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Partial Sums of Analytic Functions Involving Generalized Cho-Kwon-Srivastava Operator

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Abstract

In the present paper, we study the class of analytic functions involving generalized Cho-Kwon-Srivastava operator denoted by $I^{\lambda,\mu,s}(a,c)f(z)$ with negative coefficients. The aim of the paper is to obtain the coefficient estimates and also partial sums of its sequence $I_n^{\lambda,\mu,s}(a,c)f(z)$.

Keywords: Analytic functions, Cho-Kwon-Srivastava operator, partial sums.

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1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. And let S denote the subclass of A consisting of univalent functions f in U. Also denote by T the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k\, z^k,\ \ a_k \geq 0.$$

Let $P(\alpha,\beta)$ denote the class of function $f\in A$ which satisfy the condition

$$\Re\{\frac{\alpha z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\}>\beta,$$

for some $\alpha\geq 0$, $0\leq\beta<1$, $\frac{f(z)}{z}\neq 0$ and $z\in U$.

The classes $P(\alpha,\beta)$ and $P(\alpha,0)$ were introduced and studied by many authors and these include for example by Obradovic and Joshi [4], Li and Owa [2], Xu and Yang [10] and Singh and Gupta [9]. We also note that whenever $\alpha=0$, the class $P(0,\beta)$ was studied by Silverman [6] and the author solved problems related to partial sums. The work here is motivated by Silverman[6] and Lashin[3].

Let $TP(\alpha,\beta)$ denote the class $P(\alpha,\beta)$ with T,

$$TP(\alpha,\beta)\ =\ P(\alpha,\beta)\cap T,$$

and it is known that the class $TP(\alpha,\beta)$ was introduced and studied by Lashin [3].

Let the lambda function [8] defined as follows:

$$\psi(z,s)=\sum_{k=0}^{\infty}\frac{z^k}{(2k+1)^s},$$

$$(z\in U; s\in\mathbb{C},\ when,\ |z|<1;\ \Re(s)>1,\ when,\ |z|=1).$$

We define $(z\psi^{(-1)}(z,s))$ as the following:

$$(z\psi(z,s))*(z\psi^{(-1)}(z,s))=\frac{z}{(1-z)^\lambda},\ \ \lambda>0$$

$$(z\psi^{(-1)}(z,s))\ =z+\sum_{k=2}^{\infty}\frac{(\lambda)_{k-1}(2k-1)^s}{(k-1)!}z^k,\quad\lambda>0,s\in\mathbb{C},\tag{1.2}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1, & n=0\\ \lambda(\lambda+1)...(\lambda+n-1), & n=\{1,2,3,...\}.\end{cases}$$

Cho et.al [1] introduced the family of linear operator $L^\mu(a,c)f(z)$ as the following:

$$L^\mu(a,c)f(z)=z+\sum_{k=2}^{\infty}\frac{(c)_{k-1}(\mu+1)_{n-1}}{(a)_{k-1}(1)_{k-1}}a_k z^k\tag{1.3}$$

$$(\, z \in U, \;\; a, c \in R/\bar{Z}_0 \,; \bar{Z}_0 = \{0, -1, -2, ...\}; \, \mu > -1).$$

We now generalize Cho-Kwon-Srivastava operator by using Hadamard product as follows:

$$I^{\lambda,\mu,s}(a,c)f(z) = L^{\mu}(a,c)f(z) * (z\psi^{(-1)}(z,s)\,)$$

$$= z + \sum_{k=2}^{\infty} \frac{(c)_{k-1}(\mu+1)_{n-1}(\lambda)_{k-1}(2k-1)^s}{(a)_{k-1}(1)_{k-1}(k-1)!} a_k z^k. \tag{1.4}$$

Let $P^*_{\lambda,\mu,s}(\alpha,\beta)$ the class of function $f \in A$ which satisfy the condition

$$\Re\{\frac{\alpha z^2(I^{\lambda,\mu,s}(a,c)f(z))''}{I^{\lambda,\mu,s}(a,c)f(z)} + \frac{z(I^{\lambda,\mu,s}(a,c)f(z))'}{I^{\lambda,\mu,s}(a,c)f(z)}\} > \beta,$$

for some $\alpha \geq 0$, $0 \leq \beta < 1$, $\frac{I^{\lambda,\mu,s}(a,c)f(z)}{z} \neq 0$ and $z \in U$. Let $TP^*_{\lambda,\mu,s}(\alpha,\beta)$ denote the class $P^*_{\lambda,\mu,s}(\alpha,\beta)$ with T.

In this article, we shall begin with our first result on coefficient estimates for $f \in TP^*_{\lambda,\mu,s}(\alpha,\beta)$. Later, we determine the sharp lower bounds for $\Re\{\frac{I^{\lambda,\mu,s}(a,c)f(z)}{I_n^{\lambda,\mu,s}(a,c)f(z)}\}$, $\Re\{\frac{I_n^{\lambda,\mu,s}(a,c)f(z)}{I^{\lambda,\mu,s}(a,c)f(z)}\}$, $\Re\{\frac{(I^{\lambda,\mu}(a,c)f(z))'}{(I_n^{\lambda,\mu,s}(a,c)f(z))'}\}$ and $\Re\{\frac{(I_n^{\lambda,\mu,s}(a,c)f(z))'}{(I^{\lambda,\mu,s}(a,c)f(z))'}\}$, where

$$I_n^{\lambda,\mu,s}(a,c)f(z) = z - \sum_{k=2}^{n} \frac{(c)_{k-1}(\mu+1)_{n-1}(\lambda)_{k-1}(2k-1)^s}{(a)_{k-1}(1)_{k-1}(k-1)!} a_k z^k,$$

is the sequence of partial sums of generalized Cho-Kwon-Srivastava operator with negative coefficients.

In this sequel, we will make use of the well-known result that

$$\Re\{\frac{1+\omega(z)}{1-\omega(z)}\} > 0, \, z \in U$$

if and only if $\omega(z) = \sum\limits_{k=1}^{\infty} c_k z^k$ satisfies the inequality $|\omega(z)| < 1$.

Recall that the function f is subordinate to g if there exists a function ω, analytic in U, with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$f(z) = g(w(z)), z \in U. \tag{1.5}$$

We denote this subordination by $f(z) \prec g(z)$. If $g(z)$ is univalent in U, then the subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [5]).

2 Coefficient Estimates

In this section we obtain a necessary and sufficient condition for functions f to be in the subclass $TP^*_{\lambda,\mu,s}(\alpha,\beta)$. But first we give a result for the class $P^*_{\lambda,\mu,s}(\alpha,\beta)$.

Theorem 2.1 *A function f of the form (1.1) is in $P^*_{\lambda,\mu,s}(\alpha,\beta)$ if*

$$\sum_{k=2}^{\infty}[(k-1)(\alpha k+1)+(1-\beta)]\,|A_{\lambda,\mu,s}(a,c)|\;|a_k| \;\le\; 1-\beta, \tag{2.1}$$

for some $\alpha \ge 0$, $0 \le \beta < 1$ and $A_{\lambda,\mu,s}(a,c) = \frac{(c)_{k-1}(\mu+1)_{n-1}(\lambda)_{k-1}(2k-1)^s}{(a)_{k-1}(1)_{k-1}(k-1)!}$.

Proof: Let condition (2.1) be satisfied and let $|z|<1$. Then we have

$$\left|\frac{\alpha z^2(I^{\lambda,\mu,s}(a,c)f(z))''}{I^{\lambda,\mu,s}(a,c)f(z)}+\frac{z(I^{\lambda,\mu,s}(a,c)f(z))'}{I^{\lambda,\mu,s}(a,c)f(z)}-1\right| = \left|\frac{\sum\limits_{k=2}^{\infty}(k-1)(\alpha k+1)A_{\lambda,\mu,s}(a,c)\,a_k z^{k-1}}{1+\sum\limits_{k=2}^{\infty}A_{\lambda,\mu,s}(a,c)\,a_k z^{k-1}}\right|$$

$$\le \frac{\sum\limits_{k=2}^{\infty}(k-1)(\alpha k+1)\,|A_{\lambda,\mu,s}(a,c)|\;|a_k|}{1-\sum\limits_{k=2}^{\infty}|A_{\lambda,\mu,s}(a,c)|\;|a_k|} \le 1-\beta.$$

The extreme-right-side expression of the above inequality would remain bounded by 1. This shows that the values of $\frac{\alpha z^2(I^{\lambda,\mu}(a,c)f(z))''}{I^{\lambda,\mu}(a,c)f(z)}+\frac{z(I^{\lambda,\mu}(a,c)f(z))'}{I^{\lambda,\mu}(a,c)f(z)}$ lie in the circle centered at 1 whose radius is $1-\beta$. Hence $f \in P^*_{\lambda,\mu,s}(\alpha,\beta)$.

Theorem 2.2 *A function $f \in T$ is in the class $TP^*_{\lambda,\mu,s}(\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty}[(k-1)(\alpha k+1)+(1-\beta)]A_{\lambda,\mu,s}(a,c)\,a_k \;\le\; 1-\beta\,, \tag{2.2}$$

for some $\alpha \ge 0$, $0 \le \beta < 1$ and $A_{\lambda,\mu,s}(a,c) = \frac{(c)_{k-1}(\mu+1)_{n-1}(\lambda)_{k-1}(2k-1)^s}{(a)_{k-1}(1)_{k-1}(k-1)!}$. The result is sharp.

Proof: In view of Theorem 2.1, we need only to prove the necessity. If $f \in TP^*_{\lambda,\mu,s}(\alpha,\beta)$ and z is real then

$$\Re\left\{\frac{\alpha z^2(I^{\lambda,\mu,s}(a,c)f(z))''}{I^{\lambda,\mu,s}(a,c)f(z)}+\frac{z(I^{\lambda,\mu,s}(a,c)f(z))'}{I^{\lambda,\mu,s}(a,c)f(z)}\right\} =$$
$$\Re\left\{\frac{1-\sum\limits_{k=2}^{\infty}(\alpha k(k-1)+k)A_{\lambda,\mu,s}(a,c)\,a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}A_{\lambda,\mu,s}(a,c)\,a_k z^{k-1}}\right\} > \beta$$

letting $z \to 1^-$ along the real axis, we obtain the desired inequality

$$\sum_{k=2}^{\infty} [(k-1)(\alpha k+1)+(1-\beta)]A_{\lambda,\mu,s}(a,c)\, a_k \;\le\; 1-\beta\,.$$

Corollary 2.3 *[3] A function $f \in T$ is in the class $TP(\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty} [(k-1)(\alpha k+1)+(1-\beta)]\, a_k \;\le\; 1-\beta\,. \tag{2.3}$$

The result is sharp.

Theorem 2.4 *The extreme points of the class $TP^*_{\lambda,\mu,s}(\alpha,\beta)$ are the functions given by $I_1^{\lambda,\mu,s}(a,c)f(z)=1$ and*

$$I_k^{\lambda,\mu,s}(a,c)f(z) = z - \frac{1-\beta}{[(k-1)(\alpha k+1)+(1-\beta)]A_{\lambda,\mu,s}(a,c)} z^k, (k \ge 2), \tag{2.4}$$

for some $\alpha \ge 0$, $0 \le \beta < 1$ and $A_{\lambda,\mu,s}(a,c) = \frac{(c)_{k-1}(\mu+1)_{n-1}(\lambda)_{k-1}(2k-1)^s}{(a)_{k-1}(1)_{k-1}(k-1)!}$.

Corollary 2.5 *Let $f \in T$ be in the class $TP(\alpha,\beta)$. Then we have*

$$a_k \le \frac{1-\beta}{[(k-1)(\alpha k+1)+(1-\beta)]A_{\lambda,\mu,s}(a,c)},\ (k \ge 2). \tag{2.5}$$

Equality in (2.4) holds true.

3 Partial Sums

In this section the partial sums of generalized Cho-Kwon-Srivastava operator in the class $TP^*_{\lambda,\mu,s}(\alpha,\beta)$ are given. We obtain sharp lower bounds for the ratio of the real part.

Theorem 3.1 *If $f \in TP^*_{\lambda,\mu,s}(\alpha,\beta)$, then*

$$\Re\{\frac{I^{\lambda,\mu,s}(a,c)f(z)}{I_n^{\lambda,\mu,s}(a,c)f(z)}\} \ge \; 1 - \frac{1}{b_{n+1}} \qquad (z \in U,\, n \in N) \tag{3.1}$$

and

$$\Re\{\frac{I_n^{\lambda,\mu,s}(a,c)f(z)}{I^{\lambda,\mu,s}(a,c)f(z)}\} \ge \; \frac{b_{n+1}}{1+b_{n+1}} \qquad (z \in U,\, n \in N), \tag{3.2}$$

where

$$(b_k = \frac{[(k-1)(\alpha k+1)+(1-\beta)]A_{\lambda,\mu,s}(a,c)}{1-\beta}). \tag{3.3}$$

The estimates in (3.1) and (3.2) are sharp.

Proof: To prove (3.1), it suffices to show that

$$b_k\{\frac{I^{\lambda,\mu,s}(a,c)f(z)}{I_n^{\lambda,\mu,s}(a,c)f(z)} - (1-\frac{1}{b_{n+1}})\} \prec \frac{1+z}{1-z}, \quad z \in U.$$

By the subordination property (1.5), we can write

$$\frac{1-\sum_{k=2}^{n} A_{\lambda,\mu,s}(a,c)\, a_k z^{k-1} - \; b_{n+1}\sum_{k=2}^{\infty} A_{\lambda,\mu,s}(a,c)\, a_k z^{k-1}}{1-\sum_{k=2}^{n} A_{\lambda,\mu,s}(a,c)\, a_k z^{k-1}} = \frac{1+\omega(z)}{1-\omega(z)}.$$

Notice that $\omega(0)=0$ and

$$|\omega(z)| \le \frac{b_{n+1}\sum_{k=n+1}^{\infty} A_{\lambda,\mu,s}(a,c)\, a_k}{2-2\sum_{k=2}^{n} A_{\lambda,\mu,s}(a,c)\, a_k - b_{n+1}\sum_{k=n+1}^{\infty} A_{\lambda,\mu,s}(a,c)\, a_k}.$$

Now $|\omega(z)| < 1$ if and only if

$$\sum_{k=2}^{n} A_{\lambda,\mu,s}(a,c)\, a_k + b_{n+1}\sum_{k=n+1}^{\infty} A_{\lambda,\mu,s}(a,c)\, a_k \le 1.$$

In view of (2.2), we equivalently show that

$$\sum_{k=2}^{n} (b_k-1)A_{\lambda,\mu,s}(a,c)\, a_k + \sum_{k=n+1}^{\infty} (b_k-b_{n+1})A_{\lambda,\mu,s}(a,c)\, a_k \ge 0.$$

The above inequality holds because b_k is a non-decreasing sequence. This completes the proof of (3.1). Finally, it is observed that equality in (3.1) is attained for the function given by (2.4) when $z = re^{2\pi i/n}$ as $r \to 1^-$. The proof of(3.2) is similar to that of (3.1), and this is omitted.

Corollary 3.2 *If $f \in TP(\alpha,\beta)$, then*

$$\Re\{\frac{f(z)}{f_n(z)}\} \ge \; 1-\frac{1}{b_{n+1}} \qquad (z \in U,\, n \in N) \tag{3.4}$$

and

$$\Re\{\frac{f_n(z)}{f(z)}\} \ge \; \frac{b_{n+1}}{1+b_{n+1}} \qquad (z \in U,\, n \in N),$$

where $(b_k = \frac{[(k-1)(\alpha k+1)+(1-\beta)]}{1-\beta})$*. The estimates are sharp.*

Similarly, we can prove the following theorem.

Theorem 3.3 *If* $f \in TP^{*}_{\lambda,\mu,s}(\alpha,\beta)$, *then*

$$\Re\{\frac{(I^{\lambda,\mu,s}(a,c)f(z))'}{(I_n^{\lambda,\mu,s}(a,c)f(z))'}\} \geq \ 1 - \frac{n+1}{b_{n+1}} \qquad (z \in U,\, n \in N) \tag{3.5}$$

and

$$\Re\{\frac{(I_n^{\lambda,\mu,s}(a,c)f(z))'}{(I^{\lambda,\mu,s}(a,c)f(z))'}\} \geq \ \frac{b_{n+1}}{n+1+b_{n+1}}\,. \tag{3.6}$$

where b_k *is given by (3.3). The result is sharp for every* n*, with extremal functions given by (2.4).*

4 Open problem

Many other properties can be obtained for the generalized operator given by (1.4). Inclusion properties similar to [7] and [1] are yet to be discussed. Can we find the univalence criteria for the generalized operator (1.4)?

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