Int. J. Open Problems Complex Analysis, Vol. 15, No. 2, November 2023
ISSN 2074-2827; Copyright ©ICSRS Publication, 2023
www.i-csrs.org

Subordinating Results for Classes of Functions Defined by Rusheweyh q-Difference Operator

B.M.Munasser, S.M.Madian and A.O.Mostafa

1-Dept of Math, Faculty of Science, Al Azhar University, Egypt
1-Dept of Math, Amran University, Yemen
2-Basic Science Department, Higher Institute of Engineering and Technology New Damietta, Egypt
3-Department of Mathematics,Faculty of Science, Mansoura University Mansoura 35516, Egypt
e-mail: basheermunassar12345@gmail.com

e-mail:samarmath@yahoo.com
e-mail:adelaeg254@yahoo.com

Received 15 June 2023; Accepted 23 October 2023

Abstract

In this paper, we investigate several interesting subordination results for classes of analytic functions defined by the Rusheweyh q-difference operator.

Keywords: Analytic functions, subordination, $q-$analogue of Ruscheweyh operator.

2010 Mathematical Subject Classification: 30C45.

1 Introduction

The class of analytic univalent functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \left(z \in \mathbb{U} = \{ z \in \mathbb{C}, \ |z| < 1 \} \right), \tag{1.1}$$

is denoted by $\mathcal{A}$. Also, denote by κ the subclass of $\mathcal{A}$ which are convex in $\mathbb{U}$. For f given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the Hadamard product (or convolution)

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

For f and g analytic in $\mathbb{U}$, f is called subordinate to g $(f \prec g)$ if there exists an analytic function ω, with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$, Furthermore, if g is univalent in $\mathbb{U}$, then (see[4] and [10]) :

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

A function $f(z) \in \mathcal{A}$ is said to be in the class of $\beta-$uniformry convex functions of order α, $UCV(\alpha, \beta)$ $(-1 \leq \alpha < 1,\ \beta \geq 0)$, if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq \beta \left|\frac{zf''(z)}{f'(z)}\right|, \tag{1.2}$$

and is in the corresponding class $UST(\alpha, \beta)$ of $\beta-$uniformry starlike functions of order α, $(-1 \leq \alpha \leq 1,\ \beta \geq 0)$, if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \beta \left|\frac{zf'(z)}{f(z)} - 1\right|. \tag{1.3}$$

One can see that

$$f(z) \in UCV(\alpha, \beta) \Leftrightarrow zf'(z) \in UST(\alpha, \beta).$$

We note that
(i) $UCV(0,1) = UCV$ (Goodman [9];
(ii)$UCV(\alpha,1) = UCV(\alpha), UST(0,1) = UST$ and $UST(\alpha,1) = UST(\alpha)$(*see* [17]);
(iii) $UCV(0,\beta) = \beta-UCV$ and $UST(0,\beta) = \beta-UST$ (*see* [13], [15] *and* [14].
(*iv*) $UCV(\alpha, 0) = \kappa^*(\alpha)$ and $UST(\alpha, 0) = S^*(\alpha)$ $(0 \leq \alpha < 1)$.

Let $0 < q < 1, f \in \mathcal{A}$, the Jackson's q- derivative operator is given by (*see* [10], [7], [8]) :

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \quad z \in \mathbb{U}, \tag{1.4}$$

where

$$[k]_q = \frac{1-q^k}{1-q}, \qquad \lim_{q\to 1-} [k]_q = k. \tag{1.5}$$

For $0 < q < 1, f \in \mathcal{A}, \ \delta > -1$, Kannas and Răducanu [12] and Aldweby and Darus [1] defined the q-analogue of Ruschewey operator $\Re_q^\delta : \mathcal{A} \to \mathcal{A}$ as follows:

$$\begin{aligned}\Re_q^\delta f(z) &= z + \sum_{k=2}^{\infty} \frac{\Gamma_q(k+\delta)}{[k-1]_q! \Gamma_q(1+\delta)} a_k z^k \\ &= z + \sum_{k=2}^{\infty} \Theta_k(q,\delta) a_k z^k \qquad (z \in \mathbb{U}),\end{aligned} \tag{1.6}$$

where

$$\Theta_k(q,\delta) = \frac{\Gamma_q(k+\delta)}{[k-1]_q! \Gamma_q(1+\delta)}, \tag{1.7}$$

$$\Gamma_q(k+1) = [k]_q \Gamma_q(k) \qquad \Gamma_q(1) = 1, \tag{1.8}$$

and

$$[k]_q! = [k]_q [k-1]_q ... [1]_q, \qquad [0]_q! = 1. \tag{1.9}$$

We obseve that $\mathrm{Lim}_{q\to 1-} \Re_q^\delta f(z) = \Re^\delta f(z)$ which is the Ruschewey operator defined by Ruschewey [16], (see[2, 3, 4, 5, 6]).

Let

$$D_q^0 \Re_q^\delta f(z) = \Re_q^\delta f(z),$$

$$D_q^1 \Re_q^\delta f(z) = z\partial_q \Re_q^\delta f(z)$$

$$D_q^n \Re_q^\delta f(z) = z\partial_q \left(D_q^{n-1} \Re_q^\delta f(z)\right), \qquad n \in \mathbb{N}.$$

It is easy to have

$$D_q^n \Re_q^\delta f(z) = z + \sum_{k=2}^{\infty} [k]^n \Theta_k(q,\delta) a_k z^k. \tag{1.10}$$

Definition1.1. For $-1 \le \alpha < 1, \beta \ge 0, \ 0 < q < 1, \delta > -1, \ n \in \mathbb{N} \cup \{0\}$, $f(z)$ of the form (1.1), let $S_{n,q}^\delta(\alpha,\beta)$ be the subclass of $\mathcal{A}$ consisting of functions satisfying

$$\mathrm{Re}\left\{ \frac{z\partial_q \left(D_q^n \Re_q^\delta f\ (z)\right)}{D_q^n \Re_q^\delta f\ (z)} - \alpha \right\} > \beta \left| \frac{z\partial_q \left(D_q^n \Re_q^\delta f\ (z)\right)}{D_q^n \Re_q^\delta f\ (z)} - 1 \right| \tag{1.11}$$

and $K_{n,q}^\delta(\alpha,\beta)$ be the subclass of $\mathcal{A}$ consisting of functions satisfying

$$\mathrm{Re}\left\{ \frac{\partial_q \left(z\partial_q \left(D_q^n \Re_q^\delta f\ (z)\right)\right)}{\partial_q \left(D_q^n \Re_q^\delta f\ (z)\right)} - \alpha \right\} > \beta \left| \frac{\partial_q \left(z\partial_q \left(D_q^n \Re_q^\delta f\ (z)\right)\right)}{\partial_q \left(D_q^n \Re_q^\delta f\ (z)\right)} - 1 \right|. \tag{1.12}$$

It follows form (1.11) and (1.12) that

$$D_q^n \Re_q^\delta f(z) \in K_{n,q}^\delta(\alpha,\beta) \to z\partial_q \left(D_q^n \Re_q^\delta f(z)\right) \in S_{n,q}^\delta(\alpha,\beta). \tag{1.13}$$

2 Main Results

Throughout this paper assume that $-1 \leq \alpha < 1,\ \beta \geq 0, 0 < q < 1,\ \delta > -1,\ n \in \mathbb{N} \cup \{0\},\ f(z)$ of the form (1.1).

To prove our main result the following definition and lemma are needed.

Definition 2.1 [17] A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever f is convex,

$$\sum_{k=1}^{\infty} c_k a_k z^k \prec f(z). \qquad (a_1 = 1) \tag{2.1}$$

Lemma 2.2. [17] The sequence $\{c_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\operatorname{Re}\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0. \tag{2.2}$$

Theorem 2.3. If $f(z)$ satisfies the following inequality

$$\sum_{k=2}^{\infty} \Big[[k]_q (1+\beta) - (\alpha+\beta)\Big] [k]_q^n \Theta_k(q,\delta) |a_k| \leq 1-\alpha, \tag{2.3}$$

then, $f(z) \in S_{n,q}^{\delta}(\alpha,\beta)$.

Proof. Suppose that (2.3) holds. Then for $z \in \mathbb{U}$, we have

$$\operatorname{Re}\left\{\frac{z\partial_q\left(D_q^n \Re_q^\delta f\ (z)\right)}{D_q^n \Re_q^\delta f\ (z)} - \alpha\right\} > \beta \left|\frac{z\partial_q\left(D_q^n \Re_q^\delta f\ (z)\right)}{D_q^n \Re_q^\delta f\ (z)} - 1\right|,$$

or

$$\operatorname{Re}\left\{\frac{z\partial_q\left(D_q^n \Re_q^\delta f\ (z)\right)}{D_q^n \Re_q^\delta f\ (z)} - \alpha\right\} - \beta \left|\frac{z\partial_q\left(D_q^n \Re_q^\delta f\ (z)\right)}{D_q^n \Re_q^\delta f\ (z)} - 1\right| > 0,$$

that is,

$$\beta \left|\frac{z\partial_q\left(D_q^n \Re_q^\delta f\ (z)\right)}{D_q^n \Re_q^\delta f\ (z)} - 1\right| - \operatorname{Re}\left\{\frac{z\partial_q\left(D_q^n \Re_q^\delta f\ (z)\right)}{D_q^n \Re_q^\delta f\ (z)} - 1\right\} < (1-\alpha).$$

We have

$$\beta \left|\frac{z\partial_q\left(D_q^n \Re_q^\delta f\ (z)\right)}{D_q^n \Re_q^\delta f\ (z)} - 1\right| - \operatorname{Re}\left\{\frac{z\partial_q\left(D_q^n \Re_q^\delta f\ (z)\right)}{D_q^n \Re_q^\delta f\ (z)} - 1\right\}$$

$$\leq (1+\beta)\left|\frac{z\partial_q\left(D_q^n \Re_q^\delta f\ (z)\right)}{D_q^n \Re_q^\delta f\ (z)} - 1\right| = (1+\beta)\left|\frac{\sum_{k=2}^{\infty}([k]-1)[k]_q^n \Theta_k(q,\delta) a_k z^{k-1}}{1+\sum_{k=2}^{\infty}[k]_q^n \Theta_k(q,\delta) a_k z^{k-1}}\right|$$

$$\leq (1+\beta)\frac{\sum_{k=2}^{\infty}([k]-1)[k]_q^n\Theta_k(q,\delta)|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}[k]_q^n\Theta_k(q,\delta)|a_k||z|^{k-1}}$$

$$\leq (1+\beta)\frac{\sum_{k=2}^{\infty}([k]-1)[k]_q^n\Theta_k(q,\delta)|a_k|r^{k-1}}{1-\sum_{k=2}^{\infty}[k]_q^n\Theta_k(q,\delta)|a_k|r^{k-1}}.$$

Letting $r\to 1^-$ we have

$$<(1+\beta)\frac{\sum_{k=2}^{\infty}([k]-1)[k]_q^n\Theta_k(q,\delta)|a_k|}{1-\sum_{k=2}^{\infty}[k]_q^n\Theta_k(q,\delta)|a_k|}.$$

The last expression is bounded by $1-\alpha$ sinc (2.3) holds.

From (1.13) and Theorem 2.3, we have

Theorem 2.4. A function $f(z)\in K_{n,q}^{\delta}(\alpha,\beta)$ if

$$\sum_{k=2}^{\infty}\Big[[k]_q(1+\beta)-(\alpha+\beta)\Big][k]_q^{n+1}\Theta_k(q,\delta)|a_k|\leq 1-\alpha. \tag{2.4}$$

Let $S_{n,q}^{*\delta}(\alpha,\beta)$ and $K_{n,q}^{*\delta}(\alpha,\beta)$ be the subclasses of $\mathcal{A}$ whose coefficients satisfy the conditions (2.3) and (2.4), respectively. We note that $S_{n,q}^{*\delta}(\alpha,\beta)\subset S_{n,q}^{\delta}(\alpha,\beta)$ and $K_{n,q}^{*\delta}(\alpha,\beta)\subset K_{n,q}^{\delta}(\alpha,\beta)$.

Theorem 2.5. Let $f\in S_{n,q}^{*\delta}(\alpha,\beta)$, $g\in\kappa$. Then

$$\left(\frac{\Big[[2]_q(1+\beta)-(\alpha+\beta)\Big][2]_q^n\Theta_2(q,\delta)}{2\Big\{\Big[[2]_q(1+\beta)-(\alpha+\beta)\Big][2]_q^n\Theta_2(q,\delta)+(1-\alpha)\Big\}}\right)(f*g)(z)\prec g(z), \tag{2.5}$$

and

$$\operatorname{Re}(f(z))>-\frac{\Big[[2]_q(1+\beta)-(\alpha+\beta)\Big][2]_q^n\Theta_2(q,\delta)+(1-\alpha)}{\Big[[2]_q(1+\beta)-(\alpha+\beta)\Big][2]_q^n\Theta_2(q,\delta)}. \tag{2.6}$$

The constant $\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)+(1-\alpha)\right\}}$ is the best estimate.

Proof. Let $f(z)\in S_{n,q}^{*\delta}(\alpha,\beta)$ and $g(z)=z+\sum_{k=2}^{\infty}c_kz^k\in\kappa$. Then

$$\frac{\Big[[2]_q(1+\beta)-(\alpha+\beta)\Big][2]_q^n\Theta_2(q,\delta)}{2\Big\{\Big[[2]_q(1+\beta)-(\alpha+\beta)\Big][2]_q^n\Theta_2(q,\delta)+(1-\alpha)\Big\}}(f*g)(z)$$

$$=\frac{\Big[[2]_q(1+\beta)-(\alpha+\beta)\Big][2]_q^n\Theta_2(q,\delta)}{2\Big\{\Big[[2]_q(1+\beta)-(\alpha+\beta)\Big][2]_q^n\Theta_2(q,\delta)+(1-\alpha)\Big\}}\left(z+\sum_{k=2}^{\infty}c_ka_kz^k\right). \tag{2.7}$$

Thus by Definition 2.1, (2.5) will be true if

$$\left\{\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)+(1-\alpha)\right\}}a_k\right\}_{k=1}^{\infty} \tag{2.8}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 2.2, this will be the case if and only if

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)+(1-\alpha)}a_kz^k\right\}>0. \tag{2.9}$$

Now since

$$\Phi(k)=\left[[k]_q(1+\beta)-(\alpha+\beta)\right][k]_q^n\Theta_k(q,\delta)$$

is an increasing function of $k \geq 2$, then, when $|z| = r < 1$, we have

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)+(1-\alpha)}a_kz^k\right\}$$

$$=\operatorname{Re}\{1+\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)+(1-\alpha)}z$$
$$+\frac{\sum_{k=2}^{\infty}\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)a_kz^k}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)+(1-\alpha)}\}$$

$$\geq 1-\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)+(1-\alpha)}r-$$
$$-\frac{\sum_{k=2}^{\infty}\left[[k]_q(1+\beta)-(\alpha+\beta)\right][k]_q^n\Theta_k(q,\delta)a_kz^k}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)+(1-\alpha)}$$

$$>1-\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)+(1-\alpha)}r-$$
$$-\frac{(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\Theta_2(q,\delta)+(1-\alpha)}r$$
$$=1-r>0.$$

This proves (2.5). The inequality (2.6) follows by taking $g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$ in (2.1). To prove the sharpness of the constant

$$\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\,\Theta_k(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\,\Theta_2(q,\delta)+(1-\alpha)\right\}},$$

consider $f_0(z) \in S_{n,q}^{*\delta}(\alpha,\beta)$ given by

$$f_0(z) = z - \frac{1-\alpha}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\,\Theta_2(q,\delta)}z^2. \tag{2.10}$$

Thus form (2.5), we have

$$\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\,\Theta_k(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\,\Theta_2(q,\delta)+(1-\alpha)\right\}}f_0(z) \prec \frac{z}{1-z}. \tag{2.11}$$

It can easily verified that

$$\min_{|z|\leq r}\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\,\Theta_k(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\,\Theta_2(q,\delta)+(1-\alpha)\right\}}f_0(z) = -\frac{1}{2}.$$

Which shows that

$$\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\,\Theta_k(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^n\,\Theta_2(q,\delta)+(1-\alpha)\right\}}$$

is the best possible.

Similarly, we can prove the following theorem for the class $K_{n,q}^{*\delta}(\alpha,\beta)$.

Theorem 2.6. Let $f(z) \in K_{n,q}^{*\delta}(\alpha,\beta)$, and $g(z) \in \kappa$. Then

$$\left(\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^{n+1}\,\Theta_k(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^{n+1}\,\Theta_2(q,\delta)+(1-\alpha)\right\}}\right)(f*g)(z) \prec g(z) \tag{2.12}$$

and

$$\operatorname{Re}(f(z)) > -\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^{n+1}\,\Theta_2(q,\delta)+(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^{n+1}\,\Theta_k(q,\delta)}. \tag{2.13}$$

The constant $\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^{n+1}\Theta_k(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q^{n+1}\Theta_2(q,\delta)+(1-\alpha)\right\}}$ is the best estimate.

Putting $n=0$ in Theorems 2.5 and 2.6, respectively, we have

Corrollary 2.7.Let $f(z)\in S_{0,q}^{*\delta}(\alpha,\beta)$ satisfies

$$\sum_{k=2}^{\infty}\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\Theta_k(q,\delta)|a_k|\leq 1-\alpha,$$

then

$$\left(\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\Theta_2(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\Theta_2(q,\delta)+(1-\alpha)\right\}}\right)(f*g)(z)\prec g(z) \quad (2.14)$$

and

$$\operatorname{Re}(f(z))>-\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\Theta_2(q,\delta)+(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\Theta_2(q,\delta)}. \quad (2.15)$$

The constant $\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\Theta_k(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\Theta_2(q,\delta)+(1-\alpha)\right\}}$ is the best estimate.

Corrollary 2.8.Let $f(z)\in K_{0,q}^{*\delta}(\alpha,\beta)$ satisfies

$$\sum_{k=2}^{\infty}\left[[k]_q(1+\beta)-(\alpha+\beta)\right][k]_q\Theta_k(q,\delta)|a_k|\leq 1-\alpha,$$

then

$$\left(\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q\Theta_2(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q\Theta_2(q,\delta)+(1-\alpha)\right\}}\right)(f*g)(z)\prec g(z) \quad (2.16)$$

and

$$\operatorname{Re}(f(z))>-\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q\Theta_2(q,\delta)+(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q\Theta_2(q,\delta)}. \quad (2.17)$$

The constant $\frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q\Theta_k(q,\delta)}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][2]_q\Theta_2(q,\delta)+(1-\alpha)\right\}}$ is the best estimate.

Putting $\beta=0$ in Corrollaries 2.7 and 2.8, respectively, we have

Corrollary 2.9. Let $f(z)\in S_q^{*\delta}(\alpha)$ satisfies

$$\sum_{k=2}^{\infty}\left[[k]_q-(\alpha)\right]\Theta_k(q,\delta)|a_k|\leq 1-\alpha,$$

then

$$\left(\frac{\left[[2]_q - \alpha \right] \Theta_2 (q, \delta)}{2 \left\{ \left[[2]_q - \alpha \right] \Theta_2 (q, \delta) + (1 - \alpha) \right\}} \right) (f * g) (z) \prec g (z)$$

and

$$\mathrm{Re} (f (z)) > - \frac{\left[[2]_q - \alpha \right] \Theta_2 (q, \delta) + (1 - \alpha)}{\left[[2]_q - \alpha \right] \Theta_2 (q, \delta)}.$$

The constant $\frac{\left[[2]_q - \alpha \right] \Theta_k (q, \delta)}{2 \left\{ \left[[2]_q - \alpha \right] \Theta_2 (q, \delta) + (1 - \alpha) \right\}}$ is the best estimate.

Corrollary 2.10. Let $f (z) \in K_q^{*\delta} (\alpha)$ satisfies

$$\sum_{k=2}^{\infty} \left[[k]_q - (\alpha) \right] [k]_q \Theta_k (q, \delta) |a_k| \leq 1 - \alpha,$$

then

$$\left(\frac{\left[[2]_q - \alpha \right] [2]_q \Theta_2 (q, \delta)}{2 \left\{ \left[[2]_q - \alpha \right] [2]_q \Theta_2 (q, \delta) + (1 - \alpha) \right\}} \right) (f * g) (z) \prec g (z),$$

and

$$\mathrm{Re} (f (z)) > - \frac{\left[[2]_q - \alpha \right] [2]_q \Theta_2 (q, \delta) + (1 - \alpha)}{\left[[2]_q - \alpha \right] [2]_q \Theta_2 (q, \delta)}.$$

The constant $\frac{\left[[2]_q - \alpha \right] [2]_q \Theta_k (q, \delta)}{2 \left\{ \left[[2]_q - \alpha \right] [2]_q \Theta_2 (q, \delta) + (1 - \alpha) \right\}}$ is the best estimate.

3 Open Problem

The authors suggest to find necessary and sufficient conditions for coefficients of function

$$\mathcal{F}(z) = z - \sum_{k=2}^{\infty} a_k z^k, \; a_k \geq 0$$

in the class

$$\mathrm{Re} \left\{ \frac{z \partial_q \left(D_q^n \Re_q^\delta f \;\; (z) \right)}{D_q^n \Re_q^\delta f \;\; (z)} - \alpha \right\} > \beta \left| \frac{z \partial_q \left(D_q^n \Re_q^\delta f \;\; (z) \right)}{D_q^n \Re_q^\delta f \;\; (z)} - 1 \right|$$

and study geometric and algebric properties.

References

[1] H. Aldweby and M. Darus, Some subordination results on q-analogue of Ruscheweyh differential operator, Abst. Appl. Anal., 2014, Article ID 958563, 1-6.

[2] M. K. Aouf, On a new criteria for univalent functions of order α,Rend. Math. Series-II, (1991), 47-59.

[3] M. K. Aouf and H. E. Darwish, On inequalities for certain analytic functions involving Ruscheweyh derivative, J. Math., 21 (1995), no.4, 387-393.

[4] M. K. Aouf and H. E. Darwish, and A. A. Attiya, A remark on certain regular functions defined by Ruscheweyh derivative, Proc. Pakistan. Acal. Sci., 37 (2000), no.1, 67-69.

[5] M. K. Aouf and A. A. Al-Dohiman, Fixed second coefficient for certain subclasses of starlike functions with negative coefficients, Demonstratio Math., 38 (2005), no. 3, 551-565.

[6] M. K. Aouf and H. M. Hossen, Notes on certain classes of analytic function defined by Ruscheweyh derivative, Taiwense. T. Math., 1 (1997), no.1, 11-19.

[7] A. K. Aouf A. O.Mostafa, Subordination results for analytic function associated with fractional q-calculuus operators with complex order, Afr. Mat., 31 (2020) , 1387-1396.

[8] A. K. Aouf A. O.Mostafa, Subordinating results for classes of functions defined by Sălăgean type q-derivative operator, Filomat., 34 (2020) , no. 7, 2283-2292.

[9] A. W. Goodman, On uniformry convex functions, Ann. Polon. Math., 59(1991), 87-92.

[10] F. H. Jackson, On q-functions and a certain difference operator, Transactions of the Royal Society of Edinburgh, 46(1908), 253-281.

[11] S. Kanas, Coefficient estimates in subclasses of the Caratheodory classrelated to conic domains, Acta Math. Univ. Comenian. (N. S.), 74(2005), 149-161.

[12] S. Kanas and D. Răducanu, Some class of analytic functions related to conic domains, Math. Slovaca 64(2014), no. 5, 1183-1196.

[13] S. Kanas and H. M. Srivastava, Linear operators associated with k-uniformly convex functions, Integral Transforms Spec. Funct., 9(2000), 121-132.

[14] S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, J. Comput. Appl. Math., 105(1999), 327-336.

[15] F. Ronning, Uniformly convex functions and a corresponding class of starlikefunctions, Proc. Amer. Soc., 118(1993), 189-196.

[16] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975), 109-115.

[17] H. S. Wilf, Subordinating factor sequences for convex maps of the unit circle, Proc. Amer. Math. Soc. 12 (1961), 689-693.