2, so the proof is complete.

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