A Note on a Subclass of Analytic Functions Defined by Multiplier Transformations

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Abstract

Let $A(p, n) = \{ f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \ z \in U \}$, with $A(1, n) = A_n$, $n \in N$. In this paper, we consider multiplier transformations

$$I(m, \lambda, l)f(z) := z + \sum_{j=n+1}^{\infty} \left( \frac{\lambda(j-1) + l + 1}{l+1} \right)^m a_j z^j,$$

where $p, n \in N$, $m \in N \cup \{0\}$, $\lambda, l \geq 0$.

By making use of the multiplier transformation we define a new class $B\Omega(m, n, \mu, \alpha, \lambda, l)$ involving functions $f \in A_n$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

Keywords: Analytic function, starlike function, convex function, multiplier transformations.

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1 Introduction and definitions

Denote by $U$ the unit disc of the complex plane, $U = \{ z \in C : |z| < 1 \}$ and $\mathcal{H}(U)$ the space of holomorphic functions in $U$.

Let

$$A(p, n) = \{ f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \ z \in U \},$$
with \( \mathcal{A}(1, n) = \mathcal{A}_n \) and
\[
\mathcal{H}[a, n] = \{ f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in U \},
\]
where \( p, n \in \mathbb{N}, a \in \mathbb{C} \).

Let \( \mathcal{S} \) denote the subclass of functions that are univalent in \( U \).
By \( \mathcal{S}^*(\alpha) \) we denote a subclass of \( \mathcal{A}_n \) consisting of starlike univalent functions of order \( \alpha \), \( 0 \leq \alpha < 1 \) which satisfies
\[
Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in U. \tag{1.1}
\]

Further, a function \( f \) belonging to \( \mathcal{S} \) is said to be convex of order \( \alpha \) in \( U \), if and only if
\[
Re \left( \frac{zf''(z)}{f'(z)} + 1 \right) > \alpha, \quad z \in U \tag{1.2}
\]
for some \( \alpha, (0 \leq \alpha < 1) \). We denote by \( \mathcal{K}(\alpha) \) the class of functions in \( \mathcal{S} \) which are convex of order \( \alpha \) in \( U \) and denote by \( \mathcal{R}(\alpha) \) the class of functions in \( \mathcal{A}_n \) which satisfy
\[
Re f'(z) > \alpha, \quad z \in U. \tag{1.3}
\]

It is well known that \( \mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S} \).

If \( f \) and \( g \) are analytic functions in \( U \), we say that \( f \) is subordinate to \( g \), written \( f \prec g \), if there is a function \( w \) analytic in \( U \), with \( w(0) = 0, |w(z)| < 1 \), for all \( z \in U \) such that \( f(z) = g(w(z)) \) for all \( z \in U \). If \( g \) is univalent, then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(U) \subseteq g(U) \).

**Definition 1.1** [5] For \( f \in \mathcal{A}(p, n), p, n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}, \lambda, l \geq 0 \), the operator \( I_p(m, \lambda, l) f(z) \) is defined by the following infinite series
\[
I_p(m, \lambda, l) f(z) := z^p + \sum_{j=p+n}^{\infty} \left( \frac{p + \lambda(j - 1) + l}{p + l} \right)^m a_j z^j.
\]

**Remark 1.2** It follows from the above definition that
\[
I_p(0, \lambda, l) f(z) = f(z),
\]
\[
(p + l) I_p(m + 1, \lambda, l) f(z) = [p(1 - \lambda) + l] I_p(m, \lambda, l) f(z) + \lambda z (I_p(m, \lambda, l) f(z))',
\]
for \( z \in U \).

**Remark 1.3** If \( p = 1 \) we have \( I_1(m, \lambda, l) f(z) = I(m, \lambda, l) \) and
\[
(l + 1) I(m + 1, \lambda, l) f(z) = [l + 1 - \lambda] I(m, \lambda, l) f(z) + \lambda z (I(m, \lambda, l) f(z))',
\]
for \( z \in U \).
Remark 1.4 If \( f \in A_n \), \( f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j \), then
\[
I(m, \lambda, l) f(z) = z + \sum_{j=n+1}^{\infty} \left( \frac{1 + \lambda(j - 1) + l}{l + 1} \right)^m a_j z^j,
\]
for \( z \in U \).

Remark 1.5 For \( l = 0 \), \( \lambda \geq 0 \), the operator \( D^m_\lambda = I(m, \lambda, 0) \) was introduced and studied by Al-Oboudi, which is reduced to the Șalagean differential operator \( S^m = I(m, 1, 0) \) for \( \lambda = 1 \). The operator \( I^m = I(m, 1, l) \) was studied recently by Cho and Srivastava [8] and Cho and Kim [9]. The operator \( I_m = I(m, 1, 1) \) was studied by Uralegaddi and Somanatha [13], the operator \( D^m_\delta = I(m, \delta, 0) \), with \( \delta \in \mathbb{R}, \delta \geq 0 \), was introduced by Acu and Owa [1].

To prove our main theorem we shall need the following lemma.

Lemma 1.6 [11] Let \( u \) be analytic in \( U \) with \( u(0) = 1 \) and suppose that
\[
\Re \left( 1 + \frac{zu'(z)}{u(z)} \right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \tag{1.4}
\]
Then \( \Re u(z) > \alpha \) for \( z \in U \) and \( 1/2 \leq \alpha < 1 \).

2 Main results

Definition 2.1 We say that a function \( f \in A_n \) is in the class \( BI(m, n, \mu, \alpha, \lambda, l) \), \( m, n \in \mathbb{N}, \mu \geq 0, \alpha \in [0, 1) \) if
\[
\left| \frac{I(m + 1, \lambda, l) f(z)}{z} \left( \frac{z}{I(m, \lambda, l) f(z)} \right)^m - 1 \right| < 1 - \alpha, \quad z \in U. \tag{2.5}
\]

Remark 2.2 The family \( BI(m, n, \mu, \alpha, \lambda, l) \) is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, \( BI(0, 1, \alpha, 1, 0) = S^*(\alpha), BI(1, 1, 1, \alpha, 0, 1, 0) = K(\alpha) \) and \( BI(0, 1, 0, \alpha, 1, 0) = R(\alpha) \). Another interesting subclass is the special case \( BI(0, 1, 2, \alpha, 1, 0) = B(\alpha) \) which has been introduced by Frasin and Darus [10] and also the class \( BI(0, 1, \mu, \alpha, 1, 0) = B(\mu, \alpha) \) which has been introduced by Frasin and Jahangiri [11]. Catas and Alb have been introduced the subclasses \( BI(m, n, \mu, \alpha, \lambda, 0) = BO(m, \mu, \alpha, \lambda) [6] \) and \( BI(m, \mu, \alpha, 1, 0) = BS(m, \mu, \alpha) [7], [4] \).

In this note we provide a sufficient condition for functions to be in the class \( BI(m, n, \mu, \alpha, \lambda, l) \). Consequently, as a special case, we show that convex functions of order \( 1/2 \) are also members of the above defined family.
Theorem 2.3  For the function $f \in A_n$, $m, n \in \mathbb{N}$, $\mu \geq 0$, $1/2 \leq \alpha < 1$ if

$$l + 1 \frac{I(m + 2, \lambda, l)f(z)}{\lambda I(m + 1, \lambda, l)f(z)} - \frac{\mu (l + 1) I(m + 1, \lambda, l)f(z)}{\lambda I(m, \lambda, l)f(z)} + \frac{(l + 1)(\mu - 1)}{\lambda} + 1 \prec 1 + \beta z, \quad z \in U,$$

where

$$\beta = \frac{3\alpha - 1}{2\alpha},$$

then $f \in B\mathcal{I}(m, n, \mu, \alpha, \lambda, l)$.

Proof  If we consider

$$u(z) = \frac{I(m + 1, \lambda, l)f(z)}{z} \left( \frac{z}{I(m, \lambda, l)f(z)} \right)^{\nu}, \quad (2.7)$$

then $u(z)$ is analytic in $U$ with $u(0) = 1$. A simple differentiation yields

$$\frac{zu'(z)}{u(z)} = \frac{l + 1 I(m + 2, \lambda, l)f(z)}{\lambda I(m + 1, \lambda, l)f(z)} - \frac{\mu (l + 1) I(m + 1, \lambda, l)f(z)}{\lambda I(m, \lambda, l)f(z)} + \frac{(l + 1)(\mu - 1)}{\lambda}. \quad (2.8)$$

Using (2.6) we get

$$\text{Re} \left( 1 + \frac{zu'(z)}{u(z)} \right) > \frac{3\alpha - 1}{2\alpha}.$$ 

Thus, from Lemma 1.6 we deduce that

$$\text{Re} \left\{ \frac{I(m + 1, \lambda, l)f(z)}{z} \left( \frac{z}{I(m, \lambda, l)f(z)} \right)^{\nu} \right\} > \alpha.$$ 

Therefore, $f \in B\mathcal{I}(m, n, \mu, \alpha, \lambda, l)$, by Definition 2.1.

As a consequence of the above theorem we have the following interesting corollaries [3].

Corollary 2.4  If $f \in A_1$ and

$$\text{Re} \left\{ \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)} \right\} > -\frac{1}{2}, \quad z \in U, \quad (2.9)$$

then

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2}, \quad z \in U. \quad (2.10)$$

That is, $f$ is convex of order $\frac{1}{2}$, or $f \in B\mathcal{I} \left( 1, 1, 1, \frac{1}{2}, 1, 0 \right)$. 

Corollary 2.5 If \( f \in A_1 \) and
\[
\text{Re} \left\{ \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)} \right\} > -\frac{1}{2}, \quad z \in U, \tag{2.11}
\]
then \( f \in BI \left( 1, 1, 0, \frac{1}{2}, 1, 0 \right) \), that is
\[
\text{Re} \{ f'(z) + zf''(z) \} > \frac{1}{2}, \quad z \in U. \tag{2.12}
\]

Corollary 2.6 If \( f \in A_1 \) and
\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2}, \quad z \in U, \tag{2.13}
\]
then
\[
\text{Re} f'(z) > \frac{1}{2}, \quad z \in U. \tag{2.14}
\]

In another words, if the function \( f \) is convex of order \( \frac{1}{2} \), then \( f \in BI(0, 1, 0, \frac{1}{2}, 1, 0) \equiv \mathcal{R} \left( \frac{1}{2} \right) \).

Corollary 2.7 If \( f \in A_1 \) and
\[
\text{Re} \left\{ \frac{zf''(z) - zf'(z)}{f'(z)} \right\} < -\frac{3}{2}, \quad z \in U, \tag{2.15}
\]
then \( f \) is starlike of order \( \frac{1}{2} \), hence \( f \in BI(0, 1, 1, \frac{1}{2}, 1, 0) \).

3 Open Problem

The open problem is to define a generic class of analytic functions such that the class \( BI(m, n, \mu, \alpha, \lambda, l) \), \( m, n \in \mathbb{N}, \mu \geq 0, \alpha \in [0, 1) \) is contained inside and is possible to obtain. Compare the new results with the results given by [11].

References


