Certain Applications of Differential Subordination to Analytic Functions

Sukhwinder Singh, Sushma Gupta and Sukhjit Singh

Department of Applied Sciences
Baba Banda Singh Bahadur Engineering College
Fatehgarh Sahib-140 407, Punjab, India
e-mail: ssbilling@gmail.com

Department of Mathematics
Sant Longowal Institute of Engineering & Technology
Deemed University, Longowal-148 106, Punjab, India
e-mail: sushmagupta1@yahoo.com
e-mail: sukhjit_d@yahoo.com

Abstract

In the present paper, we study two subclasses of analytic functions recently introduced by Owa et al. [3] and extend their results. Mathematica 5.2 is used to plot the extended regions. We also obtain some results in regard of problems left open by them.

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1 Introduction

Let \( \mathcal{A} \) be the class of functions \( f \), analytic in the open unit disk \( E = \{ z : |z| < 1 \} \) and normalized by the conditions \( f(0) = f'(0) - 1 = 0 \).

Let \( f \) be analytic in \( E \), \( g \) analytic and univalent in \( E \) and \( f(0) = g(0) \). Then, by the symbol \( f(z) \prec g(z) \) (\( f \) subordinate to \( g \)) in \( E \), we shall mean \( f(E) \subset g(E) \).
Let $\psi : C \times C \to C$ be an analytic function, $p$ be an analytic function in $E$, with $(p(z),zp'(z)) \in C \times C$ for all $z \in E$ and $h$ be univalent in $E$, then the function $p$ is said to satisfy first order differential subordination if

$$\psi(p(z),zp'(z)) < h(z), \psi(p(0),0) = h(0).$$

(1)

A univalent function $q$ is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p < q$ for all $p$ satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} < q$ for all dominants $q$ of (1), is said to be the best dominant of (1). The best dominant is unique up to a rotation of $E$.

A function $f \in A$ is said to be close-to-convex if there is a real number $\alpha, -\pi/2 < \alpha < \pi/2$, and a convex function $g$ (not necessarily normalized) such that

$$\Re\left( e^{i\alpha} \frac{f'(z)}{g'(z)} \right) > 0, z \in E.$$

It is well-known that every close-to-convex function is univalent. In 1934/35, Noshiro [2] and Warchawski [4] obtained a simple but interesting criterion for univalence of analytic functions. They proved that if an analytic function $f$ satisfies the condition $\Re f'(z) > 0$ for all $z$ in $E$, then $f$ is close-to-convex and hence univalent.

A function $f \in A$ is said to belong to the class $S_{\delta}(\alpha)$ if it satisfies the condition

$$(f'(z))^\delta < \frac{\alpha(1 - z)}{\alpha - z}, z \in E,$$

for some real numbers $\alpha > 1$ and $\delta > 0$. We notice that the functions in $S_1(\alpha)$ satisfy the condition $\Re f'(z) > 0$ and therefore, are close-to-convex and hence univalent.

A function $f \in A$ is said to belong to the class $T_{\delta}(\alpha)$ if it satisfies the condition

$$\left( \frac{1}{f'(z)} \right)^\delta < \frac{\alpha(1 - z)}{\alpha - z}, z \in E,$$

for some real numbers $\alpha > 1$ and $\delta > 0$.

The above defined classes are introduced by Owa et al. [3]. They proved the following results for these classes.

**Theorem 1.1** If $f \in A$ satisfies

$$\Re \frac{zf''(z)}{f'(z)} < \frac{\alpha - 1}{2\delta(\alpha + 1)}, z \in E,$$

for some real numbers $\alpha > 1$ and $\delta > 0$, then $f \in S_{\delta}(\alpha)$. 
Theorem 1.2 If $f \in A$ satisfies
\[ \Re \frac{zf''(z)}{f'(z)} > \frac{1 - \alpha}{2\delta (\alpha + 1)}, \quad z \in E, \]
for some real numbers $\alpha > 1$ and $\delta > 0$, then $f \in T_\delta(\alpha)$.

They also left one problem open with each of the above classes. The main objective of this paper is to extend the results of Owa et al. [3] and obtain some results in regard of open problems raised by them.

2 Preliminaries

We shall use the following lemma to prove our main result.

Lemma 2.1 ([1], p.132, Theorem 3.4 h) Let $q$ be univalent in $E$ and let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(E)$, with $\phi(w) \neq 0$, when $w \in q(E)$. Let $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either
(i) $h$ is convex, or
(ii) $Q$ is starlike.
In addition, assume that
(iii) $\Re \frac{zh'(z)}{Q(z)} > 0, \quad z \in E.$
If $p$ is analytic in $E$, with $p(0) = q(0), p(E) \subset D$ and
\[ \theta[p(z)] + zp'(z)\phi[p(z)] < \theta[q(z)] + zq'(z)\phi[q(z)], \]
then $p \prec q$ and $q$ is the best dominant.

3 Main Results

The following result is essentially due to Miller and Mocanu [1,p.76]. However, we present an alternative proof of the same by using Lemma 2.1.

Theorem 3.1 Let $q$ ($q(z) \neq 0$) be a univalent function in $E$ such that $\frac{zq'(z)}{q(z)}$ is starlike in $E$. If an analytic function $p$ ($p(z) \neq 0$) satisfies the differential subordination
\[ \frac{zp'(z)}{p(z)} < \frac{zq'(z)}{q(z)}, \quad z \in E, \]
then $p(z) \prec q(z)$ and $q$ is the best dominant.
Proof. Let us define the functions $\theta$ and $\phi$ as follows:

$$\theta(w) = 0,$$

and

$$\phi(w) = \frac{1}{w}.$$ 

Obviously, the functions $\theta$ and $\phi$ are analytic in domain $D = C \setminus \{0\}$ and $\phi(w) \neq 0$ in $D$.

Now, define the functions $Q$ and $h$ as follows:

$$Q(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = \frac{zq'(z)}{q(z)}.$$ 

Given that, $Q$ is starlike in $E$ and we also have $\Re \frac{z}{Q(z)} h'(z) > 0, \ z \in E$.

Thus conditions (ii) and (iii) of Lemma 2.1, are satisfied.

In view of (2), we have

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof, now, follows from Lemma 2.1.

On writing $p(z) = (f'(z))^\delta, \ f \in A$ in Theorem 3.1, we obtain the following result.

**Theorem 3.2** Suppose $\delta > 0$ is a real number. Let $q (q(z) \neq 0)$ be a univalent function in $E$ such that $\frac{zq'(z)}{q(z)}$ is starlike in $E$. If $f \in A (f'(z) \neq 0)$ satisfies the differential subordination

$$\frac{zf''(z)}{f'(z)} \prec \frac{1}{\delta} \frac{zq'(z)}{q(z)}, \ z \in E,$$

then $(f'(z))^\delta \prec q(z)$ and $q$ is the best dominant.

By setting $p(z) = \left(\frac{1}{f'(z)}\right)^\delta, \ f \in A$ in Theorem 3.1, we have the following result.
Theorem 3.3 Let $\delta > 0$ be a real number. Let $q$ ($q(z) \neq 0$) be a univalent function in $E$ such that $\frac{zq'(z)}{q(z)}$ is starlike in $E$. If $f \in A$ ($f'(z) \neq 0$) satisfies the differential subordination

$$zf''(z) < -\frac{1}{\delta} \frac{zq'(z)}{q(z)}, \quad z \in E,$$

then $\left(\frac{1}{f'(z)}\right)^\delta < q(z)$ and $q$ is the best dominant.

4 Applications to Analytic Functions

If we select the dominant $q(z) = \frac{\alpha(1-z)}{\alpha - z}$, a little calculation yields that $\frac{zq'(z)}{q(z)}$ is starlike in $E$ for real $\alpha > 1$. From Theorem 3.2, we have the following result.

Theorem 4.1 If $f \in A$ ($f'(z) \neq 0$) satisfies the differential subordination

$$zf''(z) < \frac{(1-\alpha)z}{\delta(\alpha - z)(1-z)} = F_1(z), \quad z \in E,$$

then $f \in S_\delta(\alpha)$, where $\alpha > 1$ and $\delta > 0$ are some real numbers.

Remark 4.1 For $\alpha = 2$, $\delta = 1$, the constant on right hand side of Theorem 1.1 reduces to $\frac{1}{6}$. In Figure 4.1, we plot the dotted line $\Re(z) = \frac{1}{6}$ and the curve $F_1(z)$, $z \in E$. According to result of Owa et al. [3], the result in Theorem 1.1 holds only if the operator $zf''(z)$ lies in the portion of the plane left to the dotted line $\Re(z) = \frac{1}{6}$. Theorem 4.1 shows that the result holds even when the operator $zf''(z)$ lies in the portion of the plane left to the plotted curve $F_1(z)$, thus extending the region of variability of the operator $zf''(z)$ for the required implication. The extended portion lies between the dotted line $\Re(z) = \frac{1}{6}$ and the curve $F_1(z)$, $z \in E$. 
By setting $q(z) = \frac{\alpha(1 - z)}{\alpha - z}$ in Theorem 3.3, we obtain the following result.

**Theorem 4.2** Suppose $f \in A$ ($f'(z) \neq 0$) satisfies the differential subordination

$$zf''(z) \prec \frac{(\alpha - 1)z}{\delta(\alpha - z)(1 - z)} = F_2(z), \ z \in E,$$

then $f \in T_\delta(\alpha)$, where $\alpha > 1$ and $\delta > 0$ are some real numbers.

**Remark 4.2** For $\alpha = 2$, $\delta = \frac{1}{2}$, the constant on right hand side of Theorem 1.2 reduces to $-\frac{1}{3}$. In Figure 4.2, we plot the dotted line $\Re(z) = -\frac{1}{3}$ and the curve $F_2(z)$. According to result of Owa et al. [3], the result in Theorem 1.2 holds only if the operator $\frac{zf''(z)}{f'(z)}$ takes values in the portion of the plane right to the dotted line $\Re(z) = -\frac{1}{3}$. Theorem 4.2 shows that the result holds when the operator $\frac{zf''(z)}{f'(z)}$ lies in the portion of the plane right to the plotted curve $F_2(z)$, thus extending the region of variability of the operator $\frac{zf''(z)}{f'(z)}$ for the required implication. In Figure 4.2, extended region lies between the dotted line and the curve.
Figure 4.2 (when $\alpha = 2$, $\delta = 1/2$)

Now we obtain the results related to the problems left open by Owa et al. [3]. These are given below in Theorem 4.3 and Theorem 4.4. For this if we take $q(z) = \frac{1 + (1 - 2\beta)z}{1 - z}$ in Theorem 3.2, then a little calculation yields that $\frac{zq'(z)}{q(z)}$ is starlike in $E$ for $0 \leq \beta < 1$ and we arrive at the following result.

**Theorem 4.3** If $f \in \mathcal{A}$ ($f'(z) \neq 0$) satisfies the differential subordination

$$zf''(z)\prec \frac{2(1 - \beta)z}{\delta(1 + (1 - 2\beta)z)(1 - z)}, z \in E,$$

then $(f'(z))^{\delta} < \frac{1 + (1 - 2\beta)z}{1 - z}$, for some real numbers $0 \leq \beta < 1$ and $\delta > 0$.

In view of Theorem 4.3, we have the following result.

**Corollary 4.1** Let $0 \leq \beta < 1$ and $\delta > 0$ be real numbers. If $f \in \mathcal{A}$ ($f'(z) \neq 0$) satisfies the condition

$$\Re \frac{zf''(z)}{f'(z)} > \begin{cases} \frac{1}{2\delta} \left(1 - \frac{1}{1 - \beta}\right), & 0 \leq \beta \leq 1/2, \\ \frac{1}{2\delta} \left(1 - \frac{1}{\beta}\right), & 1/2 \leq \beta < 1, \end{cases}$$

then $\Re (f'(z))^{\delta} > \beta$.

**Remark 4.3** Setting $\beta = \delta = 1/2$ in above Corollary, we obtain the following result of Miller and Mocanu [1,p.57]:

- Setting $\beta = \delta = 1/2$ in above Corollary, we obtain the following result of Miller and Mocanu [1,p.57]:
For \( f \in A \) \((f'(z) \neq 0)\),
\[
\Re \left( \frac{zf''(z)}{f'(z)} + 1 \right) > 0 \Rightarrow \Re \sqrt{f'(z)} > 1/2, z \in E.
\]

By setting \( q(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \) in Theorem 3.3, we obtain the following result.

**Theorem 4.4** Suppose \( f \in A \) \((f'(z) \neq 0)\) satisfies the differential subordination
\[
\frac{zf''(z)}{f'(z)} < \frac{2(\beta - 1)z}{\delta(1 + (1 - 2\beta)z)(1 - z)}, \quad z \in E,
\]
then \[
\left( \frac{1}{f'(z)} \right)^\delta < \frac{1 + (1 - 2\beta)z}{1 - z}, \quad \text{where } 0 \leq \beta < 1 \text{ and } \delta > 0 \text{ are some real numbers.}
\]

In view of Theorem 4.4, we obtain the following result.

**Corollary 4.2** Let \( 0 \leq \beta < 1 \) and \( \delta > 0 \) be real numbers. If \( f \in A \) \((f'(z) \neq 0)\), satisfies the condition
\[
\Re \left( \frac{zf''(z)}{f'(z)} \right) < \left\{ \begin{array}{ll}
\frac{1}{2\delta} \left( \frac{1}{1-\beta} - 1 \right), & 0 \leq \beta \leq 1/2, \\
\frac{1}{2\delta} \left( \frac{1}{\beta} - 1 \right), & 1/2 \leq \beta < 1,
\end{array} \right.
\]
then \[
\Re \left( \frac{1}{f'(z)} \right)^\delta > \beta.
\]

**5 Open Problem**

It would be of interest to raise the following problems.

(i) For \( 0 \leq \beta < 1, \ \delta > 0 \) and \( f \in A \), does there exist a best value of \( \gamma > 0 \) such that the condition
\[
\Re \frac{zf''(z)}{f'(z)} < \gamma, \quad z \in E,
\]
implies that \( \Re (f'(z))^{\delta} > \beta \) ?

(ii) For \( 0 \leq \beta < 1, \ \delta > 0 \) and \( f \in A \), does there exist a best value of \( \gamma < 0 \) such that the condition
\[
\Re \frac{zf''(z)}{f'(z)} > \gamma, \quad z \in E,
\]
implies that \( \Re \left( \frac{1}{f'(z)} \right)^{\delta} > \beta \) ?

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