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Coefficient Bounds for New Subclass of Bi-univalent Functions Associated with Faber polynomials

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Abstract

In our present investigation, we introduce a new subclass of biunivalent and analytic functions, using new differential operator in the open unit desk $\mathbb{U}$. We determine estimates for the general coefficient bounds $|a_n|$ and initial coefficient estimates for functions in the new subclass by using Faber polynomial coefficient techniques.

Keywords: *Analytic univalent function; Starlike function; Quantum calculus, Faber Polynomials.*

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1 Introduction

We begin by letting $\mathcal{A}$ be denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$f(z) = a + \sum_{k=2}^{\infty} a_k z^k. \quad (1)$$

Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ that are analytic in $\mathbb{U}$. It is well known that every univalent function $f \in \mathcal{A}$ has an inverse f^{-1}, which satisfies

$$\begin{aligned} f^{-1}(f(z)) &= z, \quad (z \in \mathbb{U}), \\ f(f^{-1}(w)) &= w, \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right), \end{aligned}$$

where

$$h(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (2)$$

A function $f \in \mathbb{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ be denote the class of all bi-univalent functions defined in $\mathbb{U}$; for more details see ([6]-[21]). For $f \in \Sigma$, the class of bi-univalent analytic functions was first introduced and studied by Lewin where it was proved to show that $|a_2| < 1.51$. Brannan and Clunie [26] proved that $|a_2| < \sqrt{2}$. Brannan and Taha [27] introduced certain subclass of the biunivalent functions class Σ.

Let f and g are analytic functions in $\mathbb{U}$, then f is said to be subordinate to g, written by $f \prec g,\ z \in \mathbb{U}$ if there exists a Shwartz function $w(z)$, analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z)),\ z \in \mathbb{U}$.

The Hadamard product (or Convolution) of two functions f and g of the form (1.1) is denoted by $f * g$ where defined as

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

The $\{p, q\}$-derivative for $f \in \mathcal{A}$ of the form (1.1) is defined as (see [5],[25])

$$\partial_{p,q} f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad p \neq q,\ z \in \mathbb{U} \quad (3)$$

It is clear that for a function f in $\mathcal{A}$ of the form (1.1)

$$\partial_{p,q} f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1}, \quad (p \neq q), \quad (4)$$

where

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}, p \neq q. \quad (5)$$

Throughout this article we will assume that p and q is fixed number between 0 and 1, and $p \neq q$.

Using the technique of convolution, we introduce new derivative operator $\mathfrak{D}^{m,n}_{\lambda_1,\lambda_2,\ell,d,p,q} : \mathbb{A} \to \mathbb{A}$ by

$$\mathfrak{D}^{m,n}_{\lambda_1,\lambda_2,\ell,d,p,q} f(z) = z + \sum_{k=2}^{\infty} \Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell,p,q) C(n,k) a_k z^k, \quad (z \in \mathbb{U}), \tag{6}$$

where

$$\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell,p,q) = \left[\frac{\ell(1+(\lambda_1+\lambda_2)([k]_{p,q}-1))+d}{\ell(1+\lambda_2([k]_{p,q}-1))+d}\right]^m \tag{7}$$

$m, d \in \mathbb{N}_0$, $0 \le \lambda_1 \le \lambda_2$, $\ell \ge 0$, and $\ell + d > 0$. where $n, m, d \in \mathbb{N}_0$, $0 \le \lambda_1 \le \lambda_2$, $\ell \ge 0$, $1 > 0$, $\ell + d > 0$,

$$C(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(k-1)!} = \frac{\Gamma(n+k)}{\Gamma(n+1)(k-1)!} \tag{8}$$

In this current work, we introduce new subclass of bi-univalent function Σ, using new differential operator $\mathfrak{D}^{m,n}_{\lambda_1,\lambda_2,\ell,d,p,q} f(z)$ that defined in (6). We determine estimates for the general coefficient bounds $|a_n|$ for $n \ge 3$ and also estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclass by using Faber polynomial techniques.
for simplification we write $\mathfrak{D}^{m,n}_{\lambda_1,\lambda_2,\ell,d,p,q} f(z)$ by $\mathfrak{D}^{m,n} f(z)$.

Definition 1.1 *A function $f \in \Sigma$, $0 \le \gamma$, $-1 \le n, 1 \le \beta$, and $0 \le \alpha < 1$; we introduce a new class of biunivalent functions $\mathcal{N}^{m,n}_{\Sigma_{p,q}}(\alpha,\beta,\gamma)$ as $f \in \mathcal{N}^{m,n}_{\Sigma_{p,q}}(\alpha,\beta,\gamma)$ if and only if:*

$$\frac{(1-\beta)\mathfrak{D}^{m,n} f(z) + \beta z \partial_{p,q} \mathfrak{D}^{m,n} f(z) + \gamma z^2 \partial^2_{p,q} \mathfrak{D}^{m,n} f(z)}{z} \prec \frac{1+Az}{1+Bz} \tag{9}$$

$$\frac{(1-\beta)\mathfrak{D}^{m,n} h(w) + \beta w \partial_{p,q} \mathfrak{D}^{m,n} h(w) + \gamma w^2 \partial^2_{p,q} \mathfrak{D}^{m,n} h(w)}{w} \prec \frac{1+Aw}{1+Bw} \tag{10}$$

where

$$A = \{1 - \alpha(1+pq)\}, \quad \textit{and } B = -pq \tag{11}$$

and $h(w)$ is defined by (1.2).

Special Cases:

- $\mathcal{N}^{m,n}_{\Sigma_{1,q}}(\alpha,\beta,\gamma) = \mathcal{N}^{q(\alpha,\beta,\gamma,n)}_{\Sigma}$; see [23].
- $\mathcal{N}^{m,0}_{\Sigma_{1,1}}(\alpha,\beta,\gamma) = \mathcal{N}^{(\alpha,\beta,\gamma)}_{\Sigma}$; see [22].
- $\mathcal{N}^{m,0}_{\Sigma_{1,1}}(\alpha,1,\gamma) = \mathcal{B}^{(\alpha,\gamma)}_{\Sigma}$; see [29].
- $\mathcal{N}^{m,0}_{\Sigma_{1,1}}(\alpha,\beta,0) = \mathcal{N}^{(\alpha,\beta)}_{\Sigma}$; see [24].
- $\mathcal{N}^{m,0}_{\Sigma_{1,1}}(\alpha,1,0) = \mathcal{N}^{(\alpha,\beta)}_{\Sigma}$; see [28].

2 Main Result

Using Faber polynomial expansion of a function $f \in \mathbf{A}$ given by (1.1), the coefficient of $h(w) = f^{-1}(w)$ expressed as:

$$h(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...)w^n$$

where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!}a_2^{n-3}a_3 + \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4$$
$$+\frac{(-n)!}{[2(-n+2)]!(n-5)!}a_2^{n-5}[a_5 + (-n+2)a_3^2]$$
$$+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6 + (-2n+5)a_3a_4] + \sum_{j\geq 7} a_2^{n-j}V_j \tag{12}$$

and V_j with $7 \leq j \leq n$ is a homogeneous polynomial of degree j in the variables $|a_2|, |a_3|, ..., |a_n|$. (for more details see [1].)
In particular, the first three terms of K_{n-1}^{-n} are given below:

$$\frac{1}{2}K_1^{-2} = -a_2, \; \frac{1}{3}K_2^{-3} = 2a_2^2 - a_3, \; \frac{1}{4}K_3^{-4} = -(5a_2^3 - 5a_2a_3 + a_4).$$

In general, an expansion of (for more details see [3, 4]), which is follows:

$$K_n^p = pa_n + \frac{p(p-1)}{2}E_n^2 + \frac{p!}{(p-3)!(3)!}E_n^3 + ... + \frac{p!}{(p-n)!(n)!}E_n^n \tag{13}$$

where $E_n^p = E_n^p(a_2, a_3, ...)$ and, by [2],

$$E_n^p(a_2, a_3, ...) = \sum_{n=2}^{\infty} \frac{p!(a_2)^{\upsilon_1}...(a_n)^{\upsilon_{n-1}}}{\upsilon_1!...\upsilon_{n-1}!}, \tag{14}$$

for $p \leq n$, while $a_1 = 1$, and the sum is taken over all nonnegative integer $\upsilon_1, ..., \upsilon_n$ satisfying

$$\upsilon_1 + \upsilon_1 + ... + \upsilon_n = p$$
$$\upsilon_1 + 2\upsilon_2 + ... + (n-1)\upsilon_{n-1} = n - 1. \tag{15}$$

Evidently, $E_{n-1}^{n-1}(a_2, a_3, ..., a_n) = a_2^{n-1}$ [2], or, equivalently

$$E_n^p(a_2, a_3, ..., a_n) = \sum_{n=2}^{\infty} \frac{p!(a_1)^{\upsilon_1}...(a_n)^{\upsilon_n}}{\upsilon_1!\upsilon_2!...\upsilon_n!} \tag{16}$$

for $p \leq n$, while $a_1 = 1$, and the sum is taken over all nonnegative integer $\upsilon_1, ..., \upsilon_n$ satisfying

$$\upsilon_1 + \upsilon_1 + ... + \upsilon_n = p$$
$$\upsilon_1 + 2\upsilon_2 + ... + n\upsilon_n = n. \tag{17}$$

Its clear that $E_n^n(a_1, ..., a_n) = E_1^n$; the first and the last terms are $E_n^n = a_1^n$, $E_n^1 = a_n$.

Theorem 2.1 *For $n > -1$, $\beta \geq 1$, $\gamma \geq 0$ and $0 \leq \alpha < 1$. If $f \in \mathcal{N}_{\Sigma_{p,q}}^{m,n}(\alpha, \beta, \gamma)$, if $a_m = 0$, $2 \leq m \leq k-1$, then*

$$|a_k| \leq \frac{(1-\alpha)(1+pq)\Gamma(n+1)(k-1)!}{\Big[(1-\beta) + \beta[k]_{p,q} + \gamma[k]_{p,q}[k-1]_{p,q}\Big]\Gamma(n+k)}, \qquad k \geq 3 \tag{18}$$

Proof. Let $f \in \mathcal{N}_{\Sigma_{p,q}}^{m,n}(\alpha, \beta, \gamma)$ of the form (1.1), we have

$$\frac{(1-\beta)\mathfrak{D}^{m,n}f(z) + \beta z\partial_{p,q}\mathfrak{D}^{m,n}f(z) + \gamma z^2\partial_{p,q}^2\mathfrak{D}^{m,n}f(z)}{z}$$
$$= \Big\{(1-\beta) + \beta[k]_{p,q} + \gamma[k]_{p,q}[k-1]_{p,q}\Big\} \times \Lambda_d^{m,k}C(n,k)a_k z^{k-1} \tag{19}$$

and, for the inverse map $h(w) = f^{-1}(z)$, we have

$$\frac{(1-\beta)\mathfrak{D}^{m,n}h(w) + \beta w\partial_{p,q}\mathfrak{D}^{m,n}h(w) + \gamma w^2\partial_{p,q}^2\mathfrak{D}^{m,n}h(w)}{w}$$
$$= \Big\{(1-\beta) + \beta[k]_{p,q} + \gamma[k]_{p,q}[k-1]_{p,q}\Big\} \times \Lambda_d^{m,k}C(n,k)b_k w^{k-1} \tag{20}$$

where $\Lambda^{m,k}$ is given by 7 and $b_n = \frac{1}{n}K_{n-1}^{-n}(a_2, a_3, ..., a_n)$. Since both function f and its inverse map $g = f^{-1}$ are in $\mathcal{N}_{\Sigma_{p,q}}^{m,n}(\alpha, \beta, \gamma)$, by the definition of subordination there exist two Schwars functions $p(z) = \sum_{k=1}^{\infty} c_k z^k$ and $q(w) = \sum_{k=1}^{\infty} c_k w^k$, where $z, w \in \mathbb{U}$. We have

$$\frac{(1-\beta)\mathfrak{D}^{m,n}f(z) + \beta z\partial_{p,q}\mathfrak{D}^{m,n}f(z) + \gamma z^2\partial_{p,q}^2\mathfrak{D}^{m,n}f(z)}{z} = \frac{1+Ap(z)}{1+Bp(z)} \tag{21}$$

where

$$\frac{1+Ap(z)}{1+Bp(z)} = 1 - \sum_{k=1}^{\infty}(A-B)K_k^{-1}(c_1, c_2, ..., c_k, B)z^k \tag{22}$$

$$\frac{(1-\beta)\mathfrak{D}^{m,n}h(w) + \beta w\partial_{p,q}\mathfrak{D}^{m,n}h(w) + \gamma w^2\partial_{p,q}^2\mathfrak{D}^{m,n}h(w)}{w} = \frac{1+Aq(w)}{1+Bq(w)} \tag{23}$$

where

$$\frac{1+Aq(w)}{1+Bq(w)} = 1-\sum_{k=1}^{\infty}(A-B)K_k^{-1}(d_1,d_2,...,d_k,B)w^k. \tag{24}$$

For the coefficients of Schwarz functions $p(z)$ and $q(w)$, therefore, $|c_k| \leq 1$ and $|d_k| \leq 1$ [12].
Comparing the corresponding coefficients of (19) and (22), we have

$$\Big\{(1-\beta)+\beta[k]_{p,q}+\gamma[k]_{p,q}[k-1]_{p,q}\Big\}\times\Lambda_d^{m,k}a_k = -(A-B)K_k^{-1}(c_1,c_2,...,c_{k-1},B)z^k \tag{25}$$

Similarly, corresponding to coefficients of (20) and (2.13), we have

$$\Big\{(1-\beta)+\beta[k]_{p,q}+\gamma[k]_{p,q}[k-1]_{p,q}\Big\}\times\Lambda_d^{m,k}C(n,k)b_k = -(A-B)K_k^{-1}(d_1,d_2,...,d_{k-1},B)z^k \tag{26}$$

Note that for $a_m = 0$, $2 \leq m \leq k-1$ we have $b_k = -a_k$

$$\Big\{(1-\beta)+\beta[k]_{p,q}+\gamma[k]_{p,q}[k-1]_{p,q}\Big\}\times\Lambda_d^{m,k}C(n,k)a_k = -(A-B)c_{k-1}, \tag{27}$$

$$\Big\{(1-\beta)+\beta[k]_{p,q}+\gamma[k]_{p,q}[k-1]_{p,q}\Big\}\times\Lambda_d^{m,k}C(n,k)a_k = -(A-B)d_{k-1}. \tag{28}$$

Taking the absolute values of (27) and (28), we have

$$|a_k| \leq \frac{|A-B|}{\Big\{(1-\beta)+\beta[k]_{p,q}+\gamma[k]_{p,q}[k-1]_{p,q}\Big\}\times\Lambda_d^{m,k}C(n,k)}|c_{k-1}| \tag{29}$$

$$|a_k| \leq \frac{|A-B|\Gamma(n+1)(k-1)!}{\Big\{(1-\beta)+\beta[k]_{p,q}+\gamma[k]_{p,q}[k-1]_{p,q}\Big\}\Gamma(n+k)}\Big[\frac{\ell(1+\lambda_2([k]_{p,q}-1))+d}{\ell(1+(\lambda_1+\lambda_2)([k]_{p,q}-1))+d}\Big]^m \tag{30}$$

since

$$\Big[\frac{\ell(1+\lambda_2([k]_{p,q}-1))+d}{\ell(1+(\lambda_1+\lambda_2)([k]_{p,q}-1))+d}\Big]^m \leq 1$$

therefore,

$$|a_k| \leq \frac{|A-B|(k-1)!}{\Big\{(1-\beta)+\beta[k]_{p,q}+\gamma[k]_{p,q}[k-1]_{p,q}\Big\}(n+1)_{k-1}} \tag{31}$$

By using equation (21) we have

$$|a_k| \leq \frac{(1-\alpha)(1+pq)\Gamma(n+1)(k-1)!}{\Big\{(1-\beta)+\beta[k]_{p,q}+\gamma[k]_{p,q}[k-1]_{p,q}\Big\}\Gamma(n+k)} \tag{32}$$

For $\beta = 1$, in Theorem 2.1, we have the following corollary.

Corollary 2.2 *For $n > -1$, $\gamma \geq 0$ and $0 \leq \alpha < 1$. If $f \in \mathcal{N}^{m,n}_{\Sigma_{p,q}}(\alpha, 1, \gamma)$, if $a_m = 0$, $2 \leq m \leq k-1$, then*

$$|a_k| \leq \frac{(1-\alpha)(1+pq)\Gamma(n+1)(k-1)!}{[k]_{p,q}\Big\{1+\gamma[k-1]_{p,q}\Big\}\Gamma(n+k)}, \qquad k \geq 3 \tag{33}$$

For $\beta = 1$, $\gamma = 0$, in Theorem 2.1, we have the following corollary.

Corollary 2.3 *For $n > -1$ and $0 \leq \alpha < 1$. If $f \in \mathcal{N}^{m,n}_{\Sigma_{p,q}}(\alpha, 1, 0)$, if $a_m = 0$, $2 \leq m \leq k-1$, then*

$$|a_k| \leq \frac{(1-\alpha)(1+pq)\Gamma(n+1)(k-1)!}{[k]_{p,q}\Gamma(n+k)}, \qquad k \geq 3 \tag{34}$$

For $n = 0$, $p, q \to 1$, in Theorem 2.1, we have the following corollary.

Corollary 2.4 *[22] For $\beta \geq 1$, $\gamma \geq 0$ and $0 \leq \alpha < 1$. If $f \in \mathcal{N}^{m,0}_{\Sigma_{1,1}}(\alpha, \beta, \gamma)$, if $a_m = 0$, $2 \leq m \leq k-1$, then*

$$|a_k| \leq \frac{2(1-\alpha)}{1-\beta(k-1)+\gamma k(k-1)}, \qquad k \geq 3 \tag{35}$$

Special Cases:

For $\beta = 1$, $n = 0$, and $p, q \to 1$, in Theorem 2.1, we have the following corollary

Corollary 2.5 *[29] For $\gamma \geq 0$ and $0 \leq \alpha < 1$. If $f \in \mathcal{N}^{m,0}_{\Sigma_{1,1}}(\alpha, 1, \gamma)$, if $a_m = 0$, $2 \leq m \leq k-1$, then*

$$|a_k| \leq \frac{2(1-\alpha)}{k\{1+\gamma(k-1)\}}, \qquad k \geq 3 \tag{36}$$

For $\beta = 1$, $n = 0$,$\gamma = 0$ and $p, q \to 1$, in Theorem 2.1, we have the following corollary

Corollary 2.6 *[24] For $0 \leq \alpha < 1$. If $f \in \mathcal{N}^{m,0}_{\Sigma_{1,1}}(\alpha, 1, 0)$, if $a_m = 0$, $2 \leq m \leq k-1$, then*

$$|a_k| \leq \frac{2(1-\alpha)}{k}, \qquad k \geq 3 \tag{37}$$

For $\gamma = 0$, $n = 0$ and $p, q \to 1$, in Theorem 2.1, we have the following corollary

Corollary 2.7 *[28] For $0 \leq \alpha < 1$. If $f \in \mathcal{N}^{m,0}_{\Sigma_{1,1}}(\alpha, \beta, 0)$, if $a_m = 0$, $2 \leq m \leq k-1$, then*

$$|a_k| \leq \frac{2(1-\alpha)}{1+(k-1)\beta}, \qquad k \geq 3 \tag{38}$$

Theorem 2.8 *For $n > -1$, $\beta \geq 1$, $\gamma \geq 0$ and $0 \leq \alpha < 1$. If $f \in \mathcal{N}^{m,n}_{\Sigma_{p,q}}(\alpha, \beta, \gamma)$, if $a_m = 0$, $2 \leq m \leq k-1$, then*

$$|a_2| \leq min\left\{\frac{(1-\alpha)(1+pq)\Gamma(n+1)}{[(1-\beta)+[2]_{p,q}(\beta+\gamma[1]_{p,q})]\Gamma(n+2)}, \sqrt{\frac{2pq(1-\alpha)(1+pq)2!\Gamma(n+1)}{\Big((1-\beta)+[3]_{p,q}(\beta+\gamma[2]_{p,q})\Big)[2]_{p,q}\Gamma(n+3)}}\right\}, \tag{39}$$

$$|a_3| \leq \left\{\frac{[2]_{p,q}}{2}\Big(\frac{(1-\alpha)(1+pq)\Gamma(n+1)}{[(1-\beta)+[2]_{p,q}(\beta+\gamma[1]_{p,q})]\Gamma(n+2)}\Big)^2, \frac{2pq(1-\alpha)(1+pq)\Gamma(n+1)}{\Big((1-\beta)+[3]_{p,q}(\beta+\gamma[2]_{p,q})\Big)\Gamma(n+3)}\right\}, \tag{40}$$

and

$$\left|a_3 - [2]_{p,q}a_2^2\right| \leq \frac{pq(1-\alpha)(1+pq)2!\Gamma(n+1)}{\Big((1-\beta)+[3]_{p,q}(\beta+\gamma[2]_{p,q})\Big)\Gamma(n+3)}. \tag{41}$$

Proof. Replacing k by (2) and (3) in (27) and (28), respectively, we have

$$\Big\{(1-\beta)+\beta[2]_{p,q}+\gamma[2]_{p,q}[1]_{p,q}\Big\} \times \Lambda_d^{m,2}C(n,2)a_2 = -(A-B)c_1, \tag{42}$$

$$\Big\{(1-\beta)+\beta[3]_{p,q}+\gamma[3]_{p,q}[2]_{p,q}\Big\} \times \Lambda_d^{m,3}C(n,3)a_3 = -(A-B)(Bc_1^2-c_2), \tag{43}$$

$$-\Big\{(1-\beta)+\beta[2]_{p,q}+\gamma[2]_{p,q}[1]_{p,q}\Big\} \times \Lambda_d^{m,2}C(n,2)a_2 = -(A-B)d_1. \tag{44}$$

$$\Big\{(1-\beta)+\beta[3]_{p,q}+\gamma[3]_{p,q}[2]_{p,q}\Big\} \times \Lambda_d^{m,3}C(n,3)\{[2]_{p,q}a_2^2-a_3\} = -(A-B)(Bd_1^2-d_2). \tag{45}$$

From (42) and (44) we have $c_1 = -d_1$, and

$$\begin{aligned} |a_2| &\leq \frac{|A-B|}{\Big\{(1-\beta)+\beta[2]_{p,q}+\gamma[2]_{p,q}[1]_{p,q}\Big\} \times \Lambda_d^{m,2}C(n,2)}|c_1|, \\ &= \frac{|A-B|}{\Big\{(1-\beta)+\beta[2]_{p,q}+\gamma[2]_{p,q}[1]_{p,q}\Big\} \times \Lambda_d^{m,2}C(n,2)}|d_1|, \\ &\leq \frac{|A-B|}{\Big\{(1-\beta)+\beta[2]_{p,q}+\gamma[2]_{p,q}[1]_{p,q}\Big\} \times \Lambda_d^{m,2}C(n,2)}, \end{aligned} \tag{46}$$

Using (8), (11) and $\Lambda_d^{m,2} \geq 1$, therefore

$$|a_2| \leq \frac{(1-\alpha)(1+pq)\Gamma(n+1)}{[(1-\beta)+[2]_{p,q}(\beta+\gamma[1]_{p,q})]\Gamma(n+2)} \tag{47}$$

Adding (43) and (45), we have

$$[2]_{p,q}\Big\{(1-\beta)+\beta[3]_{p,q}+\gamma[3]_{p,q}[2]_{p,q}\Big\}\times\Lambda_d^{m,3}C(n,3)a_2^2 = -(A-B)\Big\{B(c_1^2+d_1^2)-(c_2+d_2)\Big\}. \tag{48}$$

Taking the absolute values for both sides of last expression and applying the estimates $|c_2| \leq 1-|c_1|^2$, $|d_2| \leq 1-|d_1|^2$ of Lemma 5 and $|c_1| \leq 1$, $|d_1| \leq 1$, we have

$$|a_2|^2 \leq \frac{|A-B||B|2!\Gamma(n+1)}{\Big[(1-\beta)+[3]_{p,q}(\beta+\gamma[2]_{p,q})\Big][2]_{p,q}\Gamma(n+3)}.$$

Using (8)and (11), we have

$$|a_2| \leq \sqrt{\frac{2pq(1-\alpha)(1+pq)2!\Gamma(n+1)}{\Big((1-\beta)+[3]_{p,q}(\beta+\gamma[2]_{p,q})\Big)[2]_{p,q}\Gamma(n+3)}}. \tag{49}$$

Now, in order to find $|a_3|$, we subtract (45) from (43) and we have

$$2a_3 = \frac{-(A-B)\{B(c_1^2-d_1^2)-(c_2-d_2)\}}{\Big\{(1-\beta)+\beta[3]_{p,q}+\gamma[3]_{p,q}[2]_{p,q}\Big\}\times\Lambda_d^{m,3}C(n,3)} + [2]_{p,q}a_2^2. \tag{50}$$

using $c_1=-d_1$, $\Lambda_d^{m,3} \leq 1$ and taking the modulus of last expression

$$|a_3| \leq \frac{|A-B|(|c_2|+|d_2|)\Gamma(n+1)}{\Big\{(1-\beta)+\beta[3]_{p,q}+\gamma[3]_{p,q}[2]_{p,q}\Big\}2\Gamma(n+3)} + \frac{[2]_{p,q}}{2}|a_2|^2. \tag{51}$$

By using the estimates $|c_2| \leq 1-|c_1|^2$, $|d_2| \leq 1-|d_1|^2$, of lemma (5), and $|c_1| \leq 1$, $|d_1| \leq 1$ in (51)

$$|a_3| \leq \frac{[2]_{p,q}}{2}|a_2|^2, \tag{52}$$

and using (47) in (52) we have

$$|a_3| \leq \frac{[2]_{p,q}}{2}\Big(\frac{(1-\alpha)(1+pq)\Gamma(n+1)}{[(1-\beta)+[2]_{p,q}(\beta+\gamma[1]_{p,q})]\Gamma(n+2)}\Big)^2, \tag{53}$$

Again using (49) in (52), we have

$$|a_3| \leq \frac{2pq(1-\alpha)(1+pq)\Gamma(n+1)}{\Big((1-\beta)+[3]_{p,q}(\beta+\gamma[2]_{p,q})\Big)\Gamma(n+3)}, \tag{54}$$

From (45) we have

$$\left|a_3 - [2]_{p,q}a_2^2\right| \le \frac{|(A-B)B||d_1|^2 + |(A-B)||d_2|}{\left\{(1-\beta) + \beta[3]_{p,q} + \gamma[3]_{p,q}[2]_{p,q}\right\} \times \Lambda_d^{m,3} C(n,3)}. \tag{55}$$

Using $|d_2| \le 1 - |d_1|^2$, $|d_1|$ of lemma (5), $\Lambda_d^{m,3} \le 1$ and Using (8)and (11), on (55), we have

$$\left|a_3 - [2]_{p,q}a_2^2\right| \le \frac{pq(1-\alpha)(1+pq)2!\Gamma(n+1)}{\Big((1-\beta) + [3]_{p,q}(\beta + \gamma[2]_{p,q})\Big)\Gamma(n+3)}. \tag{56}$$

For $\beta = 1$, $\gamma = 0$ and $p \to 1$ in Theorem 2, we obtain the following corollary

Corollary 2.9 *For $n > -1$ and $0 \le \alpha < 1$. If $f \in \mathcal{N}^{m,n}_{\Sigma_{1,q}}(\alpha, 1, 0)$, if $a_m = 0$, $2 \le m \le k-1$, then*

$$|a_2| \le min\left\{\frac{(1-\alpha)(1+q)\Gamma(n+1)}{[2]_q\Gamma(n+2)}, \sqrt{\frac{2q(1-\alpha)(1+q)2!\Gamma(n+1)}{[2]_q[3]_q\Gamma(n+3)}}\right\}, \tag{57}$$

$$|a_3| \le \left\{\frac{[2]_q}{2}\Big(\frac{(1-\alpha)(1+q)\Gamma(n+1)}{[2]_q\Gamma(n+2)}\Big)^2, \frac{2q(1-\alpha)(1+q)\Gamma(n+1)}{[3]_q\Gamma(n+3)}\right\}, \tag{58}$$

and

$$\left|a_3 - [2]_q a_2^2\right| \le \frac{q(1-\alpha)(1+q)2!\Gamma(n+1)}{[3]_q\Gamma(n+3)}. \tag{59}$$

For $n = 0$, $q \to 1$ and $p \to 1$ in Theorem 2, we obtain the following corollary

Corollary 2.10 *([23]) For $0 \le \alpha < 1$. If $f \in \mathcal{N}^{m,0}_{\Sigma_{1,1}}(\alpha, 1, 0)$, if $a_m = 0$, $2 \le m \le k-1$, then*

$$|a_2| \le min\left\{\frac{2(1-\alpha)}{1+\beta+2\gamma}, \sqrt{\frac{2(1-\alpha)}{1+2\beta+6\gamma}}\right\}, \tag{60}$$

$$|a_3| \le \frac{2(1-\alpha)}{1+2\beta+6\gamma}, \tag{61}$$

and

$$\left|a_3 - [2]_q a_2^2\right| \le \frac{2(1-\alpha)}{1+2\beta+6\gamma}. \tag{62}$$

For $n = 0$, $\gamma = 0$ $q \to 1$ and $p \to 1$ in Theorem 2, we obtain the following corollary

Corollary 2.11 *([24].) For $0 \leq \alpha < 1$. If $f \in \mathcal{N}^{m,0}_{\Sigma_{1,1}}(\alpha, 1, 0)$, if $a_m = 0$, $2 \leq m \leq k-1$, then*

$$|a_2| \leq min\left\{\frac{2(1-\alpha)}{1+\beta}, \sqrt{\frac{2(1-\alpha)}{1+2\beta}}\right\}, \tag{63}$$

$$|a_3| \leq \frac{2(1-\alpha)}{1+2\beta}, \tag{64}$$

and

$$\left|a_3 - 2a_2^2\right| \leq \frac{2(1-\alpha)}{1+2\beta}. \tag{65}$$

For $n = 0$, $\gamma = 0$, $\beta = 1$ $q \to 1$ and $p \to 1$ in Theorem 2, we obtain the following corollary

Corollary 2.12 *([28]) For $\beta \geq 1$, $0 \leq \alpha < 1$. If $f \in \mathcal{N}^{m,0}_{\Sigma_{1,1}}(\alpha, 1, 0)$, if $a_m = 0$, $2 \leq m \leq k-1$, then*

$$|a_2| \leq min\left\{\frac{2(1-\alpha)}{2}, \sqrt{\frac{2(1-\alpha)}{3}}\right\}, \tag{66}$$

$$|a_3| \leq \frac{2(1-\alpha)}{3}, \tag{67}$$

and

$$\left|a_3 - 2a_2^2\right| \leq \frac{2(1-\alpha)}{3}. \tag{68}$$

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