Extremal Function and Coefficient Inequalities
For Certain Analytic Functions

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Abstract
For analytic functions \( f(z) \) in the open unit disk \( U \), an interesting subclass \( R_\alpha \) with \( |2\alpha - 1| < \frac{\text{Re}(\alpha)}{|\alpha|} \) of analytic functions is introduced. The object of the present paper is to discuss an extremal function and some coefficient inequalities for the class \( R_\alpha \).

Keywords: analytic, extremal function, coefficient inequality.

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1 Introduction and Definitions
Let \( \mathcal{A} \) be the class of functions \( f(z) \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} ; |z| < 1 \} \).

If \( f(z) \in \mathcal{A} \) satisfies the following inequality

\[
\text{Re}(f'(z)) > \alpha \quad (z \in \mathbb{D})
\]
for some real $\alpha$ ($0 \leq \alpha < 1$), then we say that $f(z) \in \mathcal{R}(\alpha)$. This class was investigated by Hayami and Owa (1). In this paper, we consider the new subclass $\mathcal{R}_\alpha$ of $\mathcal{A}$ defined by some complex number $\alpha$.

**Definition 1.1** If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| < \Re\left( \frac{1}{2\alpha} \right) \quad (z \in \mathbb{U})$$

for some complex number $\alpha \left( |2\alpha - 1| < \frac{\Re(\alpha)}{|\alpha|} \right)$, then we say that $f(z) \in \mathcal{R}_\alpha$.

If $0 < \alpha < 1$, then the class $\mathcal{R}_\alpha$ is equivalent to the class $\mathcal{R}(\alpha)$.

We first introduce the following remark to think about the extremal function for the class $\mathcal{R}_\alpha$.

**Remark 1.2** Let $M(z)$ be defined by

$$M(z) = \frac{a - mz}{1 - \frac{a}{m}z} \quad (a \in \mathbb{C} \text{ and } m > 0).$$

Then, we know that $M(0) = a$ and $M(z)$ maps the open unit disk $\mathbb{U}$ onto the following entire circular domain

$$\mathbb{D} = \{ w \in \mathbb{C} ; |w| < m \}.$$

This assertion has been investigated by Miller and Mocanu (2). Using this result, we consider the extremal function for the class $\mathcal{R}_\alpha$.

**Theorem 1.3** The extremal function for the class $\mathcal{R}_\alpha$ is $f(z)$ defined by

$$f(z) = \frac{B}{A} z + \left( 1 - \frac{B}{A} \right) \frac{1}{A} \log (1 + Az)$$

where $A = \frac{(\Im(\alpha))^2 - 2\bar{\alpha}|\alpha|^2}{2\Re(\alpha)|\alpha|^2}$, $B = \frac{\alpha - 2|\alpha|^2}{\Re(\alpha)}$. 
Proof: Noting that \( f(z) \in \mathcal{R}_\alpha \) satisfies
\[
\left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| < \text{Re} \left( \frac{1}{2\alpha} \right),
\]
If we define the function \( M(z) \) by
\[
M(z) = \frac{1}{f'(z)} - \frac{1}{2\alpha},
\]
then it is clear that \( M(0) = 1 - \frac{1}{2\alpha} \) and \( |M(z)| < \text{Re} \left( \frac{1}{2\alpha} \right) \). Hence, from Remark 1.2, we can write
\[
M(z) = \left( 1 - \frac{1}{2\alpha} \right) - \text{Re} \left( \frac{1}{2\alpha} \right) z.
\]
A simple computation gives us that
\[
f'(z) = \frac{1 + \alpha - 2|\alpha|^2}{\text{Re}(\alpha)} z - \frac{(\text{Im}(\alpha))^2 - 2\bar{\alpha} |\alpha|^2}{2\text{Re}(\alpha) |\alpha|^2} z.
\]
Integrating both sides from 0 to \( 2\pi \) on \( \theta \), we have that
\[
f(z) = \frac{B}{A} z + \left( 1 - \frac{B}{A} \right) \frac{1}{A} \log (1 + Az).
\]
Thus, the above function \( f(z) \) is the extremal function for the class \( \mathcal{R}_\alpha \).

Remark 1.4 The extremal function \( f(z) \) for the class \( \mathcal{R}_\alpha \) has the following Taylor expansion of the form
\[
f(z) = \frac{B}{A} z + \left( 1 - \frac{B}{A} \right) \frac{1}{A} \log (1 + Az) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} A^{n-2}(A - B)}{n} z^n,
\]
where \( A = \frac{(\text{Im}(\alpha))^2 - 2\bar{\alpha} |\alpha|^2}{2\text{Re}(\alpha) |\alpha|^2} \), \( B = \frac{\alpha - 2|\alpha|^2}{\text{Re}(\alpha)} \).
2 Coefficient inequalities

We now consider the coefficient inequalities for \( f(z) \) belonging to the class \( \mathcal{R}_\alpha \).

**Theorem 2.1** If a function \( f(z) \in \mathcal{A} \) satisfies the following inequality

\[
\sum_{n=2}^{\infty} n|a_n| \leq \frac{\text{Re}(\alpha) - |\alpha||2\alpha - 1|}{\text{Re}(\alpha) + |\alpha|}
\]

for some complex number \( \alpha \left( |2\alpha - 1| < \frac{\text{Re}(\alpha)}{|\alpha|} \right) \), then \( f(z) \in \mathcal{R}_\alpha \).

**Proof :** Noting that

\[
\left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| = \frac{1}{2|\alpha|} \left| \frac{2\alpha - 1 - \sum_{n=2}^{\infty} na_n z^{n-1}}{1 + \sum_{n=2}^{\infty} na_n z^{n-1}} \right| \leq \frac{1}{2|\alpha|} \frac{|2\alpha - 1| + \sum_{n=2}^{\infty} n|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1}}
\]

\[
< \frac{1}{2|\alpha|} \frac{|2\alpha - 1| + \sum_{n=2}^{\infty} n|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|},
\]

if \( f(z) \) satisfies the following inequality

\[
|2\alpha - 1| + \sum_{n=2}^{\infty} n|a_n| \leq \frac{\text{Re}(\alpha)}{|\alpha|} \left( 1 - \sum_{n=2}^{\infty} n|a_n| \right),
\]

that is,

\[
\sum_{n=2}^{\infty} n|a_n| \leq \frac{\text{Re}(\alpha) - |\alpha||2\alpha - 1|}{\text{Re}(\alpha) + |\alpha|},
\]

then we see

\[
\left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| < \frac{\text{Re}(\alpha)}{2|\alpha|^2}.
\]

This completes the proof of the theorem.

Letting \( 0 < \alpha < 1 \) in Theorem 2.1, we obtain the following corollary.
Corollary 2.2 If a function \( f(z) \in A \) satisfies the following inequality

\[
\sum_{n=2}^{\infty} n|a_n| \leq \begin{cases} 
\alpha & (0 < \alpha \leq \frac{1}{2}) \\
1 - \alpha & (\frac{1}{2} < \alpha < 1) 
\end{cases}
\]

for some real number \( \alpha \) \((0 < \alpha < 1)\), then \( f(z) \in R(\alpha) \).

Next we derive the following necessary condition for the class \( R_\alpha \).

Theorem 2.3 If a function \( f(z) \in R_\alpha \) with \( a_n = |a_n|e^{i((n-1)\theta+\pi)} \) \((n = 2, 3, 4 \ldots)\), then

\[
\sum_{n=2}^{\infty} n|a_n| \leq \begin{cases} 
1 - \alpha & (0 < \alpha < 1) \\
1 - 2|\alpha|^2 \left( \frac{\text{Re}(\alpha) - \sqrt{\text{Re}(\alpha)^2 - (\text{Im}(\alpha))^2}}{(\text{Im}(\alpha))^2} \right) & (\alpha \not\in \mathbb{R}) 
\end{cases}
\]

Proof : By using the same method with Theorem 2.1, we obtain that

\[
\left| \frac{2\alpha - 1 - \sum_{n=2}^{\infty} na_n z^{n-1}}{1 + \sum_{n=2}^{\infty} na_n z^{n-1}} \right| < \frac{\text{Re}(\alpha)}{|\alpha|} \quad (z \in \mathbb{U})
\]

for \( f(z) \in R_\alpha \). Since \( a_n = |a_n|e^{i((n-1)\theta+\pi)} \), if we take \( z = |z|e^{-i\theta} \), then we know that

\[
\left| \frac{2\text{Re}(\alpha) - 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} + 2i\text{Im}(\alpha)}{1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1}} \right| < \frac{\text{Re}(\alpha)}{|\alpha|} \quad (z \in \mathbb{U})
\]

Letting \(|z| \to 1\) and squaring both sides, we obtain that

\[
\frac{(2\text{Re}(\alpha) - 1)^2 + 2(2\text{Re}(\alpha) - 1)\beta + \beta^2 + 4(\text{Im}(\alpha))^2}{1 - 2\beta + \beta^2} \leq \frac{(\text{Re}(\alpha))^2}{|\alpha|^2},
\]

that is, that
\[(\text{Im}(\alpha))^2 \beta^2 + 2(2\text{Re}(\alpha)|\alpha|^2 - (\text{Im}(\alpha))^2)\beta \]
\[+ 4|\alpha|^4 - 4\text{Re}(\alpha)|\alpha|^2 + (\text{Im}(\alpha))^2 \leq 0. \quad (2.1)\]

where \(\beta = \sum_{n=2}^{\infty} n|a_n|\). If \(0 < \alpha < 1\), then we see that the inequality (2.1) is equivalent to
\[
\beta \leq 1 - \alpha.
\]

If \(\alpha \not\in \mathbb{R}\), then solving the inequality (2.1), we have that
\[
\beta \leq \frac{-(2\text{Re}(\alpha)|\alpha|^2 - (\text{Im}(\alpha))^2) + 2|\alpha|^2 \sqrt{(\text{Re}(\alpha))^2 - (\text{Im}(\alpha))^2}}{\text{(Im}(\alpha))^2},
\]
which is the desired result.

Furthermore, we state about the following coefficient inequality.

**Theorem 2.4** If a function \(f(z) \in \mathcal{R}_\alpha \ (\alpha \not\in \mathbb{R})\), then
\[
\sum_{n=2}^{\infty} n^2|a_n|^2 \leq \frac{(\text{Re}(\alpha))^2 - |\alpha(2\alpha - 1)|^2}{(\text{Im}(\alpha))^2}.
\]

**Proof** : From the definition of the class \(\mathcal{R}_\alpha\), we note that
\[
|\alpha|^2|2\alpha - f'(z)|^2 < (\text{Re}(\alpha))^2|f'(z)|^2.
\]

Setting \(z = re^{i\theta} (0 \leq r < 1, \ 0 \leq \theta < 2\pi)\) and integrating both sides from 0 to \(2\pi\) on \(\theta\), we have that
\[
|\alpha|^2 \int_0^{2\pi} |2\alpha - f'(re^{i\theta})|^2 d\theta < (\text{Re}(\alpha))^2 \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta.
\]

A simple calculation gives us that
\[
2\pi|\alpha|^2 \left( |2\alpha - 1|^2 + \sum_{n=2}^{\infty} n^2|a_n|^2r^{2(n-1)} \right) < 2\pi(\text{Re}(\alpha))^2 \left( 1 + \sum_{n=2}^{\infty} n^2|a_n|^2r^{2(n-1)} \right).
\]

Therefore, letting \(r \to 1\), we obtain that
\[
\sum_{n=2}^{\infty} n^2|a_n|^2 \leq \frac{(\text{Re}(\alpha))^2 - |\alpha(2\alpha - 1)|^2}{(\text{Im}(\alpha))^2},
\]
which completes the proof of the theorem.
3 Open problem

In view of Theorem 2.3, we have that

\[ |a_n| \leq \frac{1 - \alpha}{n} \quad (0 < \alpha < 1; n = 2, 3, 4, \cdots) \]

and

\[ |a_n| \leq \frac{1}{n} \left( 1 - \frac{2|\alpha|^2(\text{Re}(\alpha) - \sqrt{\text{Re}(\alpha)^2 - (\text{Im}(\alpha))^2})}{(\text{Im}(\alpha))^2} \right) \]

where \( \alpha \notin \mathbb{R}; n = 2, 3, 4, \cdots \).

Also, from Theorem 2.4, we have that

\[ |a_n| \leq \frac{1}{n} \left( \frac{(\text{Re}(\alpha))^2 - |\alpha(2\alpha - 1)|^2}{(\text{Im}(\alpha))^2} \right)^{\frac{1}{2}}. \]

But, we know that the extremal function \( f(z) \) for the class \( \mathcal{R}_\alpha \) in Theorem 1.3 satisfies

\[ |a_n| = \frac{A^n - 2(A - B)}{n} \quad (n = 2, 3, 4, \cdots). \]

Therefore, we guess that the function \( f(z) \in \mathcal{R}_\alpha \) satisfies

\[ |a_n| \leq \frac{A^n - 2(A - B)}{n} \quad (n = 2, 3, 4, \cdots). \]  \hspace{1cm} (3.1)

How can we prove the coefficient inequality (3.1) for \( f(z) \in \mathcal{R}_\alpha \)?

References
