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Subordination Properties for a Class of Analytic Functions with Complex Order Defined by q-Derivative Operator

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Abstract

In this paper we obtain coefficients estimate theorem and prove subordination relationships for an analytic function class with complex order defined by $q-$difference operator and its subclasses.

Keywords: Analytic functions, functions of complex order, starlike function, subordinating factor sequence, Hadamard product (or convolution), coefficient bounds, $q-$analogue operator.
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1 Introduction

The class of univalent analytic functions of the form

$$F(z) = z + \sum_{k=2}^{\infty} a_k z^k, (a_k \geq 0),\ z \in \mathcal{D} = \{z \in \mathbb{C} :\mid z \mid< 1\}, \tag{1}$$

is denoted by $\mathcal{S}$.

The class of convex functions $\mathcal{K}$ satisfy

$$\operatorname{Re}\left\{1+\frac{zF''(z)}{F'(z)}\right\}>0.$$

If F, g are analytic in $\mathcal{D}$,we say that F is subordinate to g, written F $\prec g$ if there exists a Schwarz function $w(z)$, is analytic in $\mathcal{D}$ with $w(0)=0$ and $|w(z)|<1$ for all $z\in\mathcal{D}$, such that $\mathrm{F}(z)=g(w(z)),\ z\in\mathcal{D}$, *see* [16].

For F given by (1) and g given by

$$g(z)=z+\sum_{k=2}^{\infty}b_kz^k, \tag{2}$$

the Hadamard product (or convolution) is defined by

$$(F*g)(z)=z+\sum_{k=2}^{\infty}a_kb_kz^k=(g*F)(z).$$

For $F\in\mathcal{S},\ 0<q<1$, the $q-$derivative operator ∇_q is given by [10] (see also [2], [3, 4], [8], [13, 14]);

$$\nabla_qF(z)=\begin{cases}\frac{F(z)-F(qz)}{(1-q)z} & ,z\neq0\\ F'(0) & ,z=0\end{cases}$$

that is

$$\nabla_qF(z)=1+\sum_{k=2}^{\infty}[k]_qa_kz^{k-1}, \tag{3}$$

where

$$[k]_q=\frac{1-q^k}{1-q},\ [0]_q=0. \tag{4}$$

As $q\to1^-$, $[k]_q=k$ and $\nabla_qF(z)=F'(z)$.

Mostafa and Saleh ([11]) defined the $q-$Frasin differential operator $\mathcal{D}^{\zeta}_{\delta,\gamma,q}F(z)$ as

$$\begin{aligned}D^{\zeta}_{\delta,\gamma,q}F(z)&=z+\sum_{k=2}^{\infty}[1+([k]_q-1)C^{\delta}_j(\gamma)]^{\zeta}a_kz^k,(\zeta\in\mathbb{N}_0=\mathbb{N}\cup\{0\})\\ &=z+\sum_{k=2}^{\infty}\chi^{\zeta}_{q,k}(\delta,\gamma)a_kz^k,\end{aligned} \tag{5}$$

where

$$\chi_{q,k}^{\zeta}(\delta,\gamma) = [1 + ([k]_q - 1)C_j^{\delta}(\gamma)]^{\zeta}, \tag{6}$$

and

$$C_j^{\delta}(\gamma) = \sum_{j=1}^{\delta} \binom{\delta}{j} (-1)^{j+1}\gamma^j. \tag{7}$$

Note that
(i) $\lim_{q\to 1-} \mathcal{D}_{\delta,\gamma,q}^{\zeta}F(z) = \mathcal{D}_{\delta,\gamma}^{\zeta}F$ ([7]);
(ii) $\mathcal{D}_{1,1,q}^{\zeta}F(z) = \mathcal{D}_q^{\zeta}F(z)$ ([9],[15],[5]);
(iii) $\mathcal{D}_{1,\gamma,q}^{\zeta}F(z) = D_{\gamma,q}^{\zeta}F(z)$ ([6]);
(iv) $\lim_{q\to 1-} \mathcal{D}_{1,\gamma,q}^{\zeta}F(z) = \mathcal{D}_{\gamma}^{\zeta}F(z)$ ([1]);
(v) $\lim_{q\to 1-} \mathcal{D}_{1,1,q}^{\zeta}F(z) = \mathcal{D}^{\zeta}F(z)$ ([12]).

Definition 1.1 *Let $\tau \in \mathbb{C}^* = \mathbb{C}/\{0\}, \delta \geq \gamma \geq 0,\ 0 < q < 1,\ \zeta \in \mathbb{N}_0$, $0 < \eta \leq 1$ and $F \in \mathcal{S}$, such that $\mathcal{D}_{\delta,\gamma,q}^{\zeta}F(z) \neq 0$ for $z \in \mathcal{D}/\{0\}$. We say that $F \in \mathbb{S}_q^{\zeta}(\tau,\delta,\gamma,\eta)$ if*

$$\left| \frac{1}{\tau}\left(\frac{z\nabla_q \mathcal{D}_{\delta,\gamma}^{\zeta}F(z)}{\mathcal{D}_{\delta,\gamma}^{\zeta}F(z)} - 1\right) \right| < \eta. \tag{8}$$

For special values of $q, \tau, \gamma, \delta, \eta$, we have:
(i) $\lim_{q\to 1-} \mathbb{S}_q^{\zeta}(\tau,\delta,\gamma,\eta) = \mathbb{S}^{\zeta}(\tau,\delta,\gamma,\eta) = \left\{ F(z) : \left| \frac{1}{\tau}\left(\frac{z(\mathcal{D}_{\delta,\gamma}^{\zeta}F(z))'}{\mathcal{D}_{\delta,\gamma}^{\zeta}F(z)} - 1\right) \right| < \eta \right\}$;
(ii) $\mathbb{S}_q^{\zeta}(\tau,1,1,\eta) = \mathbb{S}_q^{\zeta}(\tau,\eta) = \left\{ F(z) : \left| \frac{1}{\tau}\left(\frac{z\nabla_q(\mathcal{D}_q^{\zeta}F(z))}{\mathcal{D}_q^{\zeta}F(z)} - 1\right) \right| < \eta \right\}$;
(iii) $\mathbb{S}_q^{\zeta}(\tau,1,\gamma,\eta) = \mathbb{S}_q^{\zeta}(\tau,\gamma,\eta) = \left\{ F(z) : \left| \frac{1}{\tau}\left(\frac{z\nabla_q(\mathcal{D}_{\gamma,q}^{\zeta}F(z))}{\mathcal{D}_{\gamma,q}^{\zeta}F(z)} - 1\right) \right| < \eta \right\}$;
(iv) $\mathbb{S}_q^{\zeta}(1-\psi,\delta,\gamma,1) = \mathbb{S}_q^{\zeta}(\psi,\delta,\gamma) = \left\{ F(z) : \mathrm{Re}\left\{ \frac{z\nabla_q(\mathcal{D}_{\delta,\gamma,q}^{\zeta}F(z))}{\mathcal{D}_{\delta,\gamma,q}^{\zeta}F(z)} \right\} > \psi, 0 \leq \psi < 1 \right\}$.

2 Main Results

Unless indicated, let $\tau \in \mathbb{C}^*$, $\delta \geq \gamma \geq 0$, $0 < q < 1, \zeta \in \mathbb{N}_0$,and $0 < \eta \leq 1$

The following definition and lemma are needed.

Definition 2.1 *[16] A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $F(z)$ of the form (1) is analytic, univalent and convex in $\mathcal{D}$, then,*

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec F(z) \qquad (z \in \mathcal{D}; a_1 = 1).$$

Lemma 2.2 *[16] The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1+2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \qquad (z \in \mathcal{D}).$$

Theorem 2.3 *If F satisfies*

$$\sum_{k=2}^{\infty}([k]_q + \eta\,|\tau| - 1)\chi_{q,k}^{\zeta}(\delta,\gamma)\,|a_k| \leq \eta\,|\tau|\,, \tag{9}$$

then, $F \in \mathbb{S}_q^{\zeta}(\tau,\delta,\gamma,\eta)$

Proof. Let (9) holds then,

$$\begin{aligned}
\left|\frac{z\nabla_q(\mathcal{D}_{\delta,\gamma,q}^{\zeta}F(z))}{\mathcal{D}_{\delta,\gamma,q}^{\zeta}F(z)} - 1\right| &= \left|\frac{\sum_{k=2}^{\infty}([k]_q - 1)\chi_{\delta,\gamma,q}^{\zeta}\,|a_k|\, z^{k-1}}{1-\sum_{k=2}^{\infty}\chi_{\delta,\gamma,q}^{\zeta}\,|a_k|\, z^{k-1}}\right| \\
&\leq \frac{\sum_{k=2}^{\infty}([k]_q - 1)\chi_{\delta,\gamma,q}^{\zeta}\,|a_k|\, |z|^{k-1}}{1-\sum_{k=2}^{\infty}\chi_{\delta,\gamma,q}^{\zeta}\,|a_k|\, |z|^{k-1}} \\
&\leq \frac{\sum_{k=2}^{\infty}([k]_q - 1)\chi_{\delta,\gamma,q}^{\zeta}\,|a_k|}{1-\sum_{k=2}^{\infty}\chi_{\delta,\gamma,q}^{\zeta}\,|a_k|} \\
&< \eta\,|\tau|\,,
\end{aligned}$$

Then F satisfies (8).Let $\mathbb{S}_q^{\zeta}(\tau,\delta,\gamma,\eta)$ be the class of functions satisfy (9) So $S_q^{*}(\tau,\delta,\gamma,\eta) \subset S_q(\tau,\delta,\gamma,\eta)$.

Theorem 2.4 *Let $F \in \mathbb{S}_q^{\zeta *}(\tau,\delta,\gamma,\eta)$ and $g \in \mathcal{K}$ .Then*

$$\left(\frac{(q+\eta\,|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)}{2\left[(q+\eta\,|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta\,|\tau|\right]}\right)(F * g)(z) \prec g(z) \tag{10}$$

and

$$\Re\{F(z)\} > -\frac{(q+\eta\,|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta\,|\tau|}{(q+\eta\,|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)}. \tag{11}$$

The constant factor $\frac{(q+\eta|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)}{2[q+\eta|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta|\tau|]}$ in(10) cannot be replaced by a larger one.

Proof.

Let $\in \mathbb{S}_q^{\zeta *}(\tau,\delta,\gamma,\eta)$ and $g(z)=z+\sum\limits_{k=2}^{\infty} b_k z^k \in \mathcal{K}$ .Then,

$$\left(\frac{(q+\eta\left|\tau\right|)\chi_{q,2}^{\zeta}(\delta,\gamma)}{2\left[(q+\eta\left|\tau\right|)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta\left|\tau\right|\right]}\right)(F*g)(z)$$

$$=\left(\frac{(q+\eta\left|\tau\right|)\chi_{q,2}^{\zeta}(\delta,\gamma)}{2\left[(q+\eta\left|\tau\right|)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta\left|\tau\right|\right]}\right)\left(z+\sum_{k=2}^{\infty}a_k b_k z^k\right). \tag{12}$$

Thus, by definition 1, (10) will be true if

$$\left\{\frac{(q+\eta\left|\tau\right|)\chi_{q,2}^{\zeta}(\delta,\gamma)}{2\left[(q+\eta\left|\tau\right|)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta\left|\tau\right|\right]}a_k\right\}_{k=1}^{\infty} \tag{13}$$

is a subordinating factor sequence, with $a_1=1$. In view of Lemma 1, this is equivalent to

$$\Re\left\{1+2\sum_{k=1}^{\infty}\frac{(q+\eta\left|\tau\right|)\chi_{q,2}^{\zeta}(\delta,\gamma)}{2\left[(q+\eta\left|\tau\right|)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta\left|\tau\right|\right]}a_k z^k\right\}>0, \tag{14}$$

where

$$\vartheta(k)=(q+\eta\left|\tau\right|)[1+([k]_q-1)C_j^{\delta}(\gamma)]^{\zeta}\quad(k\geq 2),$$

is an increasing function of $k\,(k\geq 2)$, when $|z|=r<1$, we have,

$$\begin{aligned}
&\Re\left\{1+\sum_{k=1}^{\infty}\frac{\vartheta(2)}{\vartheta(2)+\eta\left|\tau\right|}a_k z^k\right\}\\
=\ &\Re\left\{1+\frac{\vartheta(2)}{\vartheta(2)+\eta\left|\tau\right|}z+\frac{\sum_{k=2}^{\infty}\vartheta(2)}{\vartheta(2)+\eta\left|\tau\right|}a_k z^k\right\}\\
\geq\ &1-\frac{\vartheta(2)}{\vartheta(2)+\eta\left|\tau\right|}r-\frac{\sum\limits_{k=2}^{\infty}\vartheta(k)|a_k|}{\vartheta(2)+\eta|\tau|}r^k\\
>\ &1-\frac{\vartheta(2)}{\vartheta(2)+\eta\left|\tau\right|}r-\frac{\eta|\tau|}{\vartheta(2)+\eta|\tau|}r\\
=\ &1-r>0\qquad(|z|=r<1).
\end{aligned}$$

By taking the convex function $g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$. To prove the sharpness of $\frac{\vartheta(2)}{\vartheta(2)+\eta|\tau|}$, the function $F_0(z)\in\mathbb{S}_q^{\zeta *}(\tau,\delta,\gamma,\eta)$ given by

$$F_0(z)=z+\frac{\eta\left|\tau\right|}{(q+\eta\left|\tau\right|)\chi_{q,2}^{\zeta}(\delta,\gamma)}z^2. \tag{15}$$

Thus from (11),we have

$$\frac{(q+\eta\,|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)}{2[(q+\eta\,|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta\,|\tau|]}_0 F_0(z) \prec \frac{z}{1-z}$$

Moreover, it can easily to verify for $F_0(z)$ given by (15) that

$$\min_{|z|\le r}\left\{\Re\frac{(q+\eta\,|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)}{2[(q+\eta\,|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta\,|\tau|]}_0 F_0(z)\right\} = -\frac{1}{2} \tag{16}$$

This shows that the $\frac{(q+\eta|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)}{2[(q+\eta|\tau|)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta|\tau|]}$ is the best possible .

Taking $\lim_{q\to 1^-}$in Theorem 3, we have

Corollary 2.5 *Let* $F \in \mathbb{S}^{\zeta *}(\tau,\delta,\gamma,\eta)$ *and* $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{K}$. *Then*

$$\left(\frac{(1+\eta\,|\tau|)\chi_{1,2}^{\zeta}(\delta,\gamma)}{2\left[(1+\eta\,|\tau|)\chi_{1,2}^{\zeta}(\delta,\gamma)+\eta\,|\tau|\right]}\right)(F*g)(z) \prec g(z) \tag{17}$$

and

$$\Re\{F(z)\} > -\frac{(1+\eta\,|\tau|)[1+(k-1)C_j^{\delta}(\gamma)]^{\zeta}+\eta\,|\tau|}{(1+\eta\,|\tau|)[1+(k-1)C_j^{\delta}(\gamma)]^{\zeta}}.$$

The factor $\frac{(1+\eta|\tau|)[1+(k-1)C_j^{\delta}(\gamma)]^{\zeta}+\eta|\tau|}{(1+\eta|\tau|)[1+(k-1)C_j^{\delta}(\gamma)]^{\zeta}}$ in (17) cannot be replaced by a larger one.

Taking $\tau = 1-\alpha$ $(0 \le \alpha < 1)$, in (9) and Theorem 3, we have

Corollary 2.6 *Let* $F \in \mathbb{S}_q^{\zeta *}(\alpha,\delta,\gamma,\eta)$ $(0 \le \alpha < 1)$ *and* $g \in \mathcal{K}$ *.Then*

$$\left(\frac{(q+\eta(1-\alpha))\chi_{q,2}^{\zeta}(\delta,\gamma)}{2\left[(q+\eta(1-\alpha))\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta(1-\alpha)\right]}\right)(F*g)(z) \prec g(z) \tag{18}$$

and

$$\Re\{F(z)\} > -\frac{(q+\eta(1-\alpha))\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta(1-\alpha)}{(q+\eta(1-\alpha))\chi_{q,2}^{\zeta}(\delta,\gamma)}.$$

The factor $\frac{(q+\eta(1-\alpha))\chi_{q,2}^{\zeta}(\delta,\gamma)}{2\left[(q+\eta(1-\alpha))\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta(1-\alpha)\right]}$ *in (18) cannot be replaced by a larger one.*

Taking $\tau = e^{-i\theta}(1-\alpha)\cos\theta$ $\left(|\theta| < \frac{\pi}{2},\ 0 \le \alpha < 1\right)$ in (9) and Theorem 3, we have

Corollary 2.7 *Let $F \in \mathbb{S}_q^{\zeta*}(\delta,\gamma,\eta,\alpha,\theta)$ and $g \in \mathcal{K}$. Then*

$$\left(\frac{(q+\eta(1-\alpha)\ e^{-i\theta}\cos\theta)\chi_{q,2}^{\zeta}(\delta,\gamma)}{2\left[(q+\eta(1-\alpha)\ e^{-i\theta}\cos\theta)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta(1-\alpha)\ e^{-i\theta}\cos\theta\right]}\right)(F*g)(z)\prec g(z) \tag{19}$$

and

$$\Re\{F(z)\} > -\frac{(q+\eta\left|e^{-i\theta}(1-\alpha)\cos\theta\right|)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta(1-\alpha)\cos\theta}{(q+\eta\left|e^{-i\theta}(1-\alpha)\cos\theta\right|)\chi_{q,2}^{\zeta}(\delta,\gamma)}.$$

The factor $\frac{(q+\eta(1-\alpha)\cos\theta)\chi_{q,2}^{\zeta}(\delta,\gamma)}{2\left[(q+\eta(1-\alpha)\cos\theta)\chi_{q,2}^{\zeta}(\delta,\gamma)+\eta(1-\alpha)\cos\theta\right]}$ in (19) cannot be replaced by a larger one.

3 Open Problem

The authors suggest to find necessary and sufficient conditions for coefficients of function

$$\mathcal{F}(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ a_k \geq 0$$

in the class

$$\left|\frac{1}{\tau}\left(\frac{z\nabla_q \mathcal{D}_q^n \mathcal{R}_q^\delta F(z)}{\mathcal{D}_q^n \mathcal{R}_q^\delta F(z)} - 1\right)\right| < \eta.$$

where

$$\mathcal{D}_q^n \mathcal{R}_q^\delta F(z) = z + \sum_{k=2}^{\infty} [k]_q^n \ominus_k (q,\delta) a_k z^k, \qquad n \in \mathbb{N},$$

$$\begin{aligned} \ominus_k(q,\delta) &= \frac{\Gamma_q(k+\delta)}{[k-1]_q!\Gamma_q(1+\delta)}, \\ \Gamma_q(k+1) &= [k]_q\Gamma_q(k) \qquad \Gamma_q(1)=1. \end{aligned}$$

and study geometric and algebraic properties.

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