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Upper Bounds for Initial Taylor-Maclaurin Coefficients of New Families of Bi-Univalent Functions

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Abstract

In the current study, firstly, two new families $\mathcal{B}_\Sigma(\lambda;\mu)$ and $\mathcal{B}^*_\Sigma(\lambda;\nu)$ of normalized holomorphic and bi-univalent functions are defined. Furthermore, upper bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in each of these families are acquired.

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1 Introduction

We indicate by $\mathcal{A}$ the family of functions which are holomorphic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and have the following normalized form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

We also indicate by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. According to the Koebe one-quarter theorem [6], every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \quad (2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ stand for the class of normalized bi-univalent functions in $\mathbb{U}$ given by (1). For a brief historical account and for several interesting examples of functions in the class Σ, see the pioneering work on this subject by Srivastava *et al.* [16], which actually revived the study of bi-univalent functions in recent years. From the work of Srivastava *et al.* [16], we choose to recall here the following examples of functions in the class Σ :

$$\frac{z}{1-z}, \quad -\log(1-z) \qquad \text{and} \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

We notice that the class Σ is not empty. However, the Koebe function is not a member of Σ.

Recently, many authors obtained the estimates on the initial coefficients in the Taylor-Maclaurin expansion (1) for several different subclasses of the bi-univalent function class Σ (see, for example, [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21]). The problem to find the general coefficient bounds on the Taylor-Maclaurin coefficients

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1, 2\};\ \mathbb{N} := \{1, 2, 3, \cdots\})$$

for functions $f \in \Sigma$ is still not completely addressed for many of the subclasses of the bi-univalent function class Σ.

We now recall the following lemma that will be used to prove our main results.

Lemma 1.1 (see [6]) *If $h \in \mathcal{P}$, then*

$$|c_k| \leqq 2 \qquad (\forall\ k \in \mathbb{N}),$$

where $\mathcal{P}$ is the family of all functions h, holomorphic in $\mathbb{U}$, for which

$$\Re\big(h(z)\big) > 0 \qquad (z \in \mathbb{U})$$

with

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

2 Coefficient Estimates for the Family $\mathcal{B}_\Sigma(\lambda;\mu)$

Definition 2.1 A function $f \in \Sigma$, given by (1), is said to be in the family $\mathcal{B}_\Sigma(\lambda;\mu)$ if it satisfies the following conditions:

$$\left|\arg\left(1+\frac{z^{2-\lambda}f''(z)}{(zf'(z))^{1-\lambda}}\right)\right| < \frac{\mu\pi}{2} \tag{3}$$

and

$$\left|\arg\left(1+\frac{w^{2-\lambda}g''(w)}{(wg'(w))^{1-\lambda}}\right)\right| < \frac{\mu\pi}{2}, \tag{4}$$

where $z, w \in \mathbb{U},\ \ 0 < \mu \leqq 1,\ \ 0 \leqq \lambda \leqq 1$ and $g = f^{-1}$ is given by (2).

Theorem 2.2 Let the function $f \in \mathcal{B}_\Sigma(\lambda;\mu)$ $(0 < \mu \leqq 1; 0 \leqq \lambda \leqq 1)$ be given by (1). Then

$$|a_2| \leqq \frac{\mu}{\sqrt{|\mu(2\lambda+1)+(1-\mu)|}}$$

and

$$|a_3| \leqq \mu^2 + \frac{\mu}{3}.$$

Proof. In light of the conditions (3) and (4), we have

$$1+\frac{z^{2-\lambda}f''(z)}{(zf'(z))^{1-\lambda}} = [p(z)]^\mu \tag{5}$$

and

$$1+\frac{w^{2-\lambda}g''(w)}{(wg'(w))^{1-\lambda}} = [q(w)]^\mu, \tag{6}$$

where $g = f^{-1}$ and the functions $p, q \in \mathcal{P}$ have the following series representations:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{7}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \tag{8}$$

By comparing the corresponding coefficients of (5) and (6), we find that

$$2a_2 = \mu p_1, \tag{9}$$

$$6a_3 - 4(1-\lambda)a_2^2 = \mu p_2 + \frac{\mu(\mu-1)}{2}\,p_1^2, \tag{10}$$

$$-2a_2 = \mu q_1 \tag{11}$$

and

$$4(\lambda+2)a_2^2 - 6a_3 = \mu q_2 + \frac{\mu(\mu-1)}{2}\, q_1^2. \tag{12}$$

Thus, by using (9) and (11), we conclude that

$$p_1 = -q_1 \tag{13}$$

and

$$8a_2^2 = \mu^2(p_1^2 + q_1^2). \tag{14}$$

If we add (10) to (12), we obtain

$$4(2\lambda+1)a_2^2 = \mu(p_2+q_2) + \frac{\mu(\mu-1)}{2}\, \left(p_1^2+q_1^2\right). \tag{15}$$

Substituting the value of $p_1^2 + q_1^2$ from (14) into the right-hand side of (15), and after some computations, we deduce that

$$a_2^2 = \frac{\mu^2(p_2+q_2)}{4\left[\mu(2\lambda+1)+(1-\mu)\right]}, \tag{16}$$

By taking the moduli of both sides of (16) and applying Lemma 1.1 for the coefficients p_2 and q_2, we have

$$|a_2| \leqq \frac{\mu}{\sqrt{|\mu(2\lambda+1)+(1-\mu)|}}.$$

Next, in order to determinate the bound on $|a_3|$, by subtracting (12) from (10), we get

$$12\left(a_3 - a_2^2\right) = \mu\left(p_2 - q_2\right) + \frac{\mu(\mu-1)}{2}\left(p_1^2 - q_1^2\right). \tag{17}$$

Now, upon substituting the value of a_2^2 from (14) into (17) and using (13), we deduce that

$$a_3 = \frac{\mu^2\left(p_1^2+q_1^2\right)}{8} + \frac{\mu\left(p_2-q_2\right)}{12}. \tag{18}$$

Finally, by taking the moduli on both sides of (18) and applying the Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, it follows that

$$|a_3| \leqq \mu^2 + \frac{\mu}{3}.$$

This completes the proof of Theorem 2.2.

3 Coefficient Estimates for the Family $\mathcal{B}^*_\Sigma(\lambda;\nu)$

Definition 3.1 A function $f \in \Sigma$, given by (1), is said to be in the family $\mathcal{B}^*_\Sigma(\lambda;\nu)$ if it satisfies the following conditions:

$$\Re\left\{1+\frac{z^{2-\lambda}f''(z)}{\left(zf'(z)\right)^{1-\lambda}}\right\}>\nu \tag{19}$$

and

$$\Re\left\{1+\frac{w^{2-\lambda}g''(w)}{\left(wg'(w)\right)^{1-\lambda}}\right\}>\nu, \tag{20}$$

where $z, w \in \mathbb{U},\ \ 0 \leqq \nu < 1,\ \ 0 \leqq \lambda \leqq 1$ and $g = f^{-1}$ is given by (2).

Theorem 3.2 Let the $f \in \mathcal{B}^*_\Sigma(\lambda;\nu)$ $(0 \leqq \nu < 1; 0 \leqq \lambda \leqq 1)$ be given by (1). Then

$$|a_2| \leqq \sqrt{\frac{1-\nu}{2\lambda+1}}$$

and

$$|a_3| \leqq (1-\nu)^2 + \frac{(1-\nu)}{3}.$$

Proof. In view of the conditions (19) and (20), there exist the functions $p, q \in \mathcal{P}$ such that

$$1+\frac{z^{2-\lambda}f''(z)}{\left(zf'(z)\right)^{1-\lambda}} = \nu + (1-\nu)p(z) \tag{21}$$

and

$$1+\frac{w^{2-\lambda}g''(w)}{\left(wg'(w)\right)^{1-\lambda}} = \nu + (1-\nu)q(w), \tag{22}$$

where $g = f^{-1}$ and the functions $p, q \in \mathcal{P}$ have the series expansions given by (7) and (8), respectively. Thus, by comparing the corresponding coefficients in (21) and (22), we get

$$2a_2 = (1-\nu)p_1, \tag{23}$$

$$6a_3 - 4(1-\lambda)a_2^2 = (1-\nu)p_2, \tag{24}$$

$$-2a_2 = (1-\nu)q_1 \tag{25}$$

and

$$4(\lambda+2)a_2^2 - 6a_3 = (1-\nu)q_2. \tag{26}$$

We now find from (23) and (25) that

$$p_1 = -q_1 \tag{27}$$

and

$$8a_2^2 = (1-\nu)^2 \left(p_1^2 + q_1^2\right). \tag{28}$$

By adding (24) and (26), we obtain

$$4(2\lambda+1)a_2^2 = (1-\nu)(p_2+q_2). \tag{29}$$

Consequently, we have

$$a_2^2 = \frac{(1-\nu)(p_2+q_2)}{4(2\lambda+1)},$$

Next, by applying Lemma 1.1 for the coefficients p_2 and q_2, we deduce that

$$|a_2| \leqq \sqrt{\frac{1-\nu}{2\lambda+1}}.$$

In order to determine the bound on $|a_3|$, by subtracting (26) from (24), we get

$$12\left(a_3 - a_2^2\right) = (1-\nu)\left(p_2 - q_2\right),$$

or, equivalently,

$$a_3 = a_2^2 + \frac{(1-\nu)\left(p_2-q_2\right)}{12}. \tag{30}$$

Substituting the value of a_2^2 from (28) into (30), it follows that

$$a_3 = \frac{(1-\nu)^2\left(p_1^2+q_1^2\right)}{8} + \frac{(1-\nu)\left(p_2-q_2\right)}{12}.$$

Finally, by applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \leqq (1-\nu)^2 + \frac{(1-\nu)}{3}.$$

We have thus completed the proof of Theorem 3.2.

4 Conclusion

In the present paper we defined two new families $\mathcal{B}_\Sigma(\lambda;\mu)$ and $\mathcal{B}_\Sigma^*(\lambda;\nu)$ of normalized holomorphic and bi-univalent functions and generated Taylor-Maclaurin coefficient inequalities for functions belonging to these families.

5 Open Problem

In this current work, two new families $\mathcal{B}_\Sigma(\lambda;\mu)$ and $\mathcal{B}_\Sigma^*(\lambda;\nu)$ of bi-starlike functions was determined. Furthermore, coefficients $|a_2|$, $|a_3|$ inequalities were obtained separately for both families by using Lemma 1.1. As an open problem, we hope that this work encourage the researchers to obtain other coefficient inequalities using by different polynomials and subordination method for these defined subfamilies.

Conflict of interest. All authors declare that there is not any conflict of interests concerning the publication of this manuscript.

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