Int. J. Open Problems Complex Analysis, Vol. 15, No. 2, November 2023
ISSN 2074-2827; Copyright ©ICSRS Publication, 2023
www.i-csrs.org

Utilizing Normalized Dini Functions in the Analysis of Analytic Functions

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Received 27 June 2023; Accepted 26 October 2023

Abstract

In this paper, we investigate some characterization of normalized Dini function of order ν of first kind, to be subclass of the various analytic functions. We consider an integral operator related to normalized Dini function.

Keywords: Univelant function, Analytic function, Normalized Dini function, Convolutions.
2010 Mathematical Subject Classification: 30C45.

1 Introduction

Consider the class $H = H(\mathbb{U})$ of analytic functions, where $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ represents the open unit disk in the complex plane. Let $H[a, n]$ denote the subclass of H defined by

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \quad (n \in \mathbb{N}, a \in \mathbb{C}),$$

and let **A** be the subclass of H comprising all functions f of the form

$$f = z + \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \in \mathbb{C}, z \in \mathbb{U}), \tag{1}$$

which are analytic in the open unit disk in the complex plane $\mathbb{U}$ and satisfy the normalization conditions: $f(0) = f'(0) - 1 = 0$. We denote the subclass of $\mathbf{A}$, characterized by the form (1), that is univalent in $\mathbb{U}$ by $\mathbf{S}$. Additionally, we denote by $\mathbf{T}$ the subclass of $\mathbf{S}$ described by

$$f = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad (z \in \mathbb{U}). \tag{2}$$

The convolution (or Hadamard product) of the functions $f_1(z) = \sum_{n=2}^{\infty} \phi_n z^n$ and $f_2(z) = \sum_{n=2}^{\infty} \xi_n z^n$ is defined by:

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} \phi_n \xi_n z^n.$$

Definition 1.1 *A function $f \in \mathbf{T}$ is said to be in the class $T(\lambda, \alpha)$ if it satisfies the condition*

$$Re\left\{\frac{zf'(z)}{\lambda z f'(z) + (1-\lambda) f(z)}\right\} > \alpha, \qquad (z \in \mathbb{U}) \tag{3}$$

where $\alpha < 1$ and $\lambda \geq 0$.

Definition 1.2 *The subclass $C(\lambda, \alpha)$ denote all functions $f \in \mathbf{T}$ if it satisfies the condition*

$$Re\left\{\frac{f'(z) + zf''(z)}{f'(z) + \lambda z f''(z)}\right\} > \alpha, \qquad (z \in \mathbb{U}) \tag{4}$$

where $\alpha < 1$ and $\lambda \geq 0$.

The subclasses $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$ was studied by Altinas and Owa [2] and the certain condition for hypergeometric function for these classes was studied by Mostafa (see [1]).

Definition 1.3 *[5] The subclass $R^{\tau}(A, B)$ represent all functions $f \in \mathbf{A}$ that satisfies the condition*

$$Re\left\{\frac{f'(z) - 1}{\tau(A-B) - B\,|f'(z) - 1|}\right\} < 1 \tag{5}$$

where A and B are fixed numbers, $-1 \leq B < A \leq 1$ and τ non-zero complex number.

Consider the second-order linear homogeneous differential equation

$$z^2y^{''}(z) + bzy^{'}(z) + (cz^2 - \nu^2 + (1-b)\nu)y(z) = 0, \quad (b, c, \nu \in \mathbb{C}), \tag{6}$$

then the function $\psi_{\nu,b,c}$ is a particular solution of (6), which called the generalizes Bessel function of order ν. Also, $\psi_{\nu,b,c}$ represented by

$$\psi_{\nu,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma\left(\nu + n + \frac{(b+1)}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu}, \qquad (z \in \mathbb{C}), \tag{7}$$

where Γ stands for the Euler gamma function. For $b = c = 1$ we obtain the familiar Bessel function of the first kind of order ν.

Remark 1.4 *:*

i. For $c = b = 1$ we obtain the familiar Bessel function of the first kind of order ν.

ii. For $c = -1, b = 1$ we obtain the modified Bessel function of the first kind of order ν.

iii. For $c = 1, b = 2$ we obtain the spherical Bessel function of the first kind of order ν.

Consider the generalized Dini function introduced by Deniz, as outlined in [4]. This function is a composite of the generalized Bessel function of the first kind, denoted by

$$D_\nu = (a - \nu)\psi_{\nu,b,c}(z) + z\psi^{'}_{\nu,b,c}(z),$$

where $\psi_{\nu,b,c}$ represents the generalized Bessel function of order ν. It is worth noting that when $a = b = c = 1$, this function reduces to the classical Dini functions of Bessel functions. In the context of this study, we specifically explore the normalized form of the Dini function defined by:

$$\begin{aligned}\omega_{\nu,a,b,c}(z) &= \frac{2^\nu}{\rho}\Gamma\left(\nu + \frac{b+1}{2}\right) z^{1-\frac{\nu}{2}} \left[(a-\nu)\psi_{\nu,b,c}(\sqrt{z}) + \sqrt{z}\psi^{'}_{\nu,b,c}(\sqrt{z})\right] \\ &= \sum_{n=0}^{\infty} \frac{(-c)^n(2n+a)\Gamma(\nu+1)}{a4^n n!\Gamma(\nu+n+1)} z^{n+1} \\ &= z + \sum_{n=1}^{\infty} \frac{(-c)^n(2n+a)}{a4^n n!(\nu+1)_n} z^{n+1},\end{aligned} \tag{8}$$

where $(\delta)_k$ stands for Pochhammer symbol, that defined for $\delta, k \in \mathbb{C}$ by

$$(\delta)_k = \frac{\Gamma(\delta+k)}{\Gamma(\delta)} = \begin{cases} 1 & if \quad k=0; \quad \delta \in \mathbb{C}\setminus\{0\} \\ \delta(\delta+1)(\delta+2)\cdots(\delta+k-1) & if \quad k \in \mathbb{N}; \quad \delta \in \mathbb{C} \end{cases}.$$

Now, we introduce a new operator $\mathfrak{R}_{a,b,c,\nu}f : \mathbf{A} \to \mathbf{A}$ defined by

$$\mathfrak{R}_{a,b,c,\nu}f(z) = (\omega_{\nu,a,b,c}(z)*f)(z) = z+\sum_{n=2}^{\infty} \frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} a_n z^n \quad (z \in \mathbb{U}), \tag{9}$$

where $\omega_{\nu,a,b,c}(z)$ is defined by (8).

The results we obtain hinge significantly upon the significance of the following lemmas.

Lemma 1.5 *[2] A function f of the form (2) is belong to the class $T(\lambda,\alpha)$, if and only if*

$$\sum_{n=2}^{\infty}\Big(n-\lambda\alpha n-\alpha+\lambda\alpha\Big)|a_n| \leq 1-\alpha.$$

Lemma 1.6 *[2] A function f of the form (2) is belong to the class $C(\lambda,\alpha)$, if and only if*

$$\sum_{n=2}^{\infty} n\Big(n-\lambda\alpha n-\alpha+\lambda\alpha\Big)|a_n| \leq 1-\alpha.$$

Lemma 1.7 *[5] if $f \in R^{\tau}(A,B)$ and of the form (1), then*

$$|a_n| \leq \frac{\tau(A-B)}{n}, \qquad (n=2,3,\cdots). \tag{10}$$

The bounds in equation (10) are sharp.

2 Main Results

Theorem 2.1 *If $c<0$ and $\nu>-1$, then the function $2z-\omega_{\nu,a,b,c}(z)$ belong to the class $T(\lambda,\alpha)$ if and only if*

$$(1-\lambda\alpha)\omega'_{\nu,a,b,c}(1) \leq (2-\alpha(1+\lambda))\omega_{\nu,a,b,c}(1), \tag{11}$$

where $\omega_{\nu,a,b,c}(z)$ is defined by (8).

Proof. From (8), we have

$$2z - \omega_{\nu,a,b,c}(z) = z - \sum_{n=2}^{\infty} \frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} z^n$$

According to Lemma 1.5, we must show that

$$\sum_{n=2}^{\infty} \Big(n - \lambda\alpha n - \alpha + \lambda\alpha\Big) \frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} \le 1 - \alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} \Big(n - \lambda\alpha n - \alpha + \lambda\alpha\Big) \frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} \\
&= \sum_{n=2}^{\infty} \Big(n(1-\lambda\alpha) - \alpha(1-\lambda)\Big) \frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} \\
&= (1-\lambda\alpha) \sum_{n=2}^{\infty} n \frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} - \alpha(1-\lambda) \sum_{n=2}^{\infty} \frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} \\
&= (1-\lambda\alpha) \sum_{n=1}^{\infty} (n+1) \frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} - \alpha(1-\lambda) \sum_{n=1}^{\infty} \frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} \\
&= (1-\lambda\alpha) \sum_{n=1}^{\infty} n \frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} + (1-\lambda\alpha) \sum_{n=1}^{\infty} \frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} - \alpha(1-\lambda) \sum_{n=1}^{\infty} \frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} \\
&= (1-\lambda\alpha) \sum_{n=1}^{\infty} n \frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} + (1-\alpha) \sum_{n=1}^{\infty} \frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} \\
&= (1-\lambda\alpha)(\omega'_{\nu,a,b,c}(1) - \omega_{\nu,a,b,c}(1)) - (1-\alpha)(\omega_{\nu,a,b,c}(1) - 1) \\
&= (1-\lambda\alpha)\omega'_{\nu,a,b,c}(1) - (2 - \alpha(1+\lambda))\omega_{\nu,a,b,c}(1) + 1 - \alpha.
\end{aligned}$$

If the condition (11) holds, then the last expression is bounded above by $1-\alpha$ and the proof end.

Theorem 2.2 *If $c < 0$ and $\nu > -1$, then the function $2z - \omega_{\nu,a,b,c}(z)$ is in $C(\lambda, \alpha)$ if and only if*

$$(1-\lambda\alpha)\omega''(1) + (3 - 4\lambda\alpha - \alpha)\omega'(1) \le 2(1 - \alpha(1+\lambda)), \tag{12}$$

where $\omega_{\nu,a,b,c}(z)$ is defined by (8).

Proof. From (8), we have

$$2z - \omega_{\nu,a,b,c}(z) = z - \sum_{n=2}^{\infty} \frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} z^n.$$

Now, to show that $2z - \omega_{\nu,a,b,c}(z) \in C(\lambda, \alpha)$, then we must show that

$$\sum_{n=2}^{\infty} n\Big(n - \lambda\alpha n - \alpha + \lambda\alpha\Big)\frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} \leq 1-\alpha.$$

According to Lemma 1.6, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\Big(n - \lambda\alpha n - \alpha + \lambda\alpha\Big)\frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} \\
&= (1-\lambda\alpha)\sum_{n=2}^{\infty} n^2\frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} - (\alpha+\lambda\alpha)\sum_{n=2}^{\infty} n\frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} \\
&= (1-\lambda\alpha)\sum_{n=2}^{\infty}(n+1)^2\frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} - (\alpha+\lambda\alpha)\sum_{n=2}^{\infty}(n+1)\frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} \\
&= (1-\lambda\alpha)\sum_{n=1}^{\infty}(n^2+2n+1)\frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} - (\alpha+\lambda\alpha)\sum_{n=1}^{\infty}(n+1)\frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} \\
&= \begin{array}{c} (1-\lambda\alpha)(\omega''(1) - \omega'(1) + \omega(1)) + 2(1-\lambda\alpha)(\omega'(1) - \omega(1)) + (1-\lambda\alpha)(\omega(1)-1) \\ -(\alpha+\lambda\alpha)(\omega'(1) - \omega(1)) - (\alpha+\lambda\alpha)(\omega(1)-1) \end{array} \\
&= (1-\lambda\alpha)\omega''(1) + (3 - 4\lambda\alpha - \alpha)\omega'(1) + \alpha(2\lambda+1) - 1.
\end{aligned}$$

The last expression is bounded above by $1-\alpha$ if and only if the condition (12) holds. then proof end.

Theorem 2.3 *Let $c < 0$ and $\nu > -1$. If $f \in R^{\tau}(A, B)$ and satisfies the condition*

$$\tau(A-B)\Big((1-\lambda\alpha)\omega'_{\nu,a,b,c}(1) - (2 - \alpha(1+\lambda))\omega_{\nu,a,b,c}(1) + 1 - \alpha\Big) \leq 1-\alpha, \tag{13}$$

then $\mathfrak{R}_{a,b,c,\nu}f(z) \in K(\alpha, \lambda)$.

Proof. Since $f \in R^{\tau}(A, B)$, then by Lemma 1.7, we have $|a_n| \leq \frac{\tau(A-B)}{n}$,

$\forall n \in \mathbb{N}$. Now,

$$\begin{aligned}
&\sum_{n=2}^{\infty}\Big(n(1-\lambda\alpha)-\alpha(1-\lambda)\Big)\left|\frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}}a_n\right| \\
&\leq \tau(A-B)\sum_{n=2}^{\infty}\left(\frac{n(1-\lambda\alpha)-\alpha(1-\lambda)}{n}\right)\frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} \\
&< \tau(A-B)\sum_{n=2}^{\infty}\Big(n(1-\lambda\alpha)-\alpha(1-\lambda)\Big)\frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} \\
&\tau(A-B)\sum_{n=1}^{\infty}\Big((n+1)(1-\lambda\alpha)-\alpha(1-\lambda)\Big)\frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} \\
&= \tau(A-B)\Big((1-\lambda\alpha)\omega'_{\nu,a,b,c}(1)-(2-\alpha(1+\lambda))\omega_{\nu,a,b,c}(1)+1-\alpha\Big),
\end{aligned}$$

the last expression is bounded above by $1-\alpha$ if and only if the condition (13) holds true.

Theorem 2.4 *Let $c<0$ and $\nu>-1$. If $f \in R^{\tau}(A,B)$ and satisfies the condition*

$$\tau(A-B)\left((1-\lambda\alpha)\omega'_{\nu,a,b,c}(1)-(2-\alpha(1+\lambda))\omega_{\nu,a,b,c}(1)+1-\alpha\right)\leq 1-\alpha, \tag{14}$$

then $\mathfrak{R}_{a,b,c,\nu}f(z) \in C(\alpha,\lambda)$.

Proof. By letting $f \in R^{\tau}(A,B)$, then by using Lemma 1.7, then we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}n\big(n(1-\lambda\alpha)-\alpha(1-\lambda)\big)\left|\frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}}a_n\right| \\
&\leq \tau(A-B)\sum_{n=2}^{\infty}\big(n(1-\lambda\alpha)-\alpha(1-\lambda)\big)\frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}} \\
&= \tau(A-B)\sum_{n=1}^{\infty}\big((n+1)(1-\lambda\alpha)-\alpha(1-\lambda)\big)\frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}} \\
&= \tau(A-B)\left((1-\lambda\alpha)\omega'_{\nu,a,b,c}(1)-(2-\alpha(1+\lambda))\omega_{\nu,a,b,c}(1)+1-\alpha\right)
\end{aligned}$$

the last expression is bounded above by $1-\alpha$ if and only if the condition (14) holds true.

3 An Integral operator

In this section, we introduced new integral operator $D_{\nu,a,b,c}(z)$ as follows:

$$D_{\nu,a,b,c}(z) = \int_0^z \frac{2\zeta - \omega_{\nu,a,b,c}(\zeta)}{\zeta} d\zeta, \tag{15}$$

where $\omega_{\nu,a,b,c}(z)$ is defined by (8).

Theorem 3.1 *Let $c < 0$ and $\nu > -1$. then $D_{\nu,a,b,c} \in C(\alpha, \lambda)$ if and only if*

$$(1 - \lambda\alpha)\omega'_{\nu,a,b,c}(1) \leq (2 - \alpha(1 + \lambda))\omega_{\nu,a,b,c}(1)$$

Proof. From equation (15), we have

$$\begin{aligned} D_{\nu,a,b,c}(z) &= \int_0^z \frac{2\zeta - \omega_{\nu,a,b,c}(\zeta)}{\zeta} d\zeta \\ &= \int_0^z 1 - \sum_{n=2}^{\infty} \frac{(-c)^{n-1}(2n + a - 2)}{a4^{n-1}(n-1)!(\nu + 1)_{n-1}} \zeta^{n-1} d\zeta \\ &= z - \sum_{n=2}^{\infty} \frac{(-c)^{n-1}(2n + a - 2)}{a4^{n-1}(n-1)!(\nu + 1)_{n-1}} \int_0^z \zeta^{n-1} d\zeta \\ &= z - \sum_{n=2}^{\infty} \frac{(-c)^{n-1}(2n + a - 2)}{a4^{n-1}(n)!(\nu + 1)_{n-1}} z^n. \end{aligned}$$

In order to establish that $D(\nu, a, c, z)$ belongs to the class $C(\lambda, \alpha)$, it is necessary to demonstrate the validity of the condition specified in Equation (1.6):

$$\sum_{n=2}^{\infty} n\Big(n - \lambda\alpha n - \alpha + \lambda\alpha\Big) \frac{(-c)^{n-1}(2n + a - 2)}{a4^{n-1}(n)!(\nu + 1)_{n-1}} \leq 1 - \alpha.$$

Now,

$$\sum_{n=2}^{\infty} n\Big(n-\lambda\alpha n-\alpha+\lambda\alpha\Big)\frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n)!(\nu+1)_{n-1}}$$
$$=\sum_{n=2}^{\infty}\Big(n-\lambda\alpha n-\alpha+\lambda\alpha\Big)\frac{(-c)^{n-1}(2n+a-2)}{a4^{n-1}(n-1)!(\nu+1)_{n-1}}$$
$$= \ (1-\lambda\alpha)\sum_{n=1}^{\infty} n\tfrac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}}+(1-\lambda\alpha)\sum_{n=1}^{\infty}\tfrac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}}-\alpha(1-\lambda)\sum_{n=1}^{\infty}\tfrac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}}$$
$$=(1-\lambda\alpha)\sum_{n=1}^{\infty} n\frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}}+(1-\alpha)\sum_{n=1}^{\infty}\frac{(-c)^{n}(2n+a)}{a4^{n}(n)!(\nu+1)_{n}}$$
$$=(1-\lambda\alpha)(\omega'_{\nu,a,b,c}(1)-\omega_{\nu,a,b,c}(1))-(1-\alpha)(\omega_{\nu,a,b,c}(1)-1)$$
$$=(1-\lambda\alpha)\omega'_{\nu,a,b,c}(1)-(2-\alpha(1+\lambda))\omega_{\nu,a,b,c}(1)+1-\alpha.$$

If the condition specified in equation (1.6) is satisfied, the final expression is limited to a maximum value of $1-\alpha$, and this concludes the proof.

References

[1] A.O. Mostafa, A study on starlike and convex properties for hypergeometric functions, J. Inequal. Pure Appl. Math., 10(3) (2009), Art., 87, 1–16.

[2] O. Altintas and S. Owa, On subclasses of univalent functions with negative coefficients, East Asian mathematical journal., 4 (1988), 41–56.

[3] A. Baricz, E. Deniz, and N. Yaqmur, (2016). Close-to-convexity of normalized Dini functions. Mathematische Nachrichten, 289(14-15), 1721-1726.

[4] E.Deniz, S.Goren, , and M. Caglar,(2017, April). Starlikeness and convexity of the generalized Dini functions. In AIP Conference Proceedings (Vol. 1833, No. 1, p. 020004).

[5] K.K. Dixit and S. K. Pal, On a class of univalent functions related to complex order. Indian J. Pure Appl. Math, 26(9) (1995), 889-896.

[6] Amourah, A.; Alsoboh, A.; Ogilat, O.; Gharib, G.M.; Saadeh, R.; Al Soudi, M. A Generalization of Gegenbauer Polynomials and Bi-Univalent Functions. Axioms 2023, 12, 128. https://doi.org/10.3390/axioms12020128

[7] A. Alsoboh, A. Amourah, M. Darus, R. I. Al Sharefeen, Applications of Neutrosophic q–Poisson Distribution Series for subclass of Analytic Functions and bi–univalent functions, Mathematics, 11(4) (2023), 868.

[8] Alsoboh, A.; Amourah, A.; Sakar, F.M.; Ogilat, O.; Gharib, G.M.; Zomot, N. Coefficient Estimation Utilizing the Faber Polynomial for a Subfamily of Bi-Univalent Functions. Axioms 2023, 12, 512. https://doi.org/10.3390/axioms12060512

[9] Al-Hawary, T.; Amourah, A.; Alsoboh, A.; Alsalhi, O. A New Comprehensive Subclass of Analytic Bi-Univalent Functions Related to Gegenbauer Polynomials. Symmetry 2023, 15, 576. https://doi.org/10.3390/sym15030576

[10] Alsoboh, A.; Amourah, A.; Darus, M.; Rudder, C.A. Investigating New Subclasses of Bi-Univalent Functions Associated with q-Pascal Distribution Series Using the Subordination Principle. Symmetry 2023, 15, 1109. https://doi.org/10.3390/sym15051109

[11] A. Alsoboh, M. Darus, *On Fekete–Szegö problems for certain subclasses of analytic functions defined by differential operator involving-Ruscheweyh Operator*. Journal of Function Spaces, 2020 (2020).

[12] A. Alsoboh, M. Darus, *On Fekete-Szego Problem Associated with $q-$derivative Operator*. Conference Series. IOP Publishing, 1212. 1 (2019).

[13] A. Alsoboh, M. Darus, A q-Starlike Class of Harmonic Meromorphic Functions Defined by q-Derivative Operator. International Conference on Mathematics and Computations. Springer Nature Singapore, 2022.

[14] A.Alsoboh, M.Darus, A.Amourah and W.G.Atshan, A certain subclass of harmonic meromorphic functions with respect to k-symmetric points, International Journal of Open Problems in Complex Analysis, 15(1) (2023),1–16.

[15] B. Alshlool, A. Abu Alasal, A. Mannaa'a, A.Alsoboh and A. Amourah, Consolidate a certain class of (p;q)-Lucas polynomial based bi-univalent functions with a specific discrete probability distribution, International Journal of Open Problems in Complex Analysis 15(1)(2023),26–37.

[16] Alsoboh, A.; Darus, M. Certain subclass of Meromorphic Functions involving q-Ruscheweyh differential operator. Transylv. J. Mech. Math. 2019, 11, 10–18.

[17] Alsoboh, A.; Darus, M. On Fekete–Szegö problems for certain subclasses of analytic functions defined by differential operator involving-Ruscheweyh Operator. J. Funct. Spaces 2020, 2020, 8459405.

[18] A.Alsoboh and M.Darus, On q-starlike functions with respect to k-symmetric points, Acta Univer. Apulensis 60 (2019),61–73.

[19] Amourah, A.; Alomari, M.; Yousef, F.; Alsoboh, A. Consolidation of a Certain Discrete Probability Distribution with a Subclass of Bi-Univalent Functions Involving Gegenbauer Polynomials. *Math. Probl. Eng.* **2022**, *2022*, 6354994.

[20] H. Aldweby and M. Darus, Some subordination results on q-analogue of Ruscheweyh differential operator, Abst. Appl. Anal., 2014, Article ID 958563, 1-6.

[21] M. K. Aouf, On a new criteria for univalent functions of order α,Rend. Math. Series-II, (1991), 47-59.

[22] M. K. Aouf and H. E. Darwish, On inequalities for certain analytic functions involving Ruscheweyh derivative, J. Math., 21 (1995), no.4, 387-393.

[23] M. K. Aouf and H. E. Darwish, and A. A. Attiya, A remark on certain regular functions defined by Ruscheweyh derivative, Proc. Pakistan. Acal. Sci., 37 (2000), no.1, 67-69.

[24] M. K. Aouf and A. A. Al-Dohiman, Fixed second coefficient for certain subclasses of starlike functions with negative coefficients, Demonstratio Math., 38 (2005), no. 3, 551-565.

[25] A. K. Aouf A. O.Mostafa, Subordinating results for classes of functions defined by Sălăgean type q-derivative operator, Filomat., 34 (2020) , no. 7, 2283-2292.

[26] A. W. Goodman, On uniformry convex functions, Ann. Polon. Math., 59(1991), 87-92.

[27] H. S. Wilf, Subordinating factor sequences for convex maps of the unit circle, Proc. Amer. Math. Soc. 12 (1961), 689-693.