A Multiplier Transformation Defined by Convolution Involving a Differential Operator

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Abstract

The object of this paper is to introduce a multiplier transformation defined by convolution involving differential operator given by Al-Oboudi. A new subclass of strongly close-to-convex functions in the open unit disk using this operator will be discussed. Our results include several previous known results as special cases.

Keywords: Analytic function, Starlike and Strongly close-to-convex functions.

AMS Mathematics Subject Classification (2000): 30C45.

1 Introduction

Let $H$ be the class of analytic functions in the open unit disk $U = \{ z : |z| < 1 \}$ and $H[a,n]$ be the subclasses of $H$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$$

Let $A$ be the subclass of $H$ consisting of functions of the form:
\[ f (z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U \quad (1) \]

which are analytic in the unit disk \( U \). Let \( F \) and \( G \) be analytic functions in the unit disk \( U \), the function \( F \) is said to be subordinate to \( G \) or \( G \) is said to be superordinate to \( F \), if there exists a function \( w \) analytic in \( U \) with \( w (0) = 0 \) and \(|w| < 1 \) for \( z \in U \) and such that \( F (z) = G (w (z)) \), \( z \in U \) in such a case, we write \( F \prec G \) or \( F (z) \prec G (z) \) if the function \( G \) is univalent in \( U \), then

\[ F \prec G \iff F (0) = G (0), \quad F (U) \subset G (U). \]

For functions \( f \) given by (1) and \( g (z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in U \). let \((f * g) (z)\) denote the Hadamard product (convolution) of \( f (z) \) and \( g (z) \), defined by :

\[ (f * g) (z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \]

For \( f \in A \), Al-Oboudi [2] introduced the following operator :

\[ D_0^0 f (z) = f (z) \quad (2) \]

\[ D_\lambda^1 f (z) = D_\lambda f (z) = (1 - \lambda) f (z) + \lambda z f' (z) \quad (3) \]

\[ D_\lambda^m f (z) = D_\lambda (D_\lambda^{m-1} f (z)), \quad \lambda > 0 \quad (4) \]

if \( f \) is given by (1), then from (3) and (4) we see that

\[ D_\lambda^m f (z) = z + \sum_{n=2}^{\infty} [1 + (n - 1) \lambda]^m a_n z^n, \quad m \geq 0, \quad \lambda > 0 \]

when \( \lambda = 1 \), we get Salagean differential operator [16].

For any complex number \( s \), we define the multiplier transformation \( I^s_\delta \) of functions \( f \in A \) by :

\[ I^s_\delta f (z) = z + \sum_{n=2}^{\infty} \left( \frac{n + \delta}{1 + \delta} \right)^s a_n z^n, \quad (\delta > -1) \]

By Hadamard product we get \( D^m_{\lambda, \delta} f (z) \) defined by :

\[ D^m_{\lambda, \delta} f (z) = z + \sum_{n=2}^{\infty} [1 + (n - 1) \lambda]^m \left( \frac{n + \delta}{1 + \delta} \right)^s a_n z^n, \]
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\[(s \in C, \lambda > 0, \delta > -1, m \geq 0, z \in U).\]

Obviously, we observe that

\[D_{\lambda, \delta}^{m,s}(D_{\lambda, \delta}^{l,k}f(z)) = D_{\lambda, \delta}^{m+l,s+k}f(z), \quad (s, k \in C, \delta > -1, l, m \geq 0, z \in U).\]

For \(s \in \mathbb{Z}, \delta = 1\) and \(m = 0\) the operator \(D_{\lambda, \delta}^{m,s}\) was studied by Uralegaddi and Somanatha [19], and for \(s \in \mathbb{Z}, m = 0\) the operator \(D_{\lambda, \delta}^{m,s}\) was closely related to multiplier transformations studied by Flett [6], also, for \(s = -1, m = 0\) the operator \(D_{\lambda, \delta}^{m,s}\) belongs to integral operator studied by Owa and Srivastava [14]. And for any negative real number \(s\) and \(\delta = 1\), \(m = 0\) the operator \(D_{\lambda, \delta}^{m,s}\) was a multiplier transformation studied by Jung et al. [7], and for any nonnegative integer \(s\) and \(\delta = 1, m = 0\), the operator \(D_{\lambda, \delta}^{m,s}\) was the differential operator given by Salagean [16]. Finally, for different choices of \(s, \delta\) and \(m\), several operators investigated earlier by other authors (see for example Ahuja [1], Cho and Kim [4], and Lin and Owa [9]) are obtained.

Now, by using \(D_{\lambda, \delta}^{m,s}\), new classes of analytic functions are defined as follows: For \(s \in C, \delta > -1\) and \(m \geq 0\), let \(K_{\lambda, \delta}^{m,s}(\gamma, \alpha, \beta, A, B)\) be the class of functions \(f \in A\) satisfying the condition:

\[
\left| \arg \left( \frac{z(D_{\lambda, \delta}^{m,s}f(z))'}{D_{\lambda, \delta}^{m,s}g(z)} \right) - \gamma \right| < \frac{\pi}{2} \alpha \quad (0 \leq \gamma < 1; 0 < \alpha \leq 1; z \in U)
\]

for some \(g \in S_{\lambda, \delta}^{m,s}(\beta, A, B)\), where

\[
S_{\lambda, \delta}^{m,s}(\beta, A, B) = \left\{ g \in A : \frac{1}{1 - \beta} \left( \frac{z(D_{\lambda, \delta}^{m,s}g(z))'}{D_{\lambda, \delta}^{m,s}g(z)} \right) - \beta < \frac{1 + Az}{1 + Bz} \right\},
\]

\[(0 \leq \beta < 1; -1 \leq B < A \leq 1; z \in U)\]

Note that \(K_{1,0}^{1,0}(\gamma, 1, \beta, 1, -1)\) and \(K_{0,0}^{0,0}(\gamma, 1, \beta, 1, -1)\) are the classes of quasiconvex and close-to-convex functions of order \(\gamma\) and type \(\beta\), respectively introduced and studied by Noor and Alkhora sani [11] and Silverman [17]. Further \(K_{0,0}^{0,1}(0, \alpha, 0, 1, -1) = K_{1,0}^{1,0}(0, \alpha, 0, 1, -1)\) is the class of strongly close-to-convex functions of order \(\alpha\) in the sense of Pommerenke [15]. Finally, notice that for integer \(s\) and \(m = 0\), the class \(K_{0,0}^{0,s}(\gamma, \alpha, \beta, A, B)\) was studied by Cho and Kim [4].

We need the following lemmas to prove our main results:
Lemma 1.1 [5] Let \( h \) be convex univalent in \( U \) with \( h(0) = 1 \) and 
\( \Re(\beta h(z) + \gamma) > 0 \), \((\beta, \gamma \in \mathbb{C})\). If \( p \) is analytic in \( U \) with \( p(0) = 1 \), then
\[
p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad (z \in U)
\]
implies
\[
p(z) \prec h(z)
\]

Lemma 1.2 [10] Let \( h \) be convex univalent in \( U \) and \( w \) be analytic in \( U \) with 
\( \Re w(z) \geq 0 \). If \( p \) is analytic in \( U \) and \( p(0) = h(0) \), then
\[
p(z) + w(z)zp'(z) \prec h(z)
\]
implies
\[
p(z) \prec h(z)
\]

Lemma 1.3 [13] Let \( p \) be analytic in \( U \) with \( p(0) = 1 \) and \( p(z) \neq 0 \) in \( U \). Suppose that there exists a point \( z_0 \in U \) such that :
\[
|\arg p(z)| < \frac{\pi}{2} \eta \quad \text{for} \quad |z| < |z_0| \quad (5)
\]
and
\[
|\arg p(z_0)| = \frac{\pi}{2} \eta \quad (0 < \eta \leq 1). \quad (6)
\]
then we have
\[
\frac{zp'(z_0)}{p(z_0)} = ik\eta, \quad (7)
\]
where
\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2} \eta \quad (8)
\]
and
\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2} \eta \quad (9)
\]
and
\[
p(z_0)^\frac{1}{a} = \pm ia \quad (a > 0). \quad (10)
\]
At first, with the help of Lemma 1.1, we obtain the following theorem :

Theorem 1.4 Let \( h \) be convex univalent in \( U \) with \( h(0) = 1 \) and 
\( \Re((1 - \beta) h(z) + \beta + \delta) > 0 \). If a function \( f \in A \) satisfies the condition
\[
\frac{1}{1 - \beta} \left( \frac{z(D_{\lambda, \delta}^m f(z))^\prime}{D_{\lambda, \delta}^{m+1} f(z)} - \beta \right) < h(z), \quad (0 \leq \beta < 1; z \in U)
\]
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then

\[
\frac{1}{1 - \beta} \left( z \left( \frac{D_{\lambda,\delta}^{m,s} f(z)}{D_{\lambda,\delta}^{m,s} f(z)} \right)' - \beta \right) < h(z), \quad (0 \leq \beta < 1; z \in U)
\]

Proof: Let

\[
p(z) = \frac{1}{1 - \beta} \left( z \left( \frac{D_{\lambda,\delta}^{m,s} f(z)}{D_{\lambda,\delta}^{m,s} f(z)} \right)' - \beta \right)
\]

where \( p \) is analytic function with \( p(0) = 1 \). By using the equation:

\[
z \left( D_{\lambda,\delta}^{m,s} f(z) \right)' = (\delta + 1) D_{\lambda,\delta}^{m,s+1} f(z) - \delta D_{\lambda,\delta}^{m,s} f(z)
\]

we get:

\[
\delta + \beta + (1 - \beta) p(z) = \frac{(\delta + 1) D_{\lambda,\delta}^{m,s+1} f(z)}{D_{\lambda,\delta}^{m,s} f(z)}
\]

(12)

taking logarithmic derivatives in both sides of (12) and multiplying by \( z \), we have

\[
p(z) + \frac{zp'(z)}{\delta + \beta + (1 - \beta) p(z)} = \frac{1}{1 - \beta} \left( z \left( \frac{D_{\lambda,\delta}^{m,s+1} f(z)}{D_{\lambda,\delta}^{m,s} f(z)} \right)' - \beta \right), \quad z \in U.
\]

Applying Lemma 1.1, it follows that \( p < h \), that is

\[
\frac{1}{1 - \beta} \left( z \left( \frac{D_{\lambda,\delta}^{m,s} f(z)}{D_{\lambda,\delta}^{m,s} f(z)} \right)' - \beta \right) < h(z).
\]

Taking \( h(z) = (1 + Az)/(1 + Bz) \), \((-1 \leq B < A \leq 1)\) in Theorem 1.4 we have

**Corollary 1.5** The inclusion relation \( S_{\lambda,\delta}^{m,s+1}(\beta, A, B) \subset S_{\lambda,\delta}^{m,s}(\beta, A, B) \) holds for \( s \in C, \delta > -1, m \geq 0 \).

Letting \( s = \delta = 0, m = 0 \) and \( h(z) = ((1 + z)/(1 - z))^\mu, \quad (0 < \mu \leq 1) \) in Theorem 1.4 we have the following inclusion relation:

**Corollary 1.6** For \( s \in C, \delta > -1, m \geq 0 \) and \( h(z) = ((1 + z)/(1 - z))^\mu, \quad (0 < \mu \leq 1) \) then we have \( C(\mu, \beta) \subset S^*(\mu, \beta) \).
Theorem 1.7  Let $h$ be convex univalent in $U$ with $h(0) = 1$ and  
$\Re \left( (1 - \beta) h(z) + \beta + \frac{1}{\lambda} - 1 \right) > 0$. If a function $f \in A$ satisfies the condition  
\[
\frac{1}{1 - \beta} \left( \frac{z(D_{\lambda, \delta}^{m+1,s} f(z))'}{D_{\lambda, \delta}^{m+1,s} f(z)} - \beta \right) < h(z), \quad (0 \leq \beta < 1; z \in U)
\]
then  
\[
\frac{1}{1 - \beta} \left( \frac{z(D_{\lambda, \delta}^{m,s} f(z))'}{D_{\lambda, \delta}^{m,s} f(z)} - \beta \right) < h(z), \quad (0 \leq \beta < 1; z \in U)
\]
for $s \in C$, $\delta > -1$, $m \geq 0$

**Proof** : Let  
\[
p(z) = \frac{1}{1 - \beta} \left( \frac{z(D_{\lambda, \delta}^{m,s} f(z))'}{D_{\lambda, \delta}^{m,s} f(z)} - \beta \right), \quad (0 \leq \beta < 1; z \in U)
\]
where $p$ is analytic function with $p(0) = 1$. By using the equation  
\[
\lambda z (D_{\lambda, \delta}^{m,s} f(z))' = D_{\lambda, \delta}^{m+1,s} f(z) - (1 - \lambda) D_{\lambda, \delta}^{m,s} f(z)
\]
we get  
\[
\beta + 1 \frac{1}{\lambda} - 1 + (1 - \beta) p(z) = \frac{D_{\lambda, \delta}^{m+1,s} f(z)}{\lambda D_{\lambda, \delta}^{m,s} f(z)}
\]
and taking logarithmic derivatives in both sides of (13) and multiplying by $z$ we get  
\[
p(z) + \frac{zp'(z)}{\beta + 1 \frac{1}{\lambda} - 1 + (1 - \beta) p(z)} = \frac{1}{1 - \beta} \left( \frac{z(D_{\lambda, \delta}^{m+1,s} f(z))'}{D_{\lambda, \delta}^{m+1,s} f(z)} - \beta \right).
\]
Applying Lemma 1.1 it follows that $p < h$, that is  
\[
\frac{1}{1 - \beta} \left( \frac{z(D_{\lambda, \delta}^{m,s} f(z))'}{D_{\lambda, \delta}^{m,s} f(z)} - \beta \right) < h(z).
\]
Taking $h(z) = (1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$ in Theorem 1.7 we have
Corollary 1.8 The inclusion relation \( S_{m, s}^{m+1, s} (\beta, A, B) \subset S_{m, s}^{m, s} (\beta, A, B) \) holds for \( s \in C, \delta > -1, m \geq 0 \).

Theorem 1.9 Let \( h \) be convex univalent in \( U \), with \( h(0) = 1 \) and \( \Re((1 - \beta) h(z) + \beta + c) > 0 \). If a function \( f \in A \) satisfies the condition

\[
\frac{1}{1 - \beta} \left( \frac{z (D_{m, s}^{m, s} f (z))'}{D_{m, s}^{m, s} f (z)} - \beta \right) < h(z), \quad (0 \leq \beta < 1; z \in U)
\]

then

\[
\frac{1}{1 - \beta} \left( \frac{z (D_{m, s}^{m, s} F_c (f) (z))'}{D_{m, s}^{m, s} F_c (f) (z)} - \beta \right) < h(z), \quad (0 \leq \beta < 1; z \in U)
\]

where \( F_c \) be the integral operator defined by

\[
F_c (f) := F_c (f) (z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f (t) \ dt, \quad (c \geq 0) \quad (14)
\]

Proof: From (14) we have

\[
z (D_{m, s}^{m, s} F_c (f) (z))' = (c + 1) D_{m, s}^{m, s} f (z) - c D_{m, s}^{m, s} F_c (f) (z) \quad (15)
\]

By using the same technique as in the proof of the Theorem 1.4 and Lemma 1.1 the required result is obtained.

Letting \( h(z) = (1 + Az)/(1 + Bz) \), \((-1 \leq B < A \leq 1)\) in Theorem 1.9 we have immediately the following

Corollary 1.10 If \( f \in S_{m, s}^{m, s} (\beta, A, B) \), then \( F_c (f) (z) \in S_{m, s}^{m, s} (\beta, A, B) \) where \( F_c \) is the integral defined by (14).

Now, we obtain the following:

Theorem 1.11 Let \( f \in A \) and \( 0 < \alpha \leq 1, \quad 0 \leq \gamma < 1 \). If

\[
\left| \arg \left( \frac{z (D_{m, s}^{m, s} f (z))'}{D_{m, s}^{m, s} g (z)} - \gamma \right) \right| < \frac{\pi}{2} \alpha
\]

for some \( g \in S_{m, s}^{m, s+1} (\beta, A, B) \), then

\[
\left| \arg \left( \frac{z (D_{m, s}^{m, s} f (z))'}{D_{m, s}^{m, s} g (z)} - \gamma \right) \right| < \frac{\pi}{2} \eta
\]
where $\eta$ ($0 < \eta \leq 1$) is the solution of the equation:

$$\alpha = \begin{cases} 
\eta + \frac{2}{\pi} \tan^{-1} \left( \frac{\eta \cos \frac{\pi}{2} t_1}{(1 - \beta)(1 + A) + \beta + \delta + \eta \sin \frac{\pi}{2} t_1} \right) & \text{for } B \neq -1 \\
\eta & \text{for } B = -1 
\end{cases}$$

(16)

and

$$t_1 = \frac{2}{\pi} \sin^{-1} \left( \frac{(1 - \beta)(A - B)}{(1 - \beta)(1 - AB) + (\beta + \delta)(1 - B^2)} \right).$$

(17)

**Proof:** Let

$$p(z) = \frac{1}{1 - \gamma} \left( \frac{z \left( D_{m,s}^{(1)} f(z) \right)' - (1 - \gamma) p(z) + \gamma z \left( D_{m,s}^{(1)} g(z) \right)'}{D_{m,s}^{(1)} g(z)} \right).$$

Using (11) and simplifying, we have

$$((1 - \gamma) p(z) + \gamma D_{m,s}^{(1)} f(z) = (\delta + 1) D_{m,s}^{(1)} f(z) - \delta D_{m,s}^{(1)} f(z).$$

(18)

Differentiating (18) and multiplying by $z$, we obtain

$$(1 - \gamma) z p'(z) D_{m,s}^{(1)} g(z) + ((1 - \gamma) p(z) + \gamma z \left( D_{m,s}^{(1)} g(z) \right)'

= (\delta + 1) z \left( D_{m,s}^{(1)} f(z) \right)' - \delta z \left( D_{m,s}^{(1)} f(z) \right)'.$$

(19)

Since $g \in S_{m,s}^{(1)} (\beta, A, B)$, by Corollary 1.5, we know that $g \in S_{m,s}^{(1)} (\beta, A, B)$. Let

$$q(z) = \frac{1}{1 - \beta} \left( \frac{z \left( D_{m,s}^{(1)} g(z) \right)' - \beta}{D_{m,s}^{(1)} g(z)} \right).$$

Then using (11) once again, we have

$$(1 - \beta) q(z) + \beta + \delta = (\delta + 1) \frac{D_{m,s}^{(1)} g(z)}{D_{m,s}^{(1)} g(z)}. $$

(20)

From (19) and (20) we obtain

$$\frac{1}{1 - \gamma} \left( \frac{z \left( D_{m,s}^{(1)} f(z) \right)'}{D_{m,s}^{(1)} g(z)} - \gamma \right) = p(z) + \frac{z p'(z)}{(1 - \beta) q(z) + \beta + \delta}.$$

While, by using the result of Silverman and Silvia [18], we have

$$\left| q(z) - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2}, \quad (z \in U; B \neq -1)$$

(21)
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\[ \mathcal{R}\{q(z)\} > \frac{1 - A}{2}, \quad (z \in U; B \neq -1) \quad (22) \]

Then from (21) and (22), we obtain

\[ (1 - \beta) q(z) + \beta + \delta = \rho e^{i\phi}, \]

where

\[ \left\{ \begin{array}{l}
\frac{(1-\beta)(1-A)}{1-B} + \beta + \delta < \rho < \frac{(1-\beta)(1+A)}{1+B} + \beta + \delta \\
-t_1 < \phi < t_2 \end{array} \right. \]

for \( B \neq -1 \).

When \( t_1 \) is given by (17), and

\[ \left\{ \begin{array}{l}
\frac{(1-\beta)(1-A)}{2} + \beta + \delta < \rho < \infty \\
-1 < \phi < 1 \end{array} \right. \]

for \( B = -1 \).

We note that \( p \) is analytic in \( U \), by applying the assumption and Lemma 1.2 with \( w(z) = 1/(1 - \beta) q(z) + \beta + \delta \). Hence \( p(z) \neq 0 \) in \( U \).

If there exists a point \( z_0 \in U \) such that the conditions (5) and (6) are satisfied, then (by Lemma 1.3) we obtain (7) under the restrictions (8), (9) and (10).

At first, suppose that \( p(z_0)^{1/2} = ia, \quad (a > 0) \). Then we obtain

\[
\arg\left( p(z_0) + \frac{z_0 p'(z_0)}{(1 - \beta) q(z_0) + \beta + \delta} \right)
\]

\[ = \frac{\pi}{2} \eta + \arg\left( 1 + i\eta k \left( \rho e^{i\pi/2} \right)^{-1} \right) \]

\[ \geq \frac{\pi}{2} \eta + \tan^{-1}\left( \frac{\eta k \sin \frac{\pi}{2} (1 - \phi)}{\rho + \eta k \cos \frac{\pi}{2} (1 - \phi)} \right) \]

\[ \geq \frac{\pi}{2} \eta + \tan^{-1}\left( \frac{\eta \cos \frac{\pi}{2} t_1}{\frac{(1-\beta)(1+A)}{1+B} + \beta + \delta + \eta \sin \frac{\pi}{2} t_1} \right) \]

\[ = \frac{\pi}{2} \alpha \]

where \( \alpha \) and \( t_1 \) given by (16) and (17), respectively. Similarly for the case \( B = -1 \) we have

\[
\arg\left( p(z_0) + \frac{z_1 p'(z_0)}{(1 - \beta) q(z_0) + \beta + \delta} \right) \geq \frac{\pi}{2} \eta.
\]

These evidently contradict the assumption of Theorem 1.11.
Next, suppose that \( p(z_0)^{\frac{1}{n}} = -ia, \quad (a > 0) \). Applying the same method as the above, we have

\[
\arg \left( p(z_0) + \frac{z_0p'(z_0)}{(1 - \beta)q(z_0) + \beta + \delta} \right) \leq -\frac{\pi}{2} \eta - \tan^{-1} \left( \frac{\eta \cos \frac{\pi}{2} t_1}{(1 - \beta)(1 + A) + \beta + \delta + \eta \sin \frac{\pi}{2} t_1} \right)
\]

\[
= -\frac{\pi}{2} \alpha,
\]

where \( \alpha \) and \( t_1 \) are given by (16) and (17), respectively. Similarly, for the case \( B = -1 \) we have

\[
\arg \left( p(z_0) + \frac{z_1p'(z_0)}{(1 - \beta)q(z_0) + \beta + \delta} \right) \leq -\frac{\pi}{2} \eta.
\]

These also contradict to the assumption of Theorem 1.11. Therefore we complete the proof of Theorem 1.11.

From Theorem 1.11, we see easily the following:

**Corollary 1.12** The inclusion relation

\[ K_{\lambda, \delta}^{m+1}(\gamma, \alpha, \beta, A, B) \subset K_{\lambda, \delta}^{m, s}(\gamma, \alpha, \beta, A, B) \] holds for \( s \in C, \delta > -1, m \geq 0 \).

Taking \( s = -1, \delta = 0 \) and \( m = \lambda = 1 \) in Theorem 1.11 we have

**Corollary 1.13** Let \( f \in A \). If

\[
\left| \arg \left( \frac{(zf'(z))'}{g'(z)} - \gamma \right) \right| < \frac{\pi}{2} \alpha, \quad (0 \leq \gamma < 1, 0 < \alpha \leq 1)
\]

for some \( g \in S_{1,0}^{1,0}(\beta, A, B) \), then

\[
\left| \arg \left( \frac{zf'(z)}{g(z)} - \gamma \right) \right| < \frac{\pi}{2} \eta
\]

where \( \eta, (0 < \eta \leq 1) \) is the solution of the equation given by (16).

**Remark** : If we put \( A = 1, B = 1 \) and \( \eta = 1 \) in Corollary 1.13 then we see that every quasi-convex function of order \( \gamma \) and type \( \beta \) is close-to-convex function of order \( \gamma \) and type \( \beta \), which reduced to the result obtained by Noor [12].
**Theorem 1.14** Let \( f \in A \) and \( 0 < \alpha \leq 1, \ 0 \leq \gamma < 1 \). If

\[
\left| \arg \left( \frac{z \left( D_{\lambda, \delta}^{m, s} f(z) \right)'}{D_{\lambda, \delta}^{m, s} g(z)} - \gamma \right) \right| < \frac{\pi}{2^\alpha}
\]

from some \( g \in S_{\lambda, \delta}^{m, s} (\beta, A, B) \), then

\[
\left| \arg \left( \frac{z \left( D_{\lambda, \delta}^{m, s} F_c(f)(z) \right)'}{D_{\lambda, \delta}^{m, s} F_c(g)(z)} - \gamma \right) \right| < \frac{\pi}{2^\eta},
\]

where \( F_c \) is defined by (14), and \( \eta, (0 < \eta \leq 1) \) is the solution of the equation given by (16).

**Proof**: Let

\[
p(z) = \frac{1}{1 - \gamma} \left( \frac{z \left( D_{\lambda, \delta}^{m, s} F_c(f)(z) \right)'}{D_{\lambda, \delta}^{m, s} F_c(g)(z)} - \gamma \right).
\]

Since \( g \in S_{\lambda, \delta}^{m, s} (\beta, A, B) \), we have from Corollary 1.10 that \( F_c(g)(z) \in S_{\lambda, \delta}^{m, s} (\beta, A, B) \). Using (15) we have

\[
((1 - \gamma) p(z) + \gamma) D_{\lambda, \delta}^{m, s} F_c(g)(z) = (c + 1) D_{\lambda, \delta}^{m, s} f(z) - c D_{\lambda, \delta}^{m, s} F_c(f)(z).
\]

Then, by a simple calculation, we get

\[
(1 - \gamma) z p'(z) + ((1 - \gamma) p(z) + \gamma) ((1 - \beta) q(z) + c + \beta) = (c + 1) \frac{z \left( D_{\lambda, \delta}^{m, s} f(z) \right)'}{D_{\lambda, \delta}^{m, s} F_c(g)(z)}
\]

where

\[
q(z) = \frac{1}{1 - \beta} \left( \frac{z \left( D_{\lambda, \delta}^{m, s} F_c(g)(z) \right)'}{D_{\lambda, \delta}^{m, s} F_c(g)(z)} - \beta \right).
\]

Hence we have

\[
\frac{1}{1 - \gamma} \left( \frac{z \left( D_{\lambda, \delta}^{m, s} f(z) \right)'}{D_{\lambda, \delta}^{m, s} g(z)} - \gamma \right) = p(z) + \frac{zp'(z)}{(1 - \beta) q(z) + \beta + c}.
\]

The remaining part of the proof in Theorem 1.14 is similar to that of Theorem 1.11 and so we omit it.

From Theorem 1.9, we see easily the following:
**Corollary 1.15** If \( f \in K_{\lambda, \delta}^{m, s} (\gamma, \alpha, \beta, A, B) \) then \( F_c (f) \in K_{\lambda, \delta}^{m, s} (\gamma, \alpha, \beta, A, B) \) where \( F_c \) is the integral operator defined by (14).

**Remark** : If we take \( s = \delta = 0, \ m = \lambda = 1 \) and \( s = \delta = m = 0 \) with \( \alpha = 1, \ A = 1 \) and \( B = -1 \) in Corollary 1.15, respectively, then we have the corresponding results obtained by Noor and Alkhorasani [11]. Furthermore, taking \( s = \delta = m = \gamma = 0, \ A = 1, \ B = -1 \) and \( \alpha = 1 \) in Corollary 1.15, we obtain the classical result by Bernardi [3], which implies the result studied by Libera [8].

**2 Open Problem**

The operator defined can be extended and can solve many new results and properties.

**Acknowledgement:** The work here was supported by UKM-ST-06-FRGS0107-2009.

**References**


A multiplier transformation defined by convolution


