Certain Differential Inequalities of Meromorphic Functions Associated With Integral Operators

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Abstract

The main object of the present paper is to derive certain differential inequalities for two integral operators $P_{\alpha}^{\beta}$ and $Q_{\alpha}^{\beta}$ which are introduced by Lashin [1].

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1 Introduction

Let $\Sigma$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k,$$

which are analytic in the punctured unit disk $U^* = \{z : 0 < |z| < 1\} = U \setminus \{0\}$, with a simple pole at the origin.

For the function $f(z) \in \Sigma$, given by (1.1) and $g(z) \in \Sigma$ defined by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \quad z \in U^*$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by
(1.3) \((f \ast g)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k =: (g \ast f)(z)\).

Analogous to the operators defined by Jung, Kim and Srivastava [2] on the analytic functions Lashin [1] defines the following integral operators \(P_{\beta}^{\alpha}, Q_{\beta}^{\alpha} : \Sigma \rightarrow \Sigma:\)

\[(1.4) \quad P_{\beta}^{\alpha} = P_{\beta}^{\alpha} f(z) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{1}{z^{\beta+1}} \int_{0}^{z} t^{\beta} \left( \log \frac{z}{t} \right)^{\alpha-1} f(t) dt \quad (\alpha > 0, \beta > 0; z \in U^*)\]

and

\[(1.5) \quad Q_{\beta}^{\alpha} = Q_{\beta}^{\alpha} f(z) = \frac{\Gamma(\beta + \alpha)}{\Gamma(\beta) \Gamma(\alpha)} \frac{1}{z^{\beta+1}} \int_{0}^{z} t^{\beta} \left( 1 - \frac{t}{z} \right)^{\alpha-1} f(t) dt \quad (\alpha > 0, \beta > 0; z \in U^*),\]

where \(\Gamma(\alpha)\) is the familiar Gamma function. Using the integral representation of the Gamma and Beta functions given by (1.4) and (1.5), it can be shown that

\[P_{\beta}^{\alpha} f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left( \frac{\beta}{k + \beta + 1} \right)^{\alpha} a_k z^k, \quad (\alpha > 0, \beta > 0)\]

and

\[Q_{\beta}^{\alpha} f(z) = \frac{1}{z} + \frac{\Gamma(\beta + \alpha)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k + \beta + 1)}{\Gamma(k + \beta + \alpha + 1)} a_k z^k, \quad (\alpha > 0, \beta > 0).\]

It is easily verified from (1.6) and (1.7) (see [1])

\[(1.8) \quad z(P_{\beta}^{\alpha} f(z))' = \beta(P_{\beta}^{\alpha-1} f(z)) - (\beta + 1)(P_{\beta}^{\alpha} f(z)) \quad (\alpha > 1, \beta > 0)\]

and

\[(1.9) \quad z(Q_{\beta}^{\alpha} f(z))' = (\beta + \alpha - 1)(Q_{\beta}^{\alpha-1} f(z)) - (\beta + \alpha)(Q_{\beta}^{\alpha} f(z)) \quad (\alpha > 1, \beta > 0).\]

**Definition 1.1.** Let \(H\) be the set of complex valued function \(h(r,s,t) : C^3 \rightarrow C\) (\(C\) is the complex plane) such that:

(i) \(h(r,s,t)\) is continuous in a domain \(D \subset C^3\),

(ii) \((1,1,1) \in D\) and \(|h(1,1,1)| < 1\),
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(1) \[ \left( h \frac{e^{i\theta} + \zeta}{\beta} + e^{i\theta}, \frac{L + 2\beta e^{i\theta}}{\beta(1 + \frac{B}{\zeta} e^{i\theta})} + e^{i\theta} \right) \geq 1, \]

whenever

\[ \left( h \frac{e^{i\theta} + \zeta}{\beta} + e^{i\theta}, \frac{L + 2\beta e^{i\theta}}{\beta(1 + \frac{B}{\zeta} e^{i\theta})} + e^{i\theta} \right) \in D \]

with \( \text{Re}(L) > 0 \) for real \( \theta \) and for \( \zeta \geq 1 \).

**Definition 1.2.** Let \( G \) be the set of complex valued function \( g(r, s, t) : \mathbb{C}^3 \rightarrow \mathbb{C} \) (\( C \) is the complex plane) such that:

(i) \( g(r, s, t) \) is continuous in a domain \( D \subset \mathbb{C}^3 \),

(ii) \( (1, 1, 1) \in D \) and \( |g(1, 1, 1)| < 1 \),

(iii) \[ \left| g \left( e^{i\theta}, \frac{\zeta}{\beta + \alpha - 2}, \frac{e^{i\theta} - 1}{\beta + \alpha - 2}, \frac{1}{\beta + \alpha - 3}, \frac{(M - 1) + 2(\beta + \alpha - 1)e^{i\theta}}{\zeta} \right) \right| \geq 1 \]

whenever

\[ \left| g \left( e^{i\theta}, \frac{\zeta}{\beta + \alpha - 2}, \frac{e^{i\theta} - 1}{\beta + \alpha - 2}, \frac{1}{\beta + \alpha - 3}, \frac{(M - 1) + 2(\beta + \alpha - 1)e^{i\theta}}{\zeta} \right) \right| \in D \] for \( \zeta \geq 1 \).

with \( \text{Re}(M) > 1 \) for real \( \theta \) and for \( \zeta \geq 1 \).
2 Main Results

In proving our main result, we shall need the following lemma due to Miller and Mocanu [3].

Lemma 2.1. Let \( w(z) = a + w_n z^n + \ldots \) be analytic in \( U \) with \( w(z) \neq 0 \) and \( n \geq 1 \).

If \( z_0 = r_0 e^{i\theta} (0 < r_0 < 1) \) and \( \max_{|z|=1} |w(z)| = |w(z_0)| \). Then

\[
(2.1) \quad z_0 w'(z_0) = \zeta w(z_0),
\]

and

\[
(2.2) \quad \Re \left\{ 1 + \frac{z_0 w''(z_0)}{w(z_0)} \right\} \geq \zeta, \text{ where } \zeta \geq 1 \text{ is a real number.}
\]

Theorem 2.1. Let \( h(r,s,t) \in H \) and \( f \in \Sigma \) satisfy

\[
(2.3) \quad \left( \begin{array}{c} (P^{\alpha-1}_\beta f(z))^{(j)} \\ (P^{\alpha}_\beta f(z))^{(j)} \\ (P^{\alpha-2}_\beta f(z))^{(j)} \end{array} \right) \in D \subset \mathbb{C}^3
\]

and

\[
(2.4) \quad \left| h \left( \begin{array}{c} (P^{\alpha-1}_\beta f(z))^{(j)} \\ (P^{\alpha}_\beta f(z))^{(j)} \\ (P^{\alpha-2}_\beta f(z))^{(j)} \end{array} \right) \right| < 1,
\]

for all \( z \in U; \alpha > 3, \beta > 0 \) for some \( \alpha \in \mathbb{R}^+ \).

Then we have

\[
\left| \frac{(P^{\alpha-1}_\beta f(z))^{(j)}}{(P^{\alpha}_\beta f(z))^{(j)}} \right| < 1 \quad (z \in U).
\]

Proof. Let

\[
(2.5) \quad \frac{(P^{\alpha-1}_\beta f(z))^{(j)}}{(P^{\alpha}_\beta f(z))^{(j)}} = w(z),
\]

then it follows that \( w(z) \) is analytic in \( U \), \( w(0) = 1 \) and \( w(z) \neq 1 \). Differentiate (2.5) logarithmically and with the aid of the identity

\[
(2.6) \quad z(P^{\alpha}_\beta f(z))^{(j+1)} = \beta(P^{\alpha-1}_\beta f(z))^{(j)} - (\beta + j + 1)(P^{\alpha}_\beta f(z))^{(j)} \quad (\alpha > 1, \beta > 0),
\]

and making some simple calculation, we obtain
\[ (2.7) \quad \frac{(P_{\beta}^{a-2} f(z))^{(j)}}{(P_{\beta}^{a-1} f(z))^{(j)}} = \frac{zw'(z) + w(z)}{\beta w(z)}. \]

Again differentiate logarithmically and using (2.6), we easily get
\[ (2.8) \quad \frac{(P_{\beta}^{a-3} f(z))^{(j)}}{(P_{\beta}^{a-2} f(z))^{(j)}} = \frac{\left(\frac{zw'(z)}{w(z)} + 1\right) + 2\beta w(z)}{\beta \left(1 + \beta \frac{w(z)}{zw(z)} w(z)\right)} + \frac{w(z)}{\beta}. \]

We claim that \(|w(z)| < 1\) for \(z \in U\). Otherwise there exist a point \(z_0 \in U\) such that \(\max_{|H| \leq |H|} |w(z)| = w(z_0) = 1\). Letting \(w(z_0) = e^{i\theta}\) and using the Lemma 2.1 with \(a = n = 1\), we can see that
\[ (P_{\beta}^{a-1} f(z_0))^{(j)} = e^{i\theta}, \]
\[ (P_{\beta}^{a-2} f(z_0))^{(j)} = \frac{\zeta}{\beta} + e^{i\theta} \]
and
\[ (P_{\beta}^{a-3} f(z_0))^{(j)} = \frac{L + 2\beta e^{i\theta}}{\beta (1 + \frac{\beta}{\zeta} e^{i\theta})} + \frac{e^{i\theta}}{\beta} \quad \text{where} \quad L = \frac{z_0 w^n(z_0)}{w'(z_0)}. \]

Since \(h(r,s,t) \in H\), we have
\[
\begin{align*}
&h \left( \frac{(P_{\beta}^{a-1} f(z_0))^{(j)}}{(P_{\beta}^{a-2} f(z_0))^{(j)}}, \frac{(P_{\beta}^{a-2} f(z_0))^{(j)}}{(P_{\beta}^{a-1} f(z_0))^{(j)}}, \frac{(P_{\beta}^{a-3} f(z_0))^{(j)}}{(P_{\beta}^{a-2} f(z_0))^{(j)}} \right) \\
&= h \left( \frac{\zeta}{\beta} + e^{i\theta}, \frac{L + 2\beta e^{i\theta}}{\beta (1 + \frac{\beta}{\zeta} e^{i\theta})} + \frac{e^{i\theta}}{\beta} \right) \geq 1
\end{align*}
\]
where \(\text{Re}(L) \geq 0\) and \(\zeta \geq 1\).

This contradicts the condition (2.4) of the Theorem 2.1. Therefore we conclude that
\[
\left| \frac{(P_{\beta}^{\alpha-1} f(z_o))^{(j)}}{(P_{\beta}^{\alpha} f(z_o))^{(j)}} \right| < 1, \ (z \in U).
\]

This completes the proof of the theorem.

**Corollary 2.1.** Let \( h(r,s,t) = s \) and \( f \in \Sigma \) satisfy the condition in Theorem 2.1.

Then

(2.9) \[
\left| \frac{(P_{\beta}^{\alpha+j-1} f(z))^{(j)}}{(P_{\beta}^{\alpha+j} f(z))^{(j)}} \right| < 1, \ (i = 0,1,2,..., \alpha > 4, i, j \in N, z \in U).
\]

**Theorem 2.2.** Let \( g(r,s,t) \in G \) and \( f \in \Sigma \) satisfy

(2.10) \[
\left( \frac{(Q_{\beta}^{\alpha-1} f(z))^{(j)}}{(Q_{\beta}^{\alpha} f(z))^{(j)}} , \frac{(Q_{\beta}^{\alpha-2} f(z))^{(j)}}{(Q_{\beta}^{\alpha-1} f(z))^{(j)}} , \frac{(Q_{\beta}^{\alpha-3} f(z))^{(j)}}{(Q_{\beta}^{\alpha-2} f(z))^{(j)}} \right) \in D \subset C^3
\]
and

(2.11) \[
\frac{g \left( \frac{(Q_{\beta}^{\alpha-1} f(z))^{(j)}}{(Q_{\beta}^{\alpha} f(z))^{(j)}} , \frac{(Q_{\beta}^{\alpha-2} f(z))^{(j)}}{(Q_{\beta}^{\alpha-1} f(z))^{(j)}} , \frac{(Q_{\beta}^{\alpha-3} f(z))^{(j)}}{(Q_{\beta}^{\alpha-2} f(z))^{(j)}} \right)}{< 1, \text{ for all } z \in U; \alpha, \beta > 0}
\]

for some \( \alpha \in N \). Then we have

(2.12) \[
\left| \frac{(Q_{\beta}^{\alpha-1} f(z))^{(j)}}{(Q_{\beta}^{\alpha} f(z))^{(j)}} \right| < 1, \ (j \in N, z \in U).
\]

**Proof.** Let

\[
\frac{(Q_{\beta}^{\alpha-1} f(z))^{(j)}}{(Q_{\beta}^{\alpha} f(z))^{(j)}} = w(z).
\]

Obviously \( w(z) \) is analytic in \( U \), \( w(0) = 1 \) and \( w(z) \neq 1 \). With the aid of the identity

\[
z(Q_{\beta}^{\alpha} f(z))^{(j+1)} = (\beta + \alpha - 1)(Q_{\beta}^{\alpha-1} f(z))^{(j)} - (\beta - j - 1)(Q_{\beta}^{\alpha} f(z))^{(j)} \ (\alpha > 1, \beta > 0),
\]

and proceed exactly same method describe in Theorem 2.1, we easily get

\[
\frac{(Q_{\beta}^{\alpha-2} f(z))^{(j)}}{(Q_{\beta}^{\alpha-1} f(z))^{(j)}} = \frac{zw(z)}{(\beta + \alpha - 2)w(z)} + \frac{(\beta + \alpha - 1)w(z) - 1}{(\beta + \alpha - 2)},
\]
and
\[
\frac{(Q^a f(z))^j}{(Q^a f(z))^j} = \frac{1}{\beta + \alpha - 3} \left[ \frac{zw^n(z) + 2(\beta + \alpha - 1)w(z)}{w'(z)} - \frac{1}{1 + (\beta + \alpha - 1)\frac{(w(z))^2}{zw'(z)} - \frac{w(z)}{zw'(z)}} - (\beta + \alpha - 1)w(z). \right]
\]

We can see that
\[
\frac{(Q^a f(z_0))^j}{(Q^a f(z_0))^j} = e^{\theta},
\]
\[
\frac{(Q^a f(z_0))^j}{(Q^a f(z_0))^j} = \frac{\zeta}{\beta + \alpha - 2} + \frac{(\beta + \alpha - 1)e^{\theta} - 1}{(\beta + \alpha - 2)},
\]
and
\[
\frac{(Q^a f(z_0))^j}{(Q^a f(z_0))^j} = \frac{1}{\beta + \alpha - 3} \left\{ \frac{(M - 1) + 2(\beta + \alpha - 1)e^{\theta}}{1 + \frac{(\beta + \alpha - 1)e^{\theta} - 1}{\zeta}} - (\beta + \alpha - 1)e^{\theta} \right\}
\]
where \( M = z_0w^n(z_0) \) and \( \zeta \geq 1 \).

Since \( g(r, s, t) \in G \), we have
\[
\left| g \left( \frac{(Q^a f(z_0))^j}{(Q^a f(z_0))^j}, \frac{(Q^a f(z_0))^j}{(Q^a f(z_0))^j}, \frac{(Q^a f(z_0))^j}{(Q^a f(z_0))^j} \right) \right| \geq 1,
\]
where \( \text{Re}(M) \geq 1 \).

This contradicts the condition (2.10) of the Theorem 2.2. Therefore we conclude that
\[
\left| \frac{(Q^a f(z_0))^j}{(Q^a f(z_0))^j} \right| < 1, \quad (j \in N, z \in U).
\]
This completes the proof of the theorem.

**Corollary 2.2.** Let \( g(r,s,t) = s \) and \( f \in \Sigma \) satisfy the condition in Theorem 2.2. Then

\[
\left| \frac{(Q_{\alpha}^{\alpha+i-1} f(z))^{(j)}}{(Q_{\alpha}^{\alpha+i} f(z))^{(j)}} \right| < 1, \quad (i = 0,1,2,\ldots,\alpha > 4, i, j \in N, z \in U).
\]

## 3 An Open Problem

In this paper, we obtain some differential inequalities for the two integral operators \( P_{\alpha}^{\alpha} f(z) \) and \( Q_{\alpha}^{\alpha} f(z) \). Is it possible to generalize these results for meromorphic multivalent functions?

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**References**

