Coefficient Estimates for Certain Class of Analytic Functions

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Abstract

In this paper, we consider some coefficient estimates for certain class of analytic functions denoted by $CL(\beta)$, and satisfying $|K^2(f, z) - (1 - \beta)| < 1 - \beta$, ($\frac{1}{2} < \beta < 1$).

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1 Introduction

Let $H$ denote the class of analytic functions which are in the open unit disc $U = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $A$ denote the subclass of $H$ consisting of functions normalized by $f(0) = 0, f'(0) = 1$.

Let functions $f$ and $g$ be analytic in $U$. Then a function $f$ is said to be subordinate to a function $g$, if there exists a function $w$, analytic in $U$ such that $w(0) = 0, |w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. We denote this subordination by ($f \prec g$). In particular, if a function $g$ is univalent in $U$ we have the following equivalence

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$
Robertson introduced in [1] the classes $S^*(\beta), C^*(\beta)$ of starlike and convex functions of order $\beta \leq 1$, in $U$ which is defined by:

$$S^*(\beta) = \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \quad (z \in U),$$

and

$$C^*(\beta) = \text{Re} \left\{ \frac{zf''(z)}{f(z)} + 1 \right\} > \beta \quad (z \in U).$$

If $(0 \leq \beta < 1)$, then a function in either of this set is univalent. If $\beta < 0$, then it may fail to be univalent.

In this way, many interesting classes of analytic functions have been studied such as the class $SL^*$ which is defined by

$$SL^* = \left\{ f \in A : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \right\}, \quad z \in U.$$  

and can be found in [5].

And also $SL^* \subset SS^*(\frac{1}{2}) \subset S^*$, is the class $SS^*(\beta)$ of strongly starlike functions of order $\beta$ and given by

$$SS^*(\beta) = \left\{ f \in A : \left| \text{Arg} \frac{zf'(z)}{f(z)} \right| < \frac{\beta \pi}{2} \right\}, \quad (0 < \beta \leq 1),$$

which was studied in [2].

Moreover, $k-ST \subset SL^*$, for $k \geq 2 + \sqrt{2}$, where $k-ST$ is the class of $k$-starlike functions introduced in [3] such that

$$k-ST := \left\{ f \in A : \text{Re} \left| \frac{zf'(z)}{f(z)} \right| > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\}, \quad k \geq 0.$$ 

Let us denote

$$K(f, z) := \frac{zf''(z)}{f'(z)}.$$ 

In this paper we consider the class $CL(\beta)$, given by

$$CL(\beta) := \left\{ f \in A : \left| K^2(f, z) - (1 - \beta) \right| < 1 - \beta \right\}. \quad (1.1)$$

Notice that, $L := \{ w \in \mathbb{C}, \Re(w) > 0 : |w^2 - (1 - \beta)| < 1 - \beta \}$ is the interior of the right half of the lemniscate of Bernoulli $\gamma : (x^2 + y^2)^2 - 2(1 - \beta)(x^2 - y^2) = 0$. It can be verified that $\{ w : |w^2 - 1| < 1 \} \subset L$, for $\frac{1}{2} < \beta < 1$, thus $SL^* \subset CL(\beta)$.

Also, it is easy to see $L \subset \{ w : |\text{Arg}w| < \frac{\beta \pi}{2} \}$, thus $CL(\beta) \subset SS^*(\beta)$. 

Corollary 1.1 If \( k \geq 2 + \sqrt{2} \), \( \frac{1}{2} < \beta < 1 \), then 
\[
k - ST \subset CL(\beta) \subset SS^*(\beta) \subset S^*.
\]

**Theorem 1.2** The function \( f \) belongs to the class \( CL(\beta) \) if and only if there exists an analytic function 
\[
q \in H, q(z) \prec q_0(z) = \sqrt{(1 - \beta)z + (1 - \beta)}, q_0(0) = \sqrt{1 - \beta},
\]
such that
\[
f(z) = \int_0^z y \exp \int_0^y \frac{q(t) - 1}{t} \, dt \, dy.
\]

Let
\[
q_1(z) = \frac{3 + 2z}{3 + z}, \quad q_2(z) = \frac{5 + 3z}{5 + z}, \quad q_3(z) = \frac{8 + 4z}{8 + z},
\]
\[
q_4(z) = \frac{8 + 4z}{8 + z} \quad \text{and} \quad q_5(z) = \frac{9 + 5z}{9 + z}.
\]

Now, since \( q_i(z) \prec q_0(z) \) for \( i = 1, 2, 3, 4 \), we have then by (1.2) that
\[
\begin{align*}
f_1(z) &= \frac{z^2 + z^3}{2 + z} , \\
f_2(z) &= \frac{1}{2} z^2 + \frac{2}{15} z^3 + \frac{z^4}{100} , \\
f_3(z) &= \frac{1}{2} z^2 + \frac{2}{8} z^3 + \frac{1}{128} z^4 + \frac{1}{2560} z^5 , \\
f_4(z) &= \frac{1}{2} z^2 + \frac{4}{27} z^3 + \frac{1}{54} z^4 + \frac{4}{3645} z^5 + \frac{1}{39366} z^6 .
\end{align*}
\]

2 Main results

**Theorem 2.1** If a function \( f(z) = z + a_2 z^2 + \ldots + a_n z^n \) belongs to the class \( CL(\beta) \), then 
\[
\sum_{k=2}^{\infty} k^2(k - 1 - 2(1 - \beta))|a_k|^2 \leq 2(1 - \beta), \quad \left( \frac{1}{2} < \beta < 1 \right).
\]

**Proof.** If a function \( f \in CL(\beta) \), then \( K(f, z) \prec q_0(z) = \sqrt{(1 - \beta)z + (1 - \beta)} \). Thus we can write \( K(f, z) = \sqrt{(1 - \beta)w(z) + (1 - \beta)} \), where \( w \) satisfies \( w(0) = 0, |w(z)| < 1 \) for \( |z| < 1 \), and
\[
K(f, z) = \frac{zf''(z)}{f'(z)}.
\]
Whence
\[(1 - \beta)(f'(z))^2 = (zf''(z))^2 - (1 - \beta)w(z)(f'(z))^2.\]

From this we can obtain
\[2(1 - \beta)\pi \sum_{k=1}^{\infty} k^2|a_k|^2r^{2k-2} = (1 - \beta) \int_{0}^{2\pi} |f'(re^{i\theta})|^2 \ d\theta\]
\[\geq (1 - \beta) \int_{0}^{2\pi} |w(re^{i\theta})||f'(re^{i\theta})|^2 \ d\theta\]
\[= \int_{0}^{2\pi} \left|(re^{\theta}f''(re^{i\theta})^2 - (1 - \beta)f'(re^{i\theta})^2\right|d\theta\]

4(1 - \beta)\pi \sum_{k=1}^{\infty} k^2|a_k|^2r^{2k-2} \geq 2\pi \sum_{k=1}^{\infty} k^2(k - 1)|a_k|^2r^{2k-2},

for 0 < r < 1,

and derive into the following inequalities
\[2(1 - \beta)\sum_{k=1}^{\infty} k^2|a_k|^2r^{2k-2} \geq \sum_{k=1}^{\infty} k^2(k - 1)|a_k|^2r^{2k-2}\]
\[2(1 - \beta)\sum_{k=1}^{\infty} k^2|a_k|^2r^{2k-2} - \sum_{k=1}^{\infty} k^2(k - 1)|a_k|^2r^{2k-2} \geq 0.\]

Eventually, if we let \(r \to 1^-\), then we obtain
\[\sum_{k=2}^{\infty} k^2(k - 1 - 2(1 - \beta))|a_k|^2 \leq 2(1 - \beta).\]

**Corollary 2.2** If a function \(f(z) = z + a_2z^2 + a_3z^3 + .... + a_nz^n\) belongs to the class \(CL(\beta)\), then
\[|a_k| \leq \sqrt{\frac{2(1 - \beta)}{k^2(k - 1 - 2(1 - \beta))}} \text{ for } k \geq 2 \text{ and } \frac{1}{2} < \beta < 1.\]

**Theorem 2.3** If a function \(f(z) = \sum_{k=1}^{\infty} a_kz^k\) belongs to the class \(CL(\beta)\), then
\[|a_2| \leq \frac{1}{4}, \quad |a_3| \leq \frac{1}{6}, \quad |a_4| \leq \frac{1}{8} \quad (2.2)\]

These results are sharp.
Proof: If \( f(z) = \sum_{k=1}^{\infty} a_k z^k \) belongs to the class \( CL(\beta) \), then \((1-\beta)(f'(z))^2 = (zf''(z))^2 - (1-\beta)w(z)f'(z)^2\), where \( w \) satisfies \( w(0) = 0, \ |w(z)| < 1 \ for \ |z| < 1 \).

Let us denote
\[
(zf''(z))^2 = \sum_{k=2}^{\infty} A_k z^k, \quad f'^2(z) = \sum_{k=0}^{\infty} B_k z^k, \quad w(z) = \sum_{k=1}^{\infty} C_k z^k.
\]

Then we have
\[
A_k = 4(a_2)^2 z^2 + 24a_2 a_3 z^3 + (48a_2 a_4 + 36a_3^2)z^4 + (80a_2 a_5 + 144a_3 a_2)z^5 + \ldots,
\]
and
\[
B_k = a_1 z^2 + 2a_2 z^3 + (2a_3 + a_2)z^4 + (2a_2 a_4 - a_3^2 - a_1 a_5)z^6 + \ldots,
\]
such that
\[
\sum_{k=2}^{\infty} A_k z^k - (1-\beta) \sum_{k=0}^{\infty} B_k z^k = \sum_{k=1}^{\infty} C_k z^k \sum_{k=0}^{\infty} B_k z^k.
\]

Thus we obtain
\[
A_2 = 4a_2, \quad A_3 = 24a_2 a_3, \quad A_4 = 48a_2 a_4 + 12a_3^2, \quad A_5 = 80a_2 a_5 + 144a_3 a_2, \quad (2.6)
\]
and
\[
B_0 = a_1^2 = 1, \quad B_1 = 4a_1 a_2 = 4a_2, \quad B_2 = 6a_3 a_1 + 4^2, \quad B_3 = 8a_1 a_4 + 12a_2 a_3. \quad (2.7)
\]

Equating second and third coefficients of (2.5), we obtain
(I) \(- (1-\beta)B_1 = (1-\beta)B_0 C_1\).

(II) \(A_2 - (1-\beta)B_2 = (1-\beta)(C_2 B_0 + C_1 B_1)\).

(III) \(A_3 - (1-\beta)B_3 = (1-\beta)(C_3 B_0 + C_2 B_1 + C_1 B_2)\).

And so by (2.5), (2.6) and (2.7), we have
\[
a_2 = \frac{1}{4} C_1, \quad a_3 = -\frac{1}{6} C_2 + \frac{1}{6} C_1^2,
\]
\[
a_4 = \frac{1}{8} C_3 + \frac{(3-2\beta)}{24(1-\beta)} C_1 C_2 - \frac{1}{32} C_1^2 + \frac{(2-\beta)}{16(1-\beta)} C_1^3
\]

Using a well-known inequality \( |C_k| \leq 1, \sum_{k=1}^{\infty} |C_k|^2 \leq 1 \), we obtain (2.2).

Other work related to coefficient problems for different class of analytic functions can be found in ([6],[7],[8]).
3 Open Problem

The method here is employed from the work done by Sokol [4]. It is interesting to see other results in a similar technique for different classes such as the starlike subclasses with respect to symmetric points. We did try, but failed to get any results and it is left for the readers to tackle this problem.

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References


