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Consolidate a certain class of $(p, q)-$Lucas polynomial based bi- univalent functions with a specific discrete probability distribution

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Abstract

Using subordination conditions between the zero-truncated Poisson distribution and the $(p, q)-$Lucas polynomial, we introduce and examine a new subclass of analytical bi-univalent functions. For functions falling within this new subclass, we will more precisely estimate the first two initial Taylor-Maclaurin coefficients and resolve the Fekete-Szegö functional problem.

Keywords: $(p, q)-$*Lucas polynomial, Analytic functions, zero-truncated Poisson distribution, bi-univalent functions, Fekete-Szegö problem.*
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1 Introduction

Orthogonal polynomials (OP) were initially discovered by Legendre in 1784 [1]. Ordinary differential equations are typically solved using (OP) when certain model constraints are met. Additionally, the (OP) [2] serve a significant role in the approximation theory.

Φ_d and Φ_t are two polynomials of order d and t, respectively, and are orthogonal if

$$\int_a^b \Phi_d(x)\Phi_t(x)\varpi(x)dx = 0, \qquad \text{for} \qquad d \neq t,$$

where $\varpi(x)$ is a properly stated function in the (a,b); as a result, the integral of all finite order polynomials $\Phi_n(x)$ is well defined.

Let $\mathcal{A}$ be the class of functions f of the form

$$f(\xi) = \xi + a_2\xi^2 + a_3\xi^3 + \cdots, \quad (\xi \in \mathbb{U}). \tag{1}$$

which are analytic in the disk $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and gratify the normalization condition $f'(0) - 1 = 0 = f(0)$. Also, we represent by $\mathcal{S}$ the subclass of $\mathcal{A}$ comprising functions of the Eq. (1) which are also univalent in $\mathfrak{U}$.

Geometric function theory can benefit greatly from the powerful tools that differential subordination of analytical functions provides. Miller and Mocanu [3] introduced the first differential subordination problem, additionally, see [4]. The majority of the developments in the field are compiled in Miller and Mocanu's book [5], along with references to the publication date.

Every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(\xi)) = \xi \qquad (\xi \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4})$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function is said to be bi-univalent in $\mathbb{U}$ if both $f(\xi)$ and $f^{-1}(\xi)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For interesting examples of functions in the class Σ, see([6]-[13]).

In probability theory, the zero-truncated Poisson distribution is a certain discrete probability distribution whose support is the set of positive integers, that is, a Poisson distribution with eliminating the random variable zero [14]. This distribution is also known as the conditional Poisson distribution [15] or the positive Poisson distribution [16]. The probability density function of the zero-truncated Poisson distribution is given by

$$P_m(X = s) = \frac{m^s}{(e^m - 1)s!}, \qquad s = 1, 2, 3, ..., m > 0.$$

Here, let us consider a power series whose coefficients are probabilities of the zero-truncated Poisson distribution, that is

$$\mathbb{P}(m, \xi) = \xi + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(e^m - 1)(n-1)!}\xi^n,\ \xi \in \mathbb{U}, \tag{3}$$

where $m>0$. By ratio test the radius of convergence of this series is infinity.

Define the linear operator $\chi:\mathcal{A}\to\mathcal{A}$ by

$$\begin{aligned}\chi_m f(\xi) &= \mathbb{P}(m,\xi)*f(\xi)=\xi+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(e^m-1)(n-1)!}a_n\xi^n, \qquad \xi\in\mathcal{U}, \quad (4)\\ &=\xi+\frac{m}{(e^m-1)}a_2\xi^2+\frac{m^2}{2(e^m-1)}a_3\xi^3+\cdots\end{aligned}$$

where $*$ denote the convolution or Hadamard product of two series, see [17].

Orthogonal polynomials have been studied extensively as early as they were discovered by Legendre in 1784 [18]. In mathematical treatment of model problems, orthogonal polynomials arise often to find solutions of ordinary differential equations under certain conditions imposed by the model. The importance of the orthogonal polynomials for the contemporary mathematics, as well as for wide range of their applications in the physics and engineering, is beyond any doubt. Recently, many researchers have been exploring bi-univalent functions associated with orthogonal polynomials, few to mention ([19]-[38]).

Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The $(p,q)-$Lucas polynomials $\mathcal{L}_{p,q,n}(x)$ are defined by the recurrence relation

$$\mathcal{L}_{p,q,n}(x)=p(x)\mathcal{L}_{p,q,n-1}(x)+q(x)\mathcal{L}_{p,q,n-2}(x) \quad (n\geq 2)\,,$$

from which the first few Lucas polynomials can be found as

$$\mathcal{L}_{p,q,0}(x)=2,\ \mathcal{L}_{p,q,1}(x)=p(x) \text{ and } \mathcal{L}_{p,q,2}(x)=p^2(x)+2q(x). \tag{5}$$

Remark 1.1 *We note that Lucas polynomials $\mathcal{L}_{x,1,n}(x)\equiv\mathcal{L}_n(x)$, Pell–Lucas polynomials $\mathcal{L}_{2x,1,n}(x)\equiv D_n(x)$, Jacobsthal–Lucas polynomials $\mathcal{L}_{1,2x,n}(x)\equiv j_n(x)$ and Chebyshev polynomials first kind $\mathcal{L}_{2x,-1,n}(x)\equiv T_n(x)$, are special cases of the $(p,q)-$Lucas polynomial.*

Lemma 1.2 (16) *Let $G\{\mathcal{L}(x)\}(\zeta)$ be the generating function of the $(p,q)-$Lucas polynomial sequence $\mathcal{L}_{p,q,n}(x)$. Then,*

$$G\{\mathcal{L}(x)\}(\zeta)=\sum_{n=0}^{\infty}\mathcal{L}_{p,q,n}(x)\zeta^n=\frac{2-p(x)\zeta}{1-p(x)\zeta-q(x)\zeta^2}$$

and

$$G_{\{\mathcal{L}(x)\}}(\zeta)=G\{\mathcal{L}(x)\}(\zeta)-1=1+\sum_{n=1}^{\infty}\mathcal{L}_{p,q,n}(x)\zeta^n=\frac{1+q(x)\zeta^2}{1-p(x)\zeta-q(x)\zeta^2}.$$

The generator of the Lucas polynomials $G_{\{\mathcal{L}(x)\}}(\zeta)$ is as follows:

$$G_{\{\mathcal{L}(x)\}}(\zeta) = \frac{1+q(x)\zeta^2}{1-p(x)\zeta-q(x)\zeta^2}. \tag{6}$$

2 The class $\zeta_\Sigma(x,\alpha,\mu)$

In this section, we introduce a new subclass of Σ involving the new constructed series (3) and Gegenbauer polynomials.

Definition 2.1 *A function $f \in \Sigma$ given by (1) is said to be in the class $\zeta_\Sigma(x,\alpha,\mu)$ if the following subordinations are satisfied:*

$$(1-\mu)\frac{\chi_m f(\xi)}{\xi} + \mu\left(\chi_m f(\xi)\right)' \prec G_{\{\mathcal{L}(x)\}}(\zeta), \tag{7}$$

and

$$(1-\mu)\frac{\chi_m g(w)}{w} + \mu\left(\chi_m g(w)\right)' \prec G_{\{\mathcal{L}(x)\}}(w), \tag{8}$$

where $\mu \geq 0$ and the function $h = f^{-1}$ is given by (2).

Example 2.2 *If $\mu = 1$, then we have, $\zeta_\Sigma(x,\alpha,1) = \zeta_\Sigma(x,\alpha)$, in which $\zeta_\Sigma(x,\alpha)$ denotes the class of functions $f \in \Sigma$ given by (1) and satisfying the following conditions.*

$$\left(\chi_m f(\xi)\right)' \prec G_{\{\mathcal{L}(x)\}}(\zeta),$$

and

$$\left(\chi_m g(w)\right)' \prec G_{\{\mathcal{L}(x)\}}(w),$$

where $\mu \geq 0$ and the function $h = f^{-1}$ is given by (2).

Example 2.3 *If $\mu = 0$, then we have, $\zeta_\Sigma(x,\alpha,0) = \zeta_\Sigma(x,\alpha,0)$, in which $\zeta_\Sigma(x,\alpha,0)$ denotes the class of functions $f \in \Sigma$ given by (1) and satisfying the following conditions.*

$$\frac{\chi_m f(\xi)}{\xi} \prec G_{\{\mathcal{L}(x)\}}(\zeta),$$

and

$$\frac{\chi_m g(w)}{w} \prec G_{\{\mathcal{L}(x)\}}(w),$$

where $\mu \geq 0$ and the function $h = f^{-1}$ is given by (2).

3 Estimates of the class $\zeta_{\Sigma}(x,\alpha,\mu)$

First, we give the coefficient estimates for the class $\zeta_{\Sigma}(x,\alpha,\mu)$ given in Definition 2.1.

Theorem 3.1 *Let $f\in\Sigma$ given by (1) belongs to the class $\zeta_{\Sigma}(x,\alpha,\mu)$. Then*

$$|a_2| \leq \frac{2p(x)(e^m-1)\sqrt{p(x)}}{m\sqrt{\left|\left[(1+2\mu)(e^m-1)-2(1+\mu)^2\right]p^2(x)-4(1+\mu)^2q(x)\right|}},$$

and

$$|a_3| \leq \frac{(e^m-1)^2\left[p(x)\right]^2}{m^2(1+\mu)^2}+\frac{(e^m-1)p(x)}{m^2(1+2\mu)}.$$

proof 3.2 *Let $f\in\zeta_{\Sigma}(x,\alpha,\mu)$. From Definition 2.1, for some analytic functions w, v such that $w(0)=v(0)=0$ and $|w(\xi)|<1$, $|v(w)|<1$ for all $\xi,w\in\mathbb{U}$, then we can write*

$$(1-\mu)\frac{\chi_m f(\xi)}{\xi}+\mu\left(\chi_m f(\xi)\right)'=G_{\{\mathcal{L}(x)\}}(u(\zeta)) \tag{9}$$

and

$$(1-\mu)\frac{\chi_m g(w)}{w}+\mu\left(\chi_m g(w)\right)'=G_{\{\mathcal{L}(x)\}}(v(w)). \tag{10}$$

From the equalities (9) and (10), we obtain that

$$(1-\mu)\frac{\chi_m f(\xi)}{\xi}+\mu\left(\chi_m f(\xi)\right)'=1+\mathcal{L}_{p,q,1}(x)u_1\zeta+\mathcal{L}_{p,q,2}(x)u_2\zeta^2+\cdots \tag{11}$$

and

$$(1-\mu)\frac{\chi_m g(w)}{w}+\mu\left(\chi_m g(w)\right)'=1+\mathcal{L}_{p,q,1}(x)v_1 w+\mathcal{L}_{p,q,2}(x)v_2 w^2+\cdots. \tag{12}$$

It is fairly well known that if

$$|u(\zeta)|=\left|u_1\zeta+u_2\zeta^2+u_3\zeta^3+\cdots\right|<1,\quad(\xi\in\mathbb{U})$$

and

$$|v(w)|=\left|v_1 w+v_2 w^2+v_3 w^3+\cdots\right|<1,\quad(w\in\mathbb{U}),$$

then, see [40]

$$|u_j|\leq 1 \text{ and } |v_j|\leq 1 \text{ for all } j\in\mathbb{N}. \tag{13}$$

Thus, upon comparing the corresponding coefficients in (11) and (12), we have

$$\frac{(1+\mu)\,m}{e^m-1}a_2 = \mathcal{L}_{p,q,1}(x)u_1, \tag{14}$$

$$\frac{(1+2\mu)\,m^2}{2\,(e^m-1)}a_3 = \mathcal{L}_{p,q,1}(x)u_2 + \mathcal{L}_{p,q,2}(x)u_1^2, \tag{15}$$

$$-\frac{(1+\mu)\,m}{e^m-1}a_2 = \mathcal{L}_{p,q,1}(x)v_1, \tag{16}$$

and

$$\frac{(1+2\mu)\,m^2}{2\,(e^m-1)}\left[2a_2^2-a_3\right] = \mathcal{L}_{p,q,1}(x)v_2 + \mathcal{L}_{p,q,2}(x)v_1^2. \tag{17}$$

It follows from (14) and (16) that

$$u_1 = -v_1, \tag{18}$$

and

$$\frac{2\,(1+\mu)^2\,m^2}{(e^m-1)^2}a_2^2 = \left(\mathcal{L}_{p,q,1}(x)\right)^2\left(u_1^2+v_1^2\right). \tag{19}$$

If we add (15) and (17), we get

$$\frac{(1+2\mu)\,m^2}{(e^m-1)}a_2^2 = \mathcal{L}_{p,q,1}(x)\left(u_2+v_2\right) + \mathcal{L}_{p,q,2}(x)\left(u_1^2+v_1^2\right). \tag{20}$$

Substituting the value of $(u_1^2+v_1^2)$ *from (19) in the right hand side of (20), we deduce that*

$$\left[(1+2\mu) - \frac{2\,(1+\mu)^2}{(e^m-1)}\frac{\mathcal{L}_{p,q,2}(x)}{\left(\mathcal{L}_{p,q,1}(x)\right)^2}\right]\frac{m^2}{(e^m-1)}a_2^2 = \mathcal{L}_{p,q,1}(x)\left(u_2+v_2\right). \tag{21}$$

Moreover, computations using (5), (13) and (21), we find that

$$|a_2| \le \frac{2p(x)(e^m-1)\sqrt{p(x)}}{m\sqrt{\left|\left[(1+2\mu)\,(e^m-1) - 2\,(1+\mu)^2\right]p^2(x) - 4\,(1+\mu)^2\,q(x)\right|}}.$$

Now, if we subtract (17) from (15), we obtain

$$\frac{(1+2\mu)\,m^2}{(e^m-1)}\left(a_3-a_2^2\right) = \mathcal{L}_{p,q,1}(x)\left(u_2-v_2\right) + \mathcal{L}_{p,q,2}(x)\left(u_1^2-v_1^2\right). \tag{22}$$

Then, in view of (5) and (19), Eq. (22) becomes

$$a_3 = \frac{(e^m-1)^2\left[\mathcal{L}_{p,q,1}(x)\right]^2}{2m^2(1+\mu)^2}\left(u_1^2+v_1^2\right) + \frac{(e^m-1)\mathcal{L}_{p,q,1}(x)}{m^2(1+2\mu)}(u_2-v_2).$$

Thus, applying (5) and (13), we conclude that

$$|a_3| \leq \frac{(e^m-1)^2\left[p(x)\right]^2}{m^2(1+\mu)^2} + \frac{(e^m-1)p(x)}{m^2(1+2\mu)}.$$

Making use of the values of a_2^2 and a_3, we prove the following Fekete–Szegö inequality for functions in the class $\zeta_\Sigma(x,\alpha,\mu)$.

Theorem 3.3 *Let $f\in\Sigma$ given by (1) belongs to the class $\zeta_\Sigma(x,\alpha,\mu)$. Then* $|a_3-\eta a_2^2| \leq \begin{cases} \frac{2p(x)(e^m-1)}{m^2(1+2\mu)}, & 0\leq|T(\eta)|\leq\frac{(e^m-1)}{m^2(1+2\mu)} \\ 2|p(x)||T(\eta)|, & |T(\eta)|\geq\frac{(e^m-1)}{m^2(1+2\mu)} \end{cases}$ *where*

$$T(\eta) = \frac{(e^m-1)^2\left[\mathcal{L}_{p,q,1}(x)\right]^2(1-\eta)}{m^2\left[(1+2\mu)(e^m-1)\left[\mathcal{L}_{p,q,1}(x)\right]^2-2(1+\mu)^2\mathcal{L}_{p,q,2}(x)\right]}.$$

proof 3.4 *From (21) and (22)*

$$\begin{aligned} a_3-\eta a_2^2 &= (1-\eta)\frac{(e^m-1)^2\left[\mathcal{L}_{p,q,1}(x)\right]^3(u_2+v_2)}{m^2\left[(1+2\mu)(e^m-1)\left[\mathcal{L}_{p,q,1}(x)\right]^2-2(1+\mu)^2\mathcal{L}_{p,q,2}(x)\right]} \\ &+ \frac{(e^m-1)\mathcal{L}_{p,q,1}(x)}{m^2(1+2\mu)}(u_2-v_2) \\ &= \mathcal{L}_{p,q,1}(x)\left[T(\eta)+\frac{(e^m-1)}{m^2(1+2\mu)}\right]u_2 \\ &+ \mathcal{L}_{p,q,1}(x)\left[T(\eta)-\frac{(e^m-1)}{m^2(1+2\mu)}\right]v_2, \end{aligned}$$

where

$$T(\eta) = \frac{(e^m-1)^2\left[\mathcal{L}_{p,q,1}(x)\right]^2(1-\eta)}{m^2\left[(1+2\mu)(e^m-1)\left[\mathcal{L}_{p,q,1}(x)\right]^2-2(1+\mu)^2\mathcal{L}_{p,q,2}(x)\right]}.$$

Then, in view of (5), we conclude that

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{2(e^m-1)|\mathcal{L}_{p,q,1}(x)|}{m^2(1+2\mu)} & 0\leq|T(\eta)|\leq\frac{(e^m-1)}{m^2(1+2\mu)}, \\ 2\left|\mathcal{L}_{p,q,1}(x)\right||T(\eta)| & |T(\eta)|\geq\frac{(e^m-1)}{m^2(1+2\mu)}, \end{cases}$$

Which completes the proof of Theorem 3.3.

Corresponding essentially to Example 2.2, Theorems 3.1 and 3.3 yield the following consequence.

Corollary 3.5 *Let $f \in \Sigma$ given by (1) belongs to the class $\zeta_\Sigma(x, \alpha, 0)$. Then*

$$|a_2| \leq \frac{2p(x)(e^m - 1)\sqrt{p(x)}}{m\sqrt{|[(e^m - 1) - 2]\, p^2(x) - 4q(x)|}},$$

$$|a_3| \leq \frac{(e^m - 1)^2 \, [p(x)]^2}{m^2} + \frac{(e^m - 1)p(x)}{m^2}.$$

and

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2p(x)(e^m-1)}{m^2(1+2\mu)}, & 0 \leq |T(\eta)| \leq \frac{(e^m-1)}{m^2(1+2\mu)} \\ 2\,|p(x)|\,|T(\eta)|\,, & |T(\eta)| \geq \frac{(e^m-1)}{m^2(1+2\mu)}, \end{cases}$$

where $T(\eta) = \frac{(e^m-1)^2[\mathcal{L}_{p,q,1}(x)]^2(1-\eta)}{m^2\left[(e^m-1)[\mathcal{L}_{p,q,1}(x)]^2 - 2\mathcal{L}_{p,q,2}(x)\right]}$.

Corollary 3.6 *Let $f \in \Sigma$ given by (1) belongs to the class $\zeta_\Sigma(x, \alpha, 1)$. Then*

$$|a_2| \leq \frac{2p(x)(e^m - 1)\sqrt{p(x)}}{m\sqrt{|[3(e^m - 1) - 8]\, p^2(x) - 16q(x)|}},$$

$$|a_3| \leq \frac{(e^m - 1)^2 \, [p(x)]^2}{4m^2} + \frac{(e^m - 1)p(x)}{3m^2}.$$

and

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2p(x)(e^m-1)}{3m^2}, & 0 \leq |T(\eta)| \leq \frac{(e^m-1)}{3m^2} \\ 2\,|p(x)|\,|T(\eta)|\,, & |T(\eta)| \geq \frac{(e^m-1)}{3m^2} \end{cases}$$

where $T(\eta) = \frac{(e^m-1)^2[\mathcal{L}_{p,q,1}(x)]^2(1-\eta)}{m^2\left[3(e^m-1)[\mathcal{L}_{p,q,1}(x)]^2 - 8\mathcal{L}_{p,q,2}(x)\right]}$.

4 Conclusions

In this study, we have created a brand new subclass of normalized analytic and bi-univalent functions called $\zeta_\Sigma(x, \alpha, \mu)$ that is connected to the zero-truncated Poisson distribution and $(p, q)-$Lucas polynomial. We have derived estimates for the Fekete-Szegö functional issue and the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions that fall under this category. Additionally, one can infer the outcome for the subclass $\zeta_\Sigma(x, \alpha)$ specified in Examples 2.2 and 2.3 by appropriately specializing the parameter.

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