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Majorization For Class Of Meromorphic Functions Defined By An Integral Operator

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Abstract

In this paper, we investigate majorization properties for the class $M^m_{\mu,j}(\alpha,\gamma;A,B)$ of meromorphic functions and the class $M^m_{\mu,j}(t,\lambda)$ of spiral-like functions. Also, some special cases of our main results in a form of corollaries are shown.

Keywords: *Meromorphic function, Subordination, Convolution, Integral operator and Majorization result.*
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1 Introduction

Let $\sum$ be the class of meromorphic functions:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in $U^* = \{z : z \in \mathbb{C},\ 0 < |z| < 1\} = U\backslash\{0\}$. For functions $f(z) \in \sum$ given by (1.1) and $g(z) \in \sum$ given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \tag{1.2}$$

the Hadamard product (or convolution) is

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

Some authors were studied classes of meromorphic functions (see [2 − 6] and [8]).

The concept of majorization was introduced by MacGregor [13] as:

Definition 1.1. *Let f and g analytic in U. We say that f is majorized by g in U and written as $f(z) \ll g(z)$, if there exist a function $\phi(z)$ analytic in U, satisfies*

$$|\phi(z)| \leq 1 \quad and \quad f(z) = \phi(z)\, g(z). \tag{1.4}$$

Definition 1.2. *We say that f is subordinate to g in U and written as $f(z) \prec g(z)$, if there exist a function $w(z)$ analytic in U, satisfies (see [7], [14]):*

$$|w(z)| < 1,\ w(0) = 0 \ and \ f(z) = g(w(z)). \tag{1.5}$$

Jung et al. [9] defined an integral operator

$$\begin{aligned} I^m f(z) &= \begin{cases} \frac{2^m}{z\Gamma(m)} \int_0^z (\log \frac{z}{t})^{m-1} f(t)dt, & m > 0 \\ f(z), & m = 0 \end{cases} \\ &= z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^m a_k z^k, m \geq 0. \end{aligned}$$

and Lashin [10] modified their operator for meromorphic functions as follows:

Definition 1.3. *For $f \in \sum$ given by (1.1), if $L_\mu^m : \sum \to \sum$ is defined by*

$$\begin{aligned} L_\mu^m f(z) &= L_\mu^m(z) * f(z) \\ &= \frac{\mu^m}{\Gamma(m)\, z^{\mu+1}} \int_0^z t^\mu \left(\log\left(\frac{z}{t}\right)\right)^{m-1} f(t)\, dt, \quad (\mu, m > 0), \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{\mu}{k+\mu+1}\right)^m a_k z^k. \end{aligned} \tag{1}$$

From (1.6) we have,

$$z\left(L_\mu^m f(z)\right)' = \mu L_\mu^{m-1} f(z) - (\mu+1) L_\mu^m f(z), \quad (\mu, m \geq 1). \tag{1.7}$$

Definition 1.4. *For $-1 \leq B < A \leq 1, \alpha \geq 0$, $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\left[\frac{(A-B)|\gamma|}{\mu(1-\alpha)} + |B|\right] < 1$. a function $f \in \sum$ is in the class $M_{\mu,j}^m(\alpha, \gamma; A, B)$ of meromorphic functions of complex order γ in U^* if and only if satisfies the subordination:*

$$1 - \frac{1}{\gamma}\left(\frac{z\left(L_\mu^m f(z)\right)^{j+1}}{\left(L_\mu^m f(z)\right)^j} + j + 1\right) - \alpha \left| -\frac{1}{\gamma}\left(\frac{z\left(L_\mu^m f(z)\right)^{j+1}}{\left(L_\mu^m f(z)\right)^j} + j + 1\right)\right| \prec \frac{1+Az}{1+Bz}. \tag{1.8}$$

In particular, for $A = 1,\ B = -1$ *and* $\alpha = 0$,

(i) $M_{\mu,j}^{m}(0,\gamma;1,-1) = M_{\mu,j}^{m}(\gamma) = \left\{ f \in \sum : Re\left[1 - \frac{1}{\gamma}\left(\frac{z\left(L_{\mu}^{m}f(z)\right)^{j+1}}{\left(L_{\mu}^{m}f(z)\right)^{j}} + j + 1\right)\right] > 0 \right\}.$

(ii) For $\gamma = (1-t)\cos\lambda\, e^{-i\lambda}\ \left[|\lambda| \le \frac{\pi}{2}\ \ (0 \le t < 1)\right]$, *the class* $M_{\mu,j}^{m}\left(0,(1-t)\cos\lambda\ e^{-i\lambda},1,-1\right) = M_{\mu,j}^{m}\left[(1-t)\cos\lambda\ e^{-i\lambda}\right] = M_{\mu,j}^{m}(t,\lambda)$, *called the generalized class of meromorphic* $\lambda-$*spiral-like functions of order* t $(0 \le t < 1)$ *if (see* [1] *and* [11])

$$Re\left[e^{i\lambda}\left(\frac{z\left(L_{\mu}^{m}f(z)\right)^{j+1}}{\left(L_{\mu}^{m}f(z)\right)^{j}} + j\right)\right] < -t\cos\lambda.$$

2 Main results

Unless otherwise mentioned we shall assume that $-1 \le B < A \le 1, \alpha \ge 0$, $j \in \mathbb{N}_0$, $\mu, m \ge 1$ and $\gamma \in \mathbb{C}^*$.

Theorem 2.1. *Let* $f \in \sum$ *and suppose that* $g \in M_{\mu,j}^{m}(\alpha,\gamma;A,B)$. *If* $\left(L_{\mu}^{m}f(z)\right)^{j}$ *is majorized by* $\left(L_{\mu}^{m}g(z)\right)^{j}$ *in* U^*, *then*

$$\left|\left(L_{\mu}^{m-1}f(z)\right)^{j}\right| \le \left|\left(L_{\mu}^{m-1}g(z)\right)^{j}\right|, \quad (|z| < r_0), \tag{2.1}$$

where $r_0 = r_0(m,\mu,\alpha,\gamma,A,B)$ is the smallest positive root of the equation

$$\mu\left[\frac{(A-B)|\gamma|}{\mu(1-\alpha)} + |B|\right]r^3 - [\mu + 2|B|]r^2 - \left\{\mu\left[\frac{(A-B)|\gamma|}{\mu(1-\alpha)} + |B|\right] + 2\right\}r + \mu = 0 \tag{2.2}$$

Proof. Since $g \in M_{\mu,j}^{m}(\alpha,\gamma;A,B)$, we have

$$1 - \frac{1}{\gamma}\left(\frac{z\left(L_{\mu}^{m}g(z)\right)^{j+1}}{\left(L_{\mu}^{m}g(z)\right)^{j}} + j + 1\right) - \alpha\left|-\frac{1}{\gamma}\left(\frac{z\left(L_{\mu}^{m}g(z)\right)^{j+1}}{\left(L_{\mu}^{m}g(z)\right)^{j}} + j + 1\right)\right| = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{2.3}$$

where $w(z) = c_1 z + c_2 z^2 + ...$ is analytic and bounded function in U with

$$|w(z)| \le 1,\ w(0) = 0\ \ (z \in U). \tag{2.4}$$

Taking

$$\varpi = 1 - \frac{1}{\gamma}\left(\frac{z\left(L_{\mu}^{m}g(z)\right)^{j+1}}{\left(L_{\mu}^{m}g(z)\right)^{j}} + j + 1\right), \tag{2.5}$$

in (2.3), we have

$$\varpi=\frac{1+\left(\frac{A-B\alpha e^{-i\theta}}{1-\alpha e^{-i\theta}}\right)w(z)}{1+Bw(z)}. \tag{2.6}$$

Using (2.6) in (2.5), we have

$$\frac{z\left(L_{\mu}^{m}g(z)\right)^{j+1}}{\left(L_{\mu}^{m}g(z)\right)^{j}}=-\frac{j+1+\left[\frac{(A-B)\gamma}{1-\alpha e^{-i\theta}}+(j+1)B\right]w(z)}{1+Bw(z)}. \tag{2.7}$$

Applying of Leibnitz's theorem that is the rule which gives the derivative on j-th order of the product of two functions to (1.7), we have

$$z\left(L_{\mu}^{m}g(z)\right)^{j+1}=\mu\left(L_{\mu}^{m-1}g(z)\right)^{j}-(\mu+j+1)\left(L_{\mu}^{m}g(z)\right)^{j}\quad(j>0). \tag{2.8}$$

By using (2.8) in (2.7) and making simple calculations, we have

$$\left(L_{\mu}^{m}g(z)\right)^{j}=\frac{1+Bw(z)}{1-\left[\frac{(A-B)\gamma}{\mu\left(1-\alpha e^{-i\theta}\right)}-B\right]w(z)}\left(L_{\mu}^{m-1}g(z)\right)^{j}. \tag{2.9}$$

Since $|w(z)|\leq|z|$, $(z\in U^{*})$ (2.9) gives us

$$\left|\left(L_{\mu}^{m}g(z)\right)^{j}\right|\leq\frac{1+|B|\,|z|}{1-\left[\frac{(A-B)|\gamma|}{\mu(1-\alpha)}+|B|\right]|z|}\left|\left(L_{\mu}^{m-1}g(z)\right)^{j}\right|. \tag{2.10}$$

Since $\left(L_{\mu}^{m}f(z)\right)^{j}$ is majorized by $\left(L_{\mu}^{m}g(z)\right)^{j}$ in U^{*}. So from (1.3), we have

$$\left(L_{\mu}^{m}f(z)\right)^{j}=\phi(z)\left(L_{\mu}^{m}g(z)\right)^{j}. \tag{2.11}$$

Differentiating (2.11) with respect to z and after simplifying , we have

$$\left(L_{\mu}^{m-1}f(z)\right)^{j}=\frac{z}{\mu}\phi'(z)\left(L_{\mu}^{m}g(z)\right)^{j}+\phi(z)\left(L_{\mu}^{m-1}g(z)\right)^{j}. \tag{2.12}$$

On the other hand, noticing that the Schwarz function $\phi(z)$ satisfies(see [12]):

$$\left|\phi'(z)\right|\leq\frac{1-|\phi(z)|^{2}}{1-|z|^{2}}\ \ (z\in U-)\ . \tag{2.13}$$

Using (2.10) and (2.13) in (2.12), we get

$$\left|\left(L_{\mu}^{m-1}f(z)\right)^{j}\right|\leq\left\{\phi(z)+\frac{|z|\left(1-|\phi(z)|^{2}\right)(1+|B|\,|z|)}{\mu\left(1-|z|^{2}\right)\left[1-\left(\frac{(A-B)|\gamma|}{\mu(1-\alpha)}+|B|\right)|z|\right]}\right\}\left|\left(L_{\mu}^{m-1}g(z)\right)^{j}\right|, \tag{2.14}$$

which, upon setting $|z| = r$ and $|\phi(z)| = \rho\ (0 \leq \rho \leq 1)$,

$$\left|\left(L_{\mu}^{m-1} f(z)\right)^{j}\right| \leq \frac{\psi_1(\rho)}{\mu(1-r^2)\left[1-\left(\frac{(A-B)|\gamma|}{\mu(1-\alpha)}+|B|\right)r\right]}\left|\left(L_{\mu}^{m-1} g(z)\right)^{j}\right|, \quad (2.15)$$

where

$$\psi_1(\rho) = \rho\mu\left(1-r^2\right)\left[1-\left(\frac{(A-B)|\gamma|}{\mu(1-\alpha)}-|B|\right)r\right]+r\left(1-\rho^2\right)(1+|B|r) \quad (2)$$

$$= -r(1+|B|r)\rho^2+\mu\left(1-r^2\right)\left[1-\left(\frac{(A-B)|\gamma|}{\mu(1-\alpha)}+|B|\right)r\right]\rho+r(1+|B|r) \quad (3)$$

takes its maximum value at $\rho = 1$ with $r = r_0(m, \mu, \alpha, \gamma, A, B)$, where r_0 is the smallest positive root of (2.2). Furthermore, if $0 \leq \delta \leq r_0(m, \mu, \alpha, \gamma, A, B)$, then the function $\varphi_1(\rho)$ defined by

$$\varphi_1(\rho) = -\delta(1+|B|\delta)\rho^2+\mu\left(1-\delta^2\right)\left[1-\left(\frac{(A-B)|\gamma|}{\mu(1-\alpha)}+|B|\right)\delta\right]\rho+\delta(1+|B|\delta)$$

is an increasing function on the interval $0 \leq \rho \leq 1$, so that

$$\varphi_1(\rho) \leq \varphi_1(1) = \mu\left(1-\delta^2\right)\left[1-\left(\frac{(A-B)|\gamma|}{\mu(1-\alpha)}+|B|\right)\delta\right].$$
$$0 \leq \rho \leq 1; 0 \leq \delta \leq r_0(m, \mu, \alpha, \gamma, A, B)$$

Then, setting $\rho = 1$ in (2.16) and use it in (2.15), we conclude that (2.1) holds true for $|z| \leq r_0(m, \mu, \alpha, \gamma, A, B)$. This completes the proof of Theorem 2.1. □

By letting $A = 1$ and $B = -1$ in Theorem 2.1, we obtain

Corollary 2.2. *Let $f \in \sum$ and $g \in M_{\mu,j}^{m}(\alpha;\gamma)$. If $\left(L_{\mu}^{m} f(z)\right)^{j}$ is majorized by $\left(L_{\mu}^{m} g(z)\right)^{j}$ in U^*, then*

$$\left|\left(L_{\mu}^{m-1} f(z)\right)^{j}\right| \leq \left|\left(L_{\mu}^{m-1} g(z)\right)^{j}\right|, \quad (|z| < r_1),$$

where $r_1 = r_1(m, \mu, \alpha, \gamma)$ is the smallest positive root of the equation,

$$\left[\frac{2|\gamma|}{1-\alpha}+\mu\right]r^3-(\mu+2)r^2-\left[\frac{2|\gamma|}{1-\alpha}+\mu+2\right]r+\mu = 0,$$

given by

$$r_1 = \frac{\eta_1-\left[\eta_1^2-\mu\left(\frac{2|\gamma|}{(1-\alpha)}+\mu\right)\right]^{\frac{1}{2}}}{\left(\frac{2|\gamma|}{(1-\alpha)}+\mu\right)} \text{ and } \eta_1 = \frac{|\gamma|}{1-\alpha}+\mu+1.$$

Taking $\alpha = 0$ in Corollary 2.1, we state the following:

Corollary 2.3. *Let* $f \in \sum$ *and* $g \in M_{\mu,j}^{m}(\gamma)$. *If* $\left(L_{\mu}^{m} f(z)\right)^{j}$ *is majorized by* $\left(L_{\mu}^{m} g(z)\right)^{j}$ *in* U^{*}, *then*

$$\left|\left(L_{\mu}^{m-1} f(z)\right)^{j}\right| \leq \left|\left(L_{\mu}^{m-1} g(z)\right)^{j}\right|, \quad (|z| < r_2),$$

where $r_2 = r_2(m, \mu, \gamma)$ *is the smallest positive root of the equation,*

$$(2|\gamma| + \mu) r^3 - (\mu + 2) r^2 - (2|\gamma| + \mu + 2) r + \mu = 0,$$

given by

$$r_2 = \frac{\eta_2 - \left[\eta_2^2 - \mu(2|\gamma| + \mu)\right]^{\frac{1}{2}}}{2|\gamma| + \mu} \text{ and } \eta_2 = |\gamma| + \mu + 1.$$

Taking $\mu = 1$, $j = 0$ *in Corollary* 2.2, *we get*

Corollary 2.4. *Let* $f \in \sum$ *and* $g \in M^{m}(\gamma)$. *If* $L^{m} f(z)$ *is majorized by* $L^{m} g(z)$ *in* U^{*}, *then*

$$\left|L^{m-1} f(z)\right| \leq \left|L^{m-1} g(z)\right|, \quad (|z| < r_3),$$

where $r_3 = r_3(m, \gamma)$ *is the smallest positive root of the equation,*

$$(2|\gamma| + 1) r^3 - 3r^2 - (2|\gamma| + 3) r + 1 = 0,$$

given by

$$r_3 = \frac{\eta_3 - \left[\eta_3^2 - (2|\gamma| + 1)\right]^{\frac{1}{2}}}{2|\gamma| + 1} \text{ and } \eta_3 = |\gamma| + 2.$$

Taking $\gamma = (1-t)\cos\lambda\ e^{-i\lambda}$ $\left[|\lambda| \leq \frac{\pi}{2}\ (0 \leq t < 1)\right]$ *in Corollary* 2.3, *we get*

Corollary 2.5. *Let* $f \in \sum$ *and* $g \in M^{m}(t, \lambda)$. *If* $L^{m} f(z)$ *is majorized by* $L^{m} g(z)$ *in* U^{*}, *then*

$$\left|L^{m-1} f(z)\right| \leq \left|L^{m-1} g(z)\right|, \quad (|z| < r_4),$$

where $r_4 = r_4(m, t, \lambda)$ *is the smallest positive root of the equation,*

$$\left(2\left|(1-t)\cos\lambda\ e^{-i\lambda}\right| + 1\right) r^3 - 3r^2 - \left(2\left|(1-t)\cos\lambda\ e^{-i\lambda}\right| + 3\right) r + 1 = 0,$$

given by

$$r_4 = \frac{\eta_4 - \left[\eta_4^2 - \left(2\left|(1-t)\cos\lambda\ e^{-i\lambda}\right| + 1\right)\right]^{\frac{1}{2}}}{2\left|(1-t)\cos\lambda\ e^{-i\lambda}\right| + 1} \text{ and } \eta_4 = \left|(1-t)\cos\lambda\ e^{-i\lambda}\right| + 2\ .$$

Taking $\lambda = 0$ *in Corollary* 2.4, *we get*

Corollary 2.6. *Let* $f \in \sum$ *and* $g \in M^m(t)$. *If* $L^m f(z)$ *is majorized by* $L^m g(z)$ *in* U^*, *then*

$$\left|L^{m-1} f(z)\right| \leq \left|L^{m-1} g(z)\right|, \quad (|z| < r_5),$$

where $r_5 = r_5(m,t)$ *is the smallest positive root of the equation,*

$$(3-2t) r^3 - 3r^2 - (5-2t) r + 1 = 0,$$

given by

$$r_5 = \frac{\eta_5 - \left[\eta_5^2 - (3-2t)\right]^{\frac{1}{2}}}{3-2t} \text{ and } \eta_5 = 3 - t.$$

Taking $\gamma = 1$ *in Corollary* 2.3, *we get*

Corollary 2.7. *Let* $f \in \sum$ *and* $g \in M(m,1)$. *If* $L^m f(z)$ *is majorized by* $L^m g(z)$ *in* U^*, *then*

$$\left|L^{m-1} f(z)\right| \leq \left|L^{m-1} g(z)\right|, \quad \left(|z| < \frac{3-\sqrt{6}}{3}\right).$$

3 Conclusion

By using principle of subordination and a meromorphic integral operator, in this paper, we have defined class of meromorphic functions and obtained majorization resuts for this class and its subclasses.

4 Open Prolem

The authors suggest studying the same properties for the class of analytic function:

$$1 + \frac{1}{\gamma}\left(\frac{zI^{\alpha+1} f(z))'}{I^{\alpha+1} f(z)} - 1\right) - \eta \left|\frac{1}{\gamma}\left(\frac{zI^{\alpha+1} f(z))'}{I^{\alpha+1} f(z)} - 1\right)\right| \prec \frac{1+Az}{1+Bz},$$

where

$$\begin{aligned} I^\alpha f(z) &= \frac{2^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt \\ &= z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^\alpha a_k z^k, \alpha \geq 0 \end{aligned}$$

and the operator $I^\alpha f(z)$ was introduce by Jung et al. [9].

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